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Anything is Possible: On the Existence and Uniqueness of Equilibria in the Shleifer-Vishny Model of Limits of Arbitrage

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Abstract. This paper characterizes equilibria in the Shleifer-Vishny model of limits of arbitrage. To prove existence, one has to consider types of equilibria ignored by Shleifer and Vishny, even if one adopts their parameter restrictions. For example, the only equilibrium may be one in which maximization of expected wealth requires an "all-or-nothing" investment strategy, with intermediate investment levels being strictly unprofitable. If one goes beyond Shleifer and Vishny's parameter restrictions, multiple equilibria may arise, and equilibrium selection may be governed by sunspots. Moreover, there may exist an equilibrium in which the asset price returns to fundamentals despite worsening noise trader sentiment.

JEL Classification: G12

1. Introduction

In an ingenious and apparently simple model, Shleifer and Vishny (1997) (henceforth: SV, see also Chapter 4 in Shleifer, 2000) show that the presence of noise traders in a financial market with performance-based arbitrage (PBA) possibly gives rise to limits of arbitrage and interesting asset price dynamics. The idea is that pessimistic noise trader sentiment deepens in the market for an already undervalued asset. On the one hand, the ensuing fall in prices makes arbitrage (a long position) even more profitable. On the other hand, if the resulting negative returns are interpreted as a signal of arbitrageurs' low abilities and money is withdrawn from them (PBA), then they lack the funds required to exploit the improved arbitrage opportunity. That is, arbitrage becomes less effective, or even breaks down completely, precisely when it is most profitable. More generally, the SV model shows that noise traders have an impact on asset prices even if rational arbitrageurs try to exploit the mispricing they cause (see also DeLong et al., 1990a, 1990b and Chapters 2 and 6 in Shleifer, 2000). Since this is necessary in order for asset prices to deviate from fundamentals

[1] For helpful comments I am indebted to Bernard Dumas, Shengzhi Liu, Susanne Steger, and an anonymous referee. Any remaining errors are my responsibility.

due to irrational trading behavior of some market participants, the SV model is a cornerstone of behavioral finance (see, e.g., Shleifer, 2000, p. 4, and Barberis and Thaler, 2003, pp. 1056–1057).

SV's Theorem 1 (p. 44) states that an equilibrium exists, in which arbitrageurs invest either part of or all the funds under their management, depending on whether expected wealth (their objective function) is flat or increasing, respectively. The present paper provides a complete characterization of equilibria in the SV model. While our analysis confirms that an equilibrium exists, it turns out that to establish existence, we have to consider several types of equilibria ignored by SV. For instance, the only equilibrium may be one in which maximization of expected wealth requires an "all-or-nothing" investment strategy. The reason why different types of equilibria have to be considered is an endogenous non-concavity of the expected wealth function. To get an idea of the problem, notice that an increase in investment has a twofold effect on an arbitrageur's expected wealth: it increases wealth if the price recovers, but, because of PBA, decreases the amount of funds under management if the noise trader shock worsens (and arbitrage is most profitable). The impact of each effect on the objective function is linear in SV's specification. Suppose that arbitrageurs invest part of, but not all, the funds under their management and that the ensuing aggregate investment yields market-clearing prices such that the two effects balance out *at the margin*. Due to linearity, they balance out for all investment levels *as long as both effects remain present*. Suppose, however, that the fall in prices in the case of worsening noise trader sentiment is sufficiently strong, and PBA is sufficiently aggressive, such that arbitrageurs would lose all the funds under their management if they chose to be fully invested. This implies that the wealth-reducing effect of an increase in investment via PBA vanishes for large investment levels (when all funds are lost anyway), and expected wealth rises with increases in investment. As a result, the expected wealth function is non-concave,[1] full investment is the wealth-maximizing investment choice, and an equilibrium with less-than-full investment does not exist. Under these circumstances, equilibrium possibly requires that some arbitrageurs are fully invested and others do not invest at all, and the respective proportions of arbitrageurs adjust such that both investment strategies are equally profitable, whereas intermediate investment levels yield lower expected wealth (i.e., the expected wealth function becomes V-shaped). The non-concavity of the expected wealth function is endogenous in that it depends on the severity of the fall in the asset's price. A natural way to overcome this non-concavity problem is to introduce a no-default constraint to the arbitrageurs' expected wealth maximization problem.[2]

[1] It has the familiar shape (first flat, then kinked upward) of the function relating the value of an option to its underlying or of the return on an indivisible investment project financed with a limited-liability fixed-interest loan to the payoff of the project.

[2] The idea to introduce a constraint that rules out default and, therefore, the non-concavity problem was suggested to me by the anonymous referee and is elaborated in detail in Section 9.

Consideration of equilibria of the types considered by SV is then sufficient to establish existence.[3]

SV do not explicitly make a parameter distinction underlying their Theorem 1 (p. 44). In their comparative-statics analysis, they add a "stability condition", which "basically says that arbitrageurs do not lose so much funds that in equilibrium they bail out of the market completely" (p. 46) if noise trader misperceptions deepen. Our analysis shows that an all-or-nothing equilibrium possibly arises for parameters which satisfy this stability condition. A simplified existence proof based on the types of equilibria considered by SV is possible only for a strict subset of the parameter combinations which satisfy SV's condition. Moreover, for parameters which violate the stability condition, multiple equilibria may arise, and contrary to SV's conjecture, arbitrageurs do *not* necessarily drop out of the market completely. The source of multiplicity of equilibria is a positive feedback effect inherent in the SV model: a higher asset price leads to an increase in funds under management because of PBA; this allows higher investments, thereby driving up the asset price. Multiplicity of equilibria naturally creates scope for equilibrium selection via sunspots. Interestingly, there possibly exists a sunspot equilibrium in which the realization of the sunspot variable happens not until period two.

SV confine attention to situations in which "the arbitrage resources are not sufficient to bring the period-two price to fundamental value, unless of course noise trader misperceptions have corrected anyway" (p. 39). Our analysis shows that this is a severe restriction of the focus of the analysis: when the stability condition is violated, there may exist an equilibrium in which the asset price returns to fundamentals despite worsening noise trader sentiment. In such an "early recovery equilibrium", the asset price returns to its fundamental value irrespective of whether the noise trader shock worsens or not. As arbitrageurs make a positive return on their investments with certainty, they choose to be fully invested. If PBA is sufficiently pronounced (in particular, more pronounced than compatible with SV's stability condition), they are in fact rewarded with the accrual of a sufficiently large amount of additional funds so as to counterbalance the worsening noise trader shock. The fact that this type of equilibrium is not needed in the existence proof implies that it generally co-exists with other types of equilibria, as one would expect given its self-fulfilling nature. Evidently, as the asset price rises with certainty, adding the no-default constraint is inessential in this regard.

The analysis can be motivated in several ways. First, as the SV model is a cornerstone of the theory of behavioral finance, a thorough equilibrium analysis is called for from the theorist's point of view. One might object that the SV model is an example, rather than a full-blown theory, so that a rigorous equilibrium

[3] To be precise, besides full investment, one has to take into account the converse corner solution with zero investment.

analysis would be dispensable (which is not our preferred interpretation of the model). Even so, it is a parameterized model, and our analysis shows that specific parameterizations possibly give rise to subtle existence problems. Second, the analysis not only establishes existence; it highlights that different parameterizations lead to qualitatively different types of equilibria with probably surprising properties, such as the equilibrium with all-or-nothing investment strategies. Third, the results may help in understanding real-world phenomena. For instance, the all-or-nothing equilibrium might contribute to the explanation of why different kinds of funds are engaged in different markets: if a fund were able to invest in two markets like the one described by the SV model and in each market maximization of expected wealth entailed either full or zero investment, it would choose to invest in only one of them. Multiplicity of equilibria and sunspots contribute to an explanation of why markets move in the absence of new information and are thus excessively volatile. Fourth, the possibility of early recovery means that there is scope for fundamental valuation in the SV model. This result is related to recent work by Loewenstein and Willard (2006), who demonstrate that mispricing can generally be ruled out in a class of infinite-horizon models including DeLong et al.'s (1990a) noise trader model.[4]

The remainder of this paper is organized as follows. Section 2 briefly recapitulates the assumptions of the SV model. Equilibrium is defined in Section 3. Section 4 illustrates the main results of the subsequent analysis by means of two numerical examples. Section 5 is concerned with those model parameters which allow an existence proof using the types of equilibria introduced by SV. Sections 6 through 8 establish existence, and characterize the different types of equilibria, for the remainder of the parameter space. Section 9 shows how the no-default constraint circumvents the non-concavity problem. Section 10 concludes.[5]

2. Model

Since it is our aim to analyze the equilibria of the SV model, we do not change any of its assumptions, so the exposition can be kept brief. Consider an asset market with three types of agents, noise traders, arbitrageurs, and investors in arbitrage

[4] Loewenstein and Willard's (2006) argument goes as follows: according to the "Fundamental Theorem of Asset Pricing" (cf., e.g., Cochrane, 2005, p. 70), the absence of arbitrage opportunities implies the existence of a stochastic discount factor, which can be used to value payment streams and calculate the fundamental value of an asset. Very mild assumptions (e.g., the presence of upper and lower bounds on its price) are sufficient to rule out deviations from fundamental value (cf. Diba and Grossman, 1988, and Santos and Woodford, 2000). This powerful argument is not directly applicable to the SV model, for the assumption of free portfolio formation (cf. Cochrane, 2005, p. 62) is explicitly violated.

[5] Details of derivations are available at the author's website www.tvwl.de.

funds. Time is discrete, and there are three time periods. The supply of the asset is inelastic and normalized to unity. The asset's fundamental value is V (>0). So the asset is undervalued or valued correctly, depending on whether nominal demand is less than or equal to V, respectively. At time 3, the asset is valued correctly.

The noise traders' nominal demand for the asset in period 1 is $V - S_1$, where $0 < S_1$. At time 2, with probability q ($0 < q < 1$), their demand is $V - S_2$, where $S_1 < S_2 < V$ (noise trader misperceptions deepen);[6] with probability $1 - q$, on the other hand, their demand is V, and the asset price returns to its fundamental value.

There is a continuum of unit length of identical arbitrageurs.[7] The amount of funds under their management in period 1, F_1, is exogenous and satisfies $0 < F_1 < S_1$. Funds under management if noise trader misperceptions deepen in period 2 are denoted F_2. The arbitrageurs' investments in the asset at times $t = 1$ and $t = 2$ are denoted D_1 and D_2, respectively, where $0 \leq D_t \leq F_t, t \in \{1, 2\}$. Non-invested funds are stored at zero interest. Let p_1, p_2, and x denote the period-1 price, the period-2 price in case of worsening noise trader expectations, and the gross return on F_1, respectively. If noise trader misperceptions deepen, then $x = 1 + (p_2/p_1 - 1)D_1/F_1$. Due to PBA, period-2 assets under control of the arbitrageurs are $F_2 = \max\{F_1(ax + 1 - a), 0\}$, where the parameter a measures the extent to which past performance affects the availability of funds.[8] The restriction that the minimum admissible value of F_2 is zero is not spelled out explicitly by SV (cf. SV, eqn. (6), p. 41), but it is implicit in their discussion of the case in which "arbitrageurs bail out of the market completely" (p. 46) and the total amount of period-2 investment coincides with the noise traders' demand, $V - S_2$.[9] We focus on the case $a > 1$.[10] In characterizing the model equilibrium, SV do not make any further parameter assumptions. In their comparative statics analysis, they add the following stability condition:

$$a < 1 + \frac{V - S_1}{F_1} \tag{1}$$

[6] SV do not assume $S_2 < V$ explicitly, but the condition is implied by positive noise trader demand for the asset.

[7] This is equivalent to SV's (p. 39) assumption of many price-taking arbitrageurs and helps simplify the notation.

[8] The most promising route for future research appears to model the underlying signal extraction problem explicitly. As said above, we do not attempt to address this or other criticisms of the SV model. We adopt this important model as it stands and analyze its formal properties.

[9] SV (p. 38) define F_2 as "cumulative funds under management including their borrowing capacity". Given this interpretation, one has to assume that the borrowing capacity is also affected by PBA. An alternative interpretation of the model is that arbitrageurs do not trade on their own account and have no other source of funds besides the investors.

[10] The case $a = 1$ will be treated in footnote 16.

(p. 46). It will turn out that this condition has an important impact, not only on the comparative statics, but also on the nature of equilibrium. So we do not impose it as an assumption, but use it as a case distinction. We say the SV case prevails if (1) holds. Arbitrageurs maximize final wealth, W, in period 2 and expected final wealth, EW, in period 1.[11]

3. Equilibrium: Definition

The period-1 asset price is

$$p_1 = V - S_1 + D_1.^{12} \tag{2}$$

If noise trader expectations deepen,

$$p_2 = V - S_2 + D_2 \tag{3}$$

$$F_2 = \max\left\{F_1 + aD_1\left(\frac{p_2}{p_1} - 1\right), 0\right\} \tag{4}$$

$$W \equiv F_2 + \left(\frac{V}{p_2} - 1\right) D_2. \tag{5}$$

If, on the other hand, noise trader expectations recover, then the period-2 price is V and $W = F_1 + aD_1(V/p_1 - 1)$. The arbitrageurs' investments D_1 and D_2 maximize (expected) wealth:

$$D_1 = \arg\max_{D_1} : EW \text{ s.t.: (4) and } 0 \leq D_1 \leq F_1 \tag{6}$$

$$D_2 = \arg\max_{D_2} : W \text{ s.t.: } 0 \leq D_2 \leq F_2. \tag{7}$$

Definition. *An equilibrium is a tuple* $(p_1, p_2, D_1, D_2, F_2, W) \in \mathbb{R}^6_+$ *that satisfies* (2)-(7).

Equations (2)-(4) correspond to Equations (3), (2), and (6), respectively, in SV (pp. 39, 41). As for (4), notice that when funds under management decrease, the change $F_2 - F_1$ (<0) can be divided into the change in the value of the arbitrageurs' portfolios, $D_1(p_2/p_1 - 1)$, and the change due to withdrawals, $(a - 1)D_1$ $(p_2/p_1 - 1)$ (cf. SV, p. 41). Our definition of an equilibrium generalizes SV's in two respects. First, SV (p. 43) allow for interior solutions as well as for a corner

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[11] In SV's (p. 39) interpretation, maximization of expected wealth is necessary to break even due to Bertrand competition between arbitrageurs. Given the alternative interpretation in footnote 9, this is equivalent to profit maximization.

[12] SV (p. 46) state their stability condition as $aF_1 < p_1$ if $D_1 = F_1$. Evidently, given (2), this is equivalent to (1).

solution with $D_1 = F_1$ in (6) but not for a corner solution with $D_1 = 0$. We do not rule out $D_1 = 0$, and it will turn out that equilibria with $D_1 = 0$ may exist. Because of their similarity to the equilibria considered by SV, we count them as SV-type equilibria. Second, SV (p. 39) assume $F_2 < S_2$. Given (3) and $D_2 \leq F_2$, this means that the focus is on "undervaluation equilibria", in which $p_2 < V$ and, hence, $D_2 = F_2$ solves (7). However, as F_2 is endogenous, the inequality $F_2 < S_2$ may or may not hold, depending on model parameters, so we allow for both possibilities. It will turn out that early recovery equilibria with $p_2 = V$ (despite worsening noise trader expectations) and $D_2 < F_2$ may exist (albeit only for parameters which do not satisfy (1)). To find an undervaluation equilibrium it suffices to determine D_1 and p_2: D_1 determines p_1 via (2); together with p_2, these two variables determine F_2 via (4); given $D_2 = F_2$, this determines W via (5).

4. Two Examples

In this section, we illustrate by means of a numerical example that an equilibrium of the types investigated by SV may fail to exist, whereas an equilibrium exists in which the only investment strategies which maximize expected wealth are full or zero investment. The example also illustrates how a no-default constraint helps overcome this problem. A second example demonstrates the possibility of an early recovery equilibrium which coexists along with two other equilibria.

Example 1. *Let* $V = 1$, $F_1 = 0.1$, $a = 1.4$, $S_1 = 0.2$, $S_2 = 0.8$, *and* $q = 0.05$. *SV's stability condition (1) is satisfied* $(1.4 = a < 1 + (V - S_1)/F_1 = 9)$. *The response of period-2 funds to period-1 performance, as measured by a, appears moderate (SV, p. 44, propose* $a = 1.2$*; cf. Example 3 below).*

Following SV, we first pose the following question: is there an equilibrium with $p_2 < V$ *and* $F_2 > 0$ *and with either full or partial investment (i.e., with* $D_1 = F_1$ *or* $D_1 < F_1$*, respectively)? The answer will be in the negative. As mentioned above,* $p_2 < V$ *implies* $D_2 = F_2$. *Using the supposition* $F_2 > 0$, *(2)-(4) become*

$$p_1 = 0.8 + D_1 \tag{8}$$

$$p_2 = 0.2 + F_2 \tag{9}$$

$$F_2 = 0.1 + 1.4 D_1 \left(\frac{p_2}{p_1} - 1 \right). \tag{10}$$

Eliminating p_1 *and* F_2 *from (8)-(10) yields*

$$p_2 = \frac{(0.3 - 1.4 D_1)(0.8 + D_1)}{0.8 - 0.4 D_1}. \tag{11}$$

Suppose there is an equilibrium in which arbitrageurs are fully invested in period 1 (i.e., $D_1 = F_1$*). From (8) and (11),* $p_1 = 0.9$ *and* $p_2 = 0.1895$. *However, from*

(10), $F_2 = -0.0105$*, a contradiction. One might suspect that, as a full-investment equilibrium does not exist, there is an equilibrium in which arbitrageurs hold back funds in period 1 (i.e.,* $D_1 < F_1$*), as would be implied by SV's Proposition 1 (p. 44). To see that this is not the case, consider the expected wealth function, again imposing the supposition* $F_2 > 0$*:*

$$EW \equiv 0.95\left[0.1 + 1.4D_1\left(\frac{1}{p_1} - 1\right)\right] + \frac{0.05}{p_2}\left[0.1 + 1.4D_1\left(\frac{p_2}{p_1} - 1\right)\right]. \quad (12)$$

In an equilibrium with $0 < D_1 < F_1$*,* EW *must be constant in* D_1*, which implies* $p_2 = p_1/(20 - 19p_1)$ *or, using (8),*

$$p_2 = \frac{0.8 + D_1}{4.8 - 19D_1}. \quad (13)$$

Solving (11) and (13) yields $D_1 = 0.0617$ *and* $p_2 = 0.2375$*. From (8) and (10),* $p_1 = 0.8617$ *and* $F_2 = 0.0375$*, respectively. So have we found an equilibrium with less-than-full investment? The answer is no. To see this, reconsider the expected wealth function. Given the equilibrium prices and now taking the assumed non-negativity of* F_2 *into account explicitly, (12) becomes*

$$EW \equiv 0.95(0.1 + 0.2248D_1) + 0.2106\max\{0.1 - 1.0142D_1, 0\}.$$

For $D_1 < 0.0986$ *(=*$\bar{D}_1$ *in the left panel of Figure 1), the max-term is positive, and* EW *is in fact constant in* D_1*. However, for* $D_1 > 0.0986$*, the max-term becomes zero, so that* $d(EW)/dD_1 = 0.2135 > 0$*. Expected wealth is non-concave in* D_1 *(see the left panel of Figure 1), and each arbitrageur's optimal choice is to be fully invested (i.e.,* $D_1 = F_1 = 0.1$*), a contradiction. This answers the question posed at the outset in the negative: an equilibrium with* $p_2 < V$ *and* $F_2 > 0$ *and with either full or partial investment* (i.e., with $D_1 = F_1$ or $D_1 < F_1$, respectively) does not exist.[13] *Non-existence of an SV-type equilibrium is due to the endogenous non-concavity of the expected wealth function illustrated in the left panel of Figure 1. In Section 5, we use a stronger parameter restriction than (1), which rules out this non-concavity problem and is thus sufficient to establish existence of an SV-type equilibrium.*

13 This result depends on the assumption that the funds under management are distributed symmetrically across arbitrageurs. To see this, suppose, by contrast, that funds under management are 0.0985 for a measure 99% of "small" arbitrageurs and 0.2485 for the remaining 1% measure of "big" arbitrageurs. If the small arbitrageurs invest 0.0598 per capita and the big ones are fully invested, aggregate investment is $D_1 = 0.0617$, and the prices are as in the main text. An equilibrium prevails, because the small arbitrageurs' expected wealth is constant over the range of feasible investment levels (as their endowment of 0.0985 is lower than the value at which expected wealth starts to rise, viz., 0.0986) and the big arbitrageurs, whose expected wealth is maximal at full investment, are fully invested. In what follows, we confine attention to the case of uniform endowments.

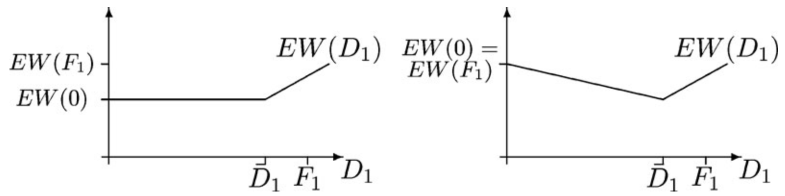

Figure 1. Expected wealth.

There is, however, an equilibrium of a different type. Consider the following all-or-nothing allocation: 63.06% of the arbitrageurs are fully invested and the remaining 36.94% do not invest at all. Aggregate investment is $D_1 = 0.0631$. *From (8),* $p_1 = 0.8631$. *Suppose* $F_2 = 0$ *for the fully invested arbitrageurs (this will be confirmed below). Then, as the non-investing arbitrageurs' funds remain unchanged at 0.1, aggregate funds under management in the case of worsening noise trader expectations equal* $F_2 = 0.0369$ *(=*$36.94\% \cdot 0.1$*), and, from (9), the period-2 price is* $p_2 = 0.2369$. *From (10), expected wealth is non-concave:* $F_2 = 0$ *for* $D_1 > 0.0985$ *(=*$\bar{D}_1$ *in the right panel of Figure 1). This confirms that the fully invested arbitrageurs lose all funds. From (12), zero investment yields expected wealth* $EW = 0.1161$. *The same level of expected wealth is obtained with full investment. As expected wealth is V-shaped in between these two extremes (*$d(EW)/dD_1 = -0.0033$ *for* $D_1 < 0.0985$*), both zero and full investment maximize expected wealth (see the right panel of Figure 1). Sections 6-8 establish existence of an equilibrium taking into account such all-or-nothing allocations.*

The existence proof in Sections 6-8 is quite intricate. A much easier way to establish existence is to modify the SV model and assume that the arbitrageurs plan their investments subject to the additional constraint that they incur no default risk. Then an SV-type equilibrium exists. In the current example, given prices $p_1 = 0.8617$ *and* $p_2 = 0.2375$ *and given* $D_1 = 0.0617$, *the value of an arbitrageur's portfolio before the investors' decision to withdraw funds is (*$F_1 + D_1(p_2/p_1 - 1) =$*)* 0.0553. *Assume investors get a zero rate of return. The amount withdrawn equals (the portfolio value before withdrawals minus* $F_2 = 0.0375$, *i.e.)* 0.0179. *Final wealth ((*$(V/p_2)F_2 =$*)* 0.1578 *is sufficient to service the remaining claims (*F_1 *minus withdrawals, i.e.) 0.0821. Hence, the no-default constraint is not binding. By contrast, whenever* $p_2 < p_1$, *any investment* D_1 *(*$>0.0986 = \bar{D}_1$*) which yields* $F_2 = 0$ *entails default in the case of worsening noise trader sentiment. This is because the portfolio value in period 1 is less than* F_1 *and there is no money left to generate positive returns in period 2. The no-default constraint rules out any such strategy. As a consequence, there is a partial-investment equilibrium in which all arbitrageurs choose* $D_1 = 0.0617$.

Example 2. *Let $V = 1$, $F_1 = 0.7$, $a = 2$, $S_1 = 0.8$, $S_2 = 0.85$, and $q = 0.5$. As $a = 2 > 1.2857 = 1 + (V - S_1)/F_1$, SV's condition (1) is violated.*

The formulas analogous to (8) and (11) are $p_1 = 0.2 + D_1$ and

$$p_2 = \frac{(0.85 - 2D_1)(0.2 + D_1)}{0.2 - D_1}, \tag{14}$$

respectively. Full investment implies $p_1 = 0.9$ and $p_2 = 0.99$ in an undervaluation equilibrium. The fact that $p_2 > p_1$ implies that arbitrageurs make a positive return with certainty, so that F_2 cannot become negative. Thus, an equilibrium with full investment exists. The formula analogous to (12) reads

$$EW \equiv 0.5\left[0.7 + 2D_1\left(\frac{1}{p_1} - 1\right)\right] + \frac{0.5}{p_2}\left[0.7 + 2D_1\left(\frac{p_2}{p_1} - 1\right)\right]. \tag{15}$$

Provided the second term in square brackets is positive, we have $d(EW)/dD_1 = 0$ if

$$p_2 = \frac{0.2 + D_1}{1.8 - D_1} \tag{16}$$

(cf. (13)). Solving (14) and (16) for D_1 yields $D_1 = 0.5816$. Given the ensuing prices, $p_1 = 0.7816$ and $p_2 = 0.6415$, the second term in square brackets in (15) is in fact positive whenever $D_1 < 1.9526$, which is implied by $D_1 \leq F_1 = 0.7$. So there also exists an equilibrium with less than full investment. Finally, besides these undervaluation equilibria, there is another equilibrium in which the price returns to its fundamental value even though the noise trader shock worsens. As the price rises with certainty in this case, arbitrageurs are fully invested in this type of equilibrium, so that $p_1 = 0.9$. The crucial requirement for early recovery is that period-2 funds under control (the second term in square brackets in (15)) are sufficient to counterbalance worsening noise trader sentiment. This condition is satisfied: $F_2 = 0.8556 > 0.85 = S_2$.

The no-default constraint is not binding in any of the three equilibria (and thus cannot be used to rule out any of these equilibria). As the price never falls, this is obvious for the two equilibria with full investment. In the equilibrium with partial investment, period-1 withdrawals are 0.1043, so that the remaining claims in period two are 0.5958, which is well below the final wealth of 0.7662. The multiplicity of equilibria creates scope for equilibrium selection via sunspots. Since there are two equilibria with the same level of period-1 investment (viz., full investment), expectations can be conditioned on a sunspot variable whose realization is observed not until period two.

5. A Special Case: SV-Type Equilibria

We now turn to the general characterization of equilibria. Our main theorems state existence of equilibrium, and the proofs highlight the possible emergence of the different types of equilibria. To make the argument as transparent as possible, we start in the present section with a simple special case, which allows us to confine attention to the types of equilibria proposed by SV:

$$a < 1 + \frac{V - S_2}{S_2 - S_1 + F_1}. \tag{17}$$

The right-hand side of (17) is strictly less than the right-hand side of (1). This means that we are focussing on a strict subset of the parameters which satisfy SV's condition (1). Section 6 is concerned with parameters which satisfy (1) but not (17), and Sections 7 and 8 will turn to parameters which violate (1). For now, we restrict attention to undervaluation equilibria, with $p_2 < V$. As noted in Section 3,

$$D_2 = F_2 \tag{18}$$

in an undervaluation equilibrium, and to find an equilibrium solution to the model, it suffices to pin down D_1 and p_2. We first show how the arbitrageurs' aggregate investment, D_1, affects p_2 given market clearing. Then we deal with the question of how p_2 affects the choice of D_1.

The Dependence of p_2 on D_1

Market clearing in periods 1 and 2 (i.e., (2)-(4) and (18)) implies

$$p_2 = \max\{A(D_1) + F(D_1)p_2, V - S_2\}, \tag{19}$$

where

$$A(D_1) \equiv V - S_2 - aD_1 + F_1,\ F(D_1) \equiv \frac{aD_1}{V - S_1 + D_1}. \tag{20}$$

There is a positive feedback effect inherent in (19): higher period-2 prices, p_2, raise period-2 funds under control, F_2, thereby driving up p_2. $F(D_1)$ is measure of the strength of this positive feedback effect. We have

$$A(D_1) \underset{<}{\overset{>}{=}} 0 \Leftrightarrow D_1 \underset{>}{\overset{<}{=}} \frac{V - S_2 + F_1}{a} \equiv D_1^0. \tag{21}$$

Furthermore, $F(D_1) > 0$ and

$$F(D_1) \underset{>}{\overset{<}{=}} 1 \Leftrightarrow D_1 \underset{>}{\overset{<}{=}} \frac{V - S_1}{a - 1} \equiv D_1^\infty. \tag{22}$$

Notice that

$$F_1 \lessgtr D_1^0 \Leftrightarrow a \lessgtr 1 + \frac{V - S_2}{F_1} \tag{23}$$

and

$$F_1 \lessgtr D_1^\infty \Leftrightarrow a \lessgtr 1 + \frac{V - S_1}{F_1}. \tag{24}$$

From (23) and (24), the assumed validity of (17), together with $S_2 > S_1$, implies $F_1 < \min\{D_1^0, D_1^\infty\}$.[14]

Let $p_2(D_1) : [0, F_1] \to \mathbb{R}$ denote the solution to (19) that obtains if the max-operator is ignored:[15]

$$p_2(D_1) \equiv \frac{A(D_1)}{1 - F(D_1)} = \frac{(V - S_2 - aD_1 + F_1)(V - S_1 + D_1)}{V - S_1 - (a-1)D_1}. \tag{25}$$

For future reference, notice that if $D_1 < D_1^\infty$ (so that the denominator is positive), (25) implies

$$p_2(D_1) \underset{<}{\overset{>}{=}} V - S_2 \Leftrightarrow aD_1(S_2 - S_1 + D_1) \underset{>}{\overset{<}{=}} F_1(V - S_1 + D_1) \tag{26}$$

or

$$p_2(D_1) \underset{<}{\overset{>}{=}} V - S_2 \Leftrightarrow a \underset{>}{\overset{<}{=}} \frac{F_1}{D_1}\left(1 + \frac{V - S_2}{S_2 - S_1 + D_1}\right). \tag{27}$$

Hence,

$$p_2(F_1) \gtrless V - S_2 \Leftrightarrow a \lessgtr 1 + \frac{V - S_2}{S_2 - S_1 + F_1}. \tag{28}$$

Condition (17) is necessary and sufficient for $p_2(F_1) > V - S_2$. Since the term on the far right-hand side of (27) is decreasing in D_1, it follows that $p_2(D_1) > V - S_2$ for all $D_1 \in [0, F_1]$. From (25), $D_1 < D_1^\infty$ further implies

$$p_2(F_1) < V. \tag{29}$$

Solutions p_2 to (19) are illustrated in Figure 2. The figure depicts $A(D_1) + F(D_1)p_2$ as a function of p_2. Since $D_1 \le F_1 < \min\{D_1^0, D_1^\infty\}$, we have, from the definitions in (21) and (22), $A(D_1) > 0$ and $F(D_1) < 1$. So the line $A(D_1) + F(D_1)p_2$ has a positive intercept and its slope is less than one. $p_2(D_1)$ is found at its intersection

[14] When making case distinctions, we neglect the non-generic cases in which weak inequalities hold as an equality.

[15] This slight abuse of notation will remind us that the mapping p_2 gives the equilibrium period-2 price level (for a certain range of parameters determined below).

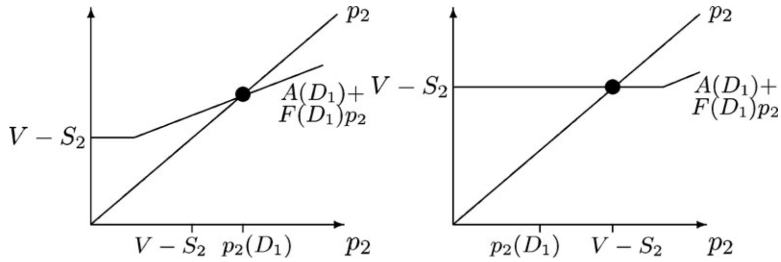

Figure 2. Equilibrium period-2 price level when (17) is satisfied.

with the 45-degree line. The right-hand side of (19) is given by the maximum of this line and the horizontal line at height $V - S_2$. p_2 is found at the intersection of this kinked line and the 45-degree line. As can be seen from Figure 2,

$$p_2 = \max\{p_2(D_1), V - S_2\}. \tag{30}$$

Since $p_2(D_1) > V - S_2$ for all $D_1 \in [0, F_1]$, it follows that

$$p_2 = p_2(D_1). \tag{31}$$

This, together with (25), is the relationship between period-1 aggregate investment and period-2 prices we are looking for.

The Dependence of D_1 on p_2

Next, we investigate how the arbitrageurs' period-1 investment choice, D_1, depends on the period-2 price level, p_2.

Expected wealth as of period 1 is

$$\begin{aligned} EW = (1-q)\left[F_1 + aD_1\left(\frac{V}{p_1} - 1\right)\right] \\ + q\frac{V}{p_2}\max\left\{F_1 + aD_1\left(\frac{p_2}{p_1} - 1\right), 0\right\} \equiv EW(D_1). \end{aligned} \tag{32}$$

Arbitrageurs maximize $EW : [0, F_1] \to \mathbb{R}$, given p_1 and p_2. An increase in D_1 has a twofold effect on expected wealth. For one thing, it raises the return in case of a return to fundamental valuation in period 2 (see the term in square brackets in (32)). For another, because of PBA, it reduces the amount of funds under management at time 2, F_2, if $p_2 < p_1$. The latter effect reduces final wealth because funds yield a positive safe return $V/p_2 - 1$ (>0) in an undervaluation equilibrium. The effect vanishes when the first term in the max operator in (32) is negative, i.e., when the arbitrageurs lose all funds if noise trader misperceptions deepen. This happens when $D_1 > \bar{D}_1$, where

$$\bar{D}_1 \equiv \frac{F_1}{a\left(1 - \frac{p_2}{p_1}\right)}. \tag{33}$$

For $D_1 < \bar{D}_1$, differentiating (32) yields

$$EW' = aV\left(\frac{1}{p_1} - \frac{1-q}{V} - \frac{q}{p_2}\right) \overset{>}{\underset{<}{=}} 0 \Leftrightarrow p_2 \overset{>}{\underset{<}{=}} \frac{q}{\frac{1}{p_1} - \frac{1-q}{V}}. \tag{34}$$

$EW' = aV(1-q)(1/p_1 - 1/V) > 0$ for $D_1 > \bar{D}_1$. If $EW' \leq 0$ for $D_1 < \bar{D}_1$ and $\bar{D}_1 < F_1$, then expected wealth is a non-concave function of period-1 investment, D_1 (see the left panel of Figure 1). Whether or not this non-concavity prevails is determined endogenously in that $\bar{D}_1$ depends on prices, p_1 and p_2.

Let $\psi(D_1) : [0, F_1] \to \mathbb{R}$ be given by

$$\psi(D_1) \equiv \frac{q}{\frac{1}{V - S_1 + D_1} - \frac{1-q}{V}}. \tag{35}$$

$\psi(D_1)$ is continuous and increasing and satisfies $0 < \psi(D_1) < V - S_1 + F_1$ $(< V)$ for all $D_1 \in [0, F_1]$. From (2), (34), and (35), we have

$$EW' \overset{>}{\underset{<}{=}} 0 \Leftrightarrow p_2 \overset{>}{\underset{<}{=}} \psi(D_1) \tag{36}$$

for $D_1 < \bar{D}_1$.

Recall that, due to the assumption that there is a unit measure of arbitrageurs, D_1 denotes both individual and aggregate investment, which have to be distinguished carefully. The aggregate investment level D_1 determines p_1 via (2) and p_2 via (25) and (31). Given the price levels, an individual arbitrageur's D_1 determines his expected wealth via (32), and (34) gives the change in his expected wealth when he changes D_1.

Existence

We are now in a position to prove the existence of equilibrium. The first lines of Table I define three different types of undervaluation equilibrium: partial-investment equilibrium (PIE), full-investment equilibrium (FIE), and zero-investment equilibrium (ZIE). In a PIE, expected wealth, EW, is independent of D_1, and the arbitrageurs invest part of, but not all, the funds under their management. In a FIE, the arbitrageurs are fully invested, as expected wealth is a non-decreasing function of D_1. In a ZIE, they do not invest any money in the asset in period 1, because expected wealth is non-increasing in D_1. From (31), $p_2 = p_2(D_1)$ in each type of equilibrium.

Table I Types of undervaluation equilibria This table summarizes the conditions that need to be fulfilled for the occurrence of the different types of undervaluation equilibria

	D_1	p_2	$\bar{D}_1$ vs. F_1	$EW(0)$ vs. $EW(F_1)$
PIE	$0 < D_1 < F_1$	$p_2 = p_2(D_1) = \psi(D_1)$	$\bar{D}_1 \geq F_1$	
FIE	$D_1 = F_1$	$p_2 = p_2(F_1) \geq \psi(F_1)$		
ZIE	$D_1 = 0$	$p_2 = p_2(0) \leq \psi(0)$	$\bar{D}_1 \geq F_1$	
FIEV	$D_1 = F_1$	$p_2 = p_2(F_1) < \psi(F_1)$	$\bar{D}_1 < F_1$	$EW(0) \leq EW(F_1)$
ZIEV	$D_1 = 0$	$p_2 = p_2(0) \leq \psi(0)$	$\bar{D}_1 < F_1$	$EW(0) \geq EW(F_1)$
TILE	$0 < D_1 < F_1$	$p_2 = V - S_2 + F_1 - D_1 < \psi(D_1)$	$\bar{D}_1 < F_1$	$EW(0) = EW(F_1)$

PIE

A PIE exists if there is D_1^* $(0 < D_1^* < F_1)$ such that $p_2 = \psi(D_1^*)$, EW is flat, and $p_2 < V$. EW is flat iff $\bar{D}_1$ does not fall short of F_1: $\bar{D}_1 \geq F_1$. The undervaluation condition $p_2 < V$ is satisfied: $p_2 = p_2(D_1^*) = \psi(D_1^*) < V$.

FIE

A FIE exists if $p_2(F_1) \geq \psi(F_1)$. From (29) and (31), the undervaluation condition $p_2 < V$ is satisfied. Whether or not $\bar{D}_1$ is in the interval $[0, F_1]$ is inessential.

ZIE

A ZIE exists if expected wealth is non-increasing in D_1, i.e., if $p_2(0) \leq \psi(0)$ and $\bar{D}_1 \geq F_1$. The undervaluation condition is satisfied: $p_2 = p_2(0) \leq \psi(0) < V$.

Theorem 1. *Given (17), an equilibrium (viz., a PIE, ZIE, or FIE) exists.*

Proof. Suppose $p_2 = p_2(D_1) < p_1$. Then, from (25) and (33),

$$\bar{D}_1 \overset{<}{\underset{>}{=}} F_1 \Leftrightarrow a\frac{S_2 - S_1 + D_1 - F_1}{V - S_1 - (a-1)D_1} \overset{>}{\underset{<}{=}} 1. \tag{37}$$

As $D_1 \leq F_1 < D_1^{\infty}$, the denominator of the fraction on the right-hand side of (37) is positive, and

$$\bar{D}_1 \overset{<}{\underset{>}{=}} F_1 \Leftrightarrow D_1 \overset{>}{\underset{<}{=}} \frac{V - S_1}{2a - 1}\left(1 - a\frac{S_2 - S_1 - F_1}{V - S_1}\right) \equiv \widetilde{D}_1. \tag{38}$$

From the definition of $\widetilde{D}_1$,

$$\widetilde{D}_1 \overset{>}{\underset{<}{=}} F_1 \Leftrightarrow a \overset{<}{\underset{>}{=}} 1 + \frac{V - S_2}{S_2 - S_1 + F_1}. \tag{39}$$

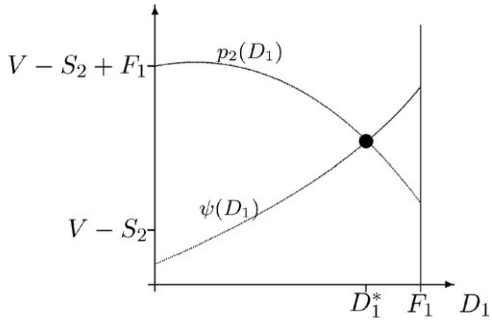

Figure 3. Existence of equilibrium when (17) is satisfied.

Condition (17) implies $\widetilde{D}_1 > F_1$ and, hence, $D_1 < \widetilde{D}_1$ for all $D_1 \in [0, F_1]$. The fact that, by (31), the supposition $p_2 = p_2(D_1)$ is satisfied implies $\bar{D}_1 > F_1$ whenever $p_2 < p_1$.

Let q rise from zero to unity. This raises $\psi(D_1)$ for all $D_1 \in [0, F_1]$ and leaves $p_2(D_1)$ unaffected. For q such that $\psi(F_1) \leq p_2(F_1)$, a FIE exists. If $\psi(F_1) > p_2(F_1)$ and $\psi(0) < p_2(0)$, by continuity of $\psi(D_1)$ and $p_2(D_1)$, there is D_1^* $(0 < D_1^* < F_1)$ such that $p_2(D_1^*) = \psi(D_1^*)$ (see Figure 3). So a PIE with $D_1 = D_1^*$ exists. Suppose $S_2 > S_1 + F_1$. Then, $\psi(0) = V - S_1 > V - S_2 + F_1 = p_2(0)$ for $q = 1$. In this case, if $\psi(0) > p_2(0)$, a ZIE exists. If $S_2 < S_1 + F_1$, q reaches unity before a ZIE emerges.[16] ■

The existence proof makes use of SV-type equilibria only: PIE, FIE, and, in the case $S_2 > S_1 + F_1$, ZIE. It can be shown that the equilibrium is unique. SV (p. 44) present an example that belongs to the case presently discussed:

Example 3. $V = 1$, $F_1 = 0.2$, $a = 1.2$, $S_1 = 0.3$*, and* $S_2 = 0.4$. *Condition (17) is satisfied:* $1.2 = a < 1 + (V - S_2)/(S_2 - S_1 + F_1) = 3$. *As shown by SV (p. 44), there is a FIE for* $0 < q < 0.3590$ *and a PIE for* $0.3590 < q < 1$. *The fact that there is not a ZIE reflects the fact that* $S_2 = 0.4 < 0.5 = S_1 + F_1$.

6. Completing the SV Case: Two-Investment-Levels Equilibria

In this section, we consider the set of parameters which satisfy SV's condition (1) but not (17), thereby completing the analysis of the SV case:

$$1 + \frac{V - S_2}{S_2 - S_1 + F_1} < a < 1 + \frac{V - S_1}{F_1}. \tag{40}$$

[16] The case $a = 1$ can be treated analogously.

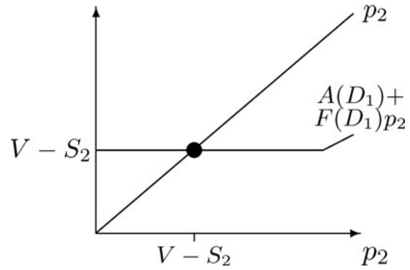

Figure 4. Equilibrium period-2 price level when (40) is satisfied.

The problem of non-concave expected wealth becomes prevalent in this case and gives rise to new types of equilibria. In particular, two-investment-levels equilibria in which arbitrageurs are either fully invested or do not invest at all emerge.

The Dependence of p_2 on D_1

From (26), there is a unique positive D_1 such that $p_2(D_1) = V - S_2$. This is because $aD_1(S_2 - S_1 + D_1)$ equals zero for $D_1 = 0$ and is convex for $D_1 \geq 0$, whereas $F_1(V - S_1 + D_1)$ is positive for $D_1 = 0$ and linear in D_1. From (28), $p_2(F_1) < V - S_2$. For $0 \leq D_1 \leq D_1^0$, by the same reasoning as above, $A(D_1) \geq 0$ and $F(D_1) < 1$, so p_2 obeys (30) (cf. Figure 2). Condition (40) implies $F_1 < D_1^\infty$. So for $D_1^0 < D_1 < F_1$, we have $A(D_1) < 0$, $F(D_1) < 1$, and $p_2 = V - S_2$ (see Figure 4). For future reference, define $t : [0, F_1] \to \mathbb{R}$ by

$$t(D_1) = \frac{1}{\frac{1}{a(V-S_2+F_1-D_1)} + \frac{1}{V-S_1+D_1}}. \tag{41}$$

Undervaluation Equilibrium with V-shaped Expected Wealth

In the proof of Theorem 1, we demonstrated existence of equilibrium, given (17), by ruling out non-concavity of the expected wealth function (i.e., $\bar{D}_1 < F_1$). PIE, FIE, and ZIE are defined as before, except that $p_2 = p_2(D_1)$ is replaced with (30) in a PIE and with $p_2 = V - S_2$ in a FIE. To prove existence under parameter restriction (40), we have to consider equilibria which entail a V-shaped expected wealth function (cf. the right panel of Figure 1). For future reference, we notice in passing that, from (32),

$$EW(0) \underset{>}{\overset{<}{=}} EW(F_1) \Leftrightarrow \psi(D_1) \underset{>}{\overset{<}{=}} \frac{1}{\frac{1}{ap_2} + \frac{1}{p_1}} \tag{42}$$

in this case. We define three types of equilibria for the case of V-shaped expected wealth (see the last three lines in Table I). First, in a full-investment equilibrium with V-shaped expected wealth (FIEV), full investment is at least as profitable as zero investment, and arbitrageurs choose full investment. Second, in a zero-investment equilibrium with V-shaped expected wealth (ZIEV), arbitrageurs choose zero investment, which is at least as profitable as full investment. In all five undervaluation equilibria defined so far, arbitrageurs choose a uniform investment level, D_1.[17] The final type of undervaluation equilibrium is a two-investment-levels equilibrium (TILE): expected wealth takes on exactly the same value for $D_1 = 0$ and for $D_1 = F_1$, so that arbitrageurs are indifferent between zero investment and full investment (see the right panel of Figure 1). That is, (42) holds with equality. As already illustrated by Example 1 in Section 4, this is a generic possibility because D_1 can adjust such that $EW(0) = EW(F_1)$.[18] We now summarize the necessary and sufficient conditions for the existence of the different types of equilibria.

PIE

A PIE exists if there is D_1^* such that $p_2 = p_2(D_1^*) = \psi(D_1^*)$ and $\bar{D}_1 \geq F_1$. By the same arguments as in the preceding section, $p_2 < V$ is satisfied.

ZIE

A FIE exists if $V - S_2 \geq \psi(F_1)$.

ZIE

A ZIE exists if $p_2 = p_2(0) \leq \psi(0)\ (<V)$ and $\bar{D}_1 \geq F_1$.

FIEV

In a FIEV, $D_1 = F_1$, $p_1 = V - S_1 + F_1$, $F_2 = 0$ (since $\bar{D}_1 < F_1$), $p_2 = V - S_2$ $(<V)$, and $p_2(F_1) \leq \psi(F_1)$. From (37) with $D_1 = F_1$ and (40), the requirement $\bar{D}_1 < F_1$ is satisfied. From (42) with $p_1 = V - S_1 + F_1$ and $p_2 = V - S_2$, $EW(0) \leq EW(F_1)$ iff $\psi(F_1) \leq t(F_1)$. Hence, a FIEV exists iff $p_2(F_1) < \psi(F_1) \leq t(F_1)$.

[17] Needless to say that if a PIE exists, unequal distributions of investment across arbitrageurs are also compatible with equilibrium.

[18] Jointly, the six types of equilibria listed in Table I comprise an exhaustive list of possible undervaluation equilibria. Being partially invested cannot be optimal unless the conditions of a PIE are satisfied. The only other case in which more than one investment levels yield the same expected wealth is that of a TILE. In order for all arbitrageurs to be fully invested, the conditions for either a FIE or a FIEV have to be fulfilled. In order for all to prefer zero investment, the conditions of a ZIE or a ZIEV have to be met.

ZIEV

A ZIEV exists if $p_2 = p_2(0) < \psi(0)$, $\bar{D}_1 < F_1$, and $EW(0) \geq EW(F_1)$. From (42) with $p_1 = V - S_1$ and $p_2 = V - S_2 + F_1$ ($<V$) and the definition of $t(D_1)$ in (41), $EW(0) \geq EW(F_1)$ iff $\psi(0) \geq t(0)$.

TILE

Let γ (>0) and $1 - \gamma$ (>0) denote the proportions of arbitrageurs who choose full and zero investment, respectively, in a TILE. Then aggregate investment is $D_1 = \gamma F_1$, and (2) gives the period-1 price level. From (4), period-2 funds under management in the case of worsening noise trader expectations are zero for the fully invested arbitrageurs (since $\bar{D}_1 < F_1$) and F_1 for the others. So aggregate funds under management are $F_2 = (1 - \gamma)F_1 = F_1 - D_1$, and, from (3), the period-2 price level is

$$p_2 = V - S_2 + F_1 - D_1. \tag{43}$$

$S_2 > S_1 > F_1$ implies $p_2 < V$. D_1 has to adjust to a level D_1^{**} that yields $EW(0) = EW(F_1)$. From (2) and (41)-(43), this requires $\psi(D_1^{**}) = t(D_1^{**})$.

Existence

Theorem 2. *Given (40), an equilibrium (viz., a PIE, ZIE, FIE, FIEV, ZIEV, or TILE) exists.*

Proof. $\widetilde{D}_1 < F_1$ because of (40). We subdivide the parameter combinations which satisfy (40) into two subsets, with $\widetilde{D}_1 > 0$ and $\widetilde{D}_1 \leq 0$, respectively. From the definition of $\widetilde{D}_1$ in (38), $\widetilde{D}_1 > 0$ if

$$S_2 < S_1 + F_1 \text{ or else } a < 1 + \frac{V - S_2 + F_1}{S_2 - S_1 - F_1}, \tag{44}$$

whereas $\widetilde{D}_1 < 0$ if

$$S_2 > S_1 + F_1 \quad \text{and} \quad a > 1 + \frac{V - S_2 + F_1}{S_2 - S_1 - F_1}. \tag{45}$$

Consider the shape of the curves $p_2(D_1)$, $V - S_2 + F_1 - D_1$, and $t(D_1)$ in the two cases. From (25), $p_2(0) = V - S_2 + F_1$. Hence, the curves $\max\{p_2(D_1), V - S_2\}$ and $V - S_2 + F_1 - D_1$ start at $V - S_2 + F_1$ for $D_1 = 0$ and terminate at $V - S_2$ for $D_1 = F_1$. Also from (25),

$$V - S_2 + F_1 - D_1 \underset{>}{\overset{<}{=}} p_2(D_1) \Leftrightarrow D_1 \underset{>}{\overset{<}{=}} \widetilde{D}_1 \tag{46}$$

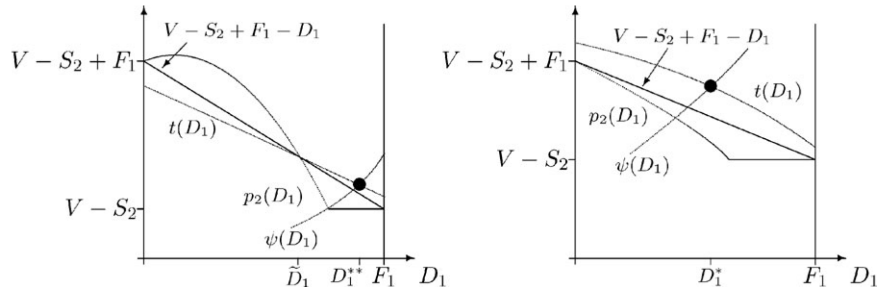

Figure 5. Existence of equilibrium when (40) is satisfied (left panel: (44) satisfied, right panel: (45) satisfied).

for $D_1 > 0$. From (41),

$$t(D_1) \underset{>}{\overset{<}{=}} V - S_2 + F_1 - D_1 \Leftrightarrow D_1 \underset{>}{\overset{<}{=}} \widetilde{D}_1. \tag{47}$$

If (44) holds, then, according to (46) and (47), $p_2(D_1)$, $V - S_2 + F_1 - D_1$, and $t(D_1)$ have a joint intersection at $D_1 = \widetilde{D}_1$ (>0), $t(D_1) < V - S_2 + F_1 - D_1 < p_2(D_1)$ for $0 < D_1 < \widetilde{D}_1$, and $p_2(D_1) < V - S_2 + F_1 - D_1 < t(D_1)$ for $\widetilde{D}_1 < D_1 < F_1$ (as in the left panel of Figure 5). If, by contrast (45) is valid, then $D_1 > \widetilde{D}_1$ for all $D_1 > 0$ (since $\widetilde{D}_1 < 0$), so that $t(D_1) > V - S_2 + F_1 - D_1 > p_2(D_1)$ (see the right panel of Figure 5).

From (38), the condition $\bar{D}_1 \geq F_1$ for a PIE reads as before if $D_1^* < D_1^\infty$: $D_1^* \leq \widetilde{D}_1$. For $D_1^* > D_1^\infty$, the inequality signs in (38) are reversed, so that $\bar{D}_1 \geq F_1$ iff $D_1^* \geq \widetilde{D}_1$. In a ZIE, $D_1 = 0 < D_1^\infty$. So, from (38), $\bar{D}_1 \geq F_1$ iff $D_1 = 0 \leq \widetilde{D}_1$. As remarked above, this condition is satisfied if (44) holds and is violated if (45) holds. In a FIEV, (38) does not hold as $p_2 = V - S_2 < p_2(F_1)$. Inserting $p_1 = V - S_1 + F_1$ and $p_2 = V - S_2$ ($<p_1$) into (33) shows that $\bar{D}_1 < F_1$ iff

$$a > 1 + \frac{V - S_2}{S_2 - S_1 + F_1}, \tag{48}$$

which is satisfied due to (40). In a ZIEV, from (38), $\bar{D}_1 < F_1$ iff $D_1 = 0 > \widetilde{D}_1$, i.e., iff (45) is satisfied. As (38) is derived under the assumption that $p_2 = p_2(D_1)$, it does not hold true when p_2 is determined by (43) (i.e., in a TILE). In this case, it follows from (2), (33), and (43) that $\bar{D}_1 < F_1$ iff $D_1^{**} > \widetilde{D}_1$.

As in the proof of Theorem 1, let q rise from zero to unity. The functions $p_2(D_1)$, $V - S_2 + F_1 - D_1$, and $t(D_1)$ are independent of q. Assume first that (44) is satisfied (see the left panel of Figure 5). For q close to zero, $\psi(F_1)$ is close to zero. As q rises, a FIE exists as long $\psi(F_1) \leq V - S_2$. When q is such that $V - S_2 < \psi(F_1) \leq t(F_1)$, a FIEV exists. As q rises further, there is D_1^{**} such that $\psi(D_1^{**}) = t(D_1^{**})$ and $D_1^{**} > \widetilde{D}_1$, so a TILE exists (as in the left panel of Figure 5). When q grows to the level such that $\psi(\widetilde{D}_1) = V - S_2 + F_1 - \widetilde{D}_1$, a PIE emerges. As q rises further, a PIE exists as long as $\psi(0) < p_2(0)$. As $D_1^* \leq \widetilde{D}_1$, the requirement $\bar{D}_1 \geq F_1$ is satisfied. $\psi(0) \geq p_2(0)$ for $q = 1$ iff $S_2 \geq S_1 + F_1$. In this case, a ZIE emerges as q approaches unity. The assumed validity of (44) ensures that $\bar{D}_1 \geq F_1$ holds. If $S_2 < S_1 + F_1$, $q = 1$ is reached before a ZIE emerges.

Similar arguments apply when (45) is satisfied (see the right panel of Figure 5). For q small, a FIE exists. For q such that $V - S_2 < \psi(F_1) \leq t(F_1)$, a FIEV exists. As q grows such that $\psi(F_1) > t(F_1)$, a TILE emerges (as in the right panel of Figure 5). When q is such that $\psi(0) \geq t(0)$, there is a ZIEV (from (35) and (41), $\psi(0) > t(0)$ for $q = 1$). The requirements $\bar{D}_1 < F_1$ and $EW(0) \geq EW(F_1)$ are satisfied (because of (38) and $\widetilde{D}_1 < 0$ and (42) and (41), respectively). ■

Theorem 2 demonstrates that the non-concavity of the expected wealth function does not preclude the existence of an equilibrium. Contrary to the proof of Theorem 1, all six types of equilibrium are needed to prove existence. In particular, the unique equilibrium is a TILE if q is too high for a FIEV but too low for a PIE (when (44) is satisfied) or a ZIEV (when (45) holds). Equilibrium aggregate investment, D_1, is continuous in the parameter q. In particular, as the market switches from a TILE to a PIE at $\widetilde{D}_1$ (when (44) holds), aggregate investment does not jump.

Example 4. *Consider the parameters introduced in Example 1:* $V = 1$, $F_1 = 0.1$, $a = 1.4$, $S_1 = 0.2$, $S_2 = 0.8$, *and* $q = 0.05$. *The example obeys condition (40):* $1.2857 < 1.4 < 9$. *There is a FIE for* $0 < q < 0.0278$, *a FIEV for* $0.0278 < q < 0.0317$, *a TILE for* $0.0317 < q < 0.0546$, *a PIE for* $0.0546 < q < 0.1071$, *and a ZIE for* $0.1071 < q < 1$.

7. Beyond the SV Case: Early Recovery

We proceed to extend the existence result in Theorems 1 and 2 to parameter combinations which do not satisfy SV's stability condition (1). This raises the possibility, ruled out by assumption in SV, that the price returns to its fundamental value even though noise trader misperceptions deepen.

Consider parameters which satisfy

$$a > 1 + \frac{V - S_1}{F_1} \tag{49}$$

and either

$$S_2 > S_1 + F_1 \tag{50}$$

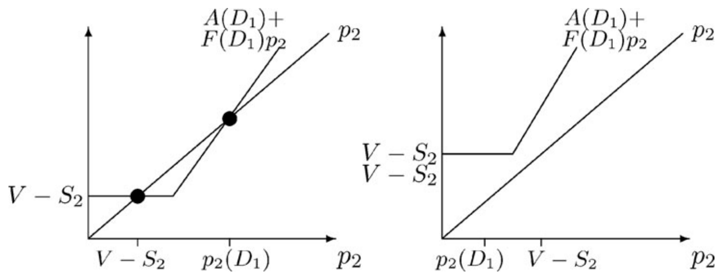

Figure 6. Equilibrium period-2 price level when (49) and (50) or (51) are satisfied and $D_1 > D_1^\infty$.

or

$$S_2 < S_1 + F_1 \quad \text{and} \quad a < 1 + \frac{V - S_1}{F_1 + S_1 - S_2}. \tag{51}$$

The Dependence of p_2 on D_1

Condition (49) jointly with either (50) or (51) implies that $D_1^0 < D_1^\infty < F_1$. So $p_2(D_1)$ diverges to minus infinity when D_1 approaches D_1^∞ from below and to plus infinity when D_1 approaches D_1^∞ from above. The equilibrium period-2 price level satisfies

$$p_2 \in \begin{cases} \max\{p_2(D_1), V - S_2\}; & \text{for } 0 \leq D_1 \leq D_1^0 \\ V - S_2; & \text{for } D_1^0 \leq D_1 < D_1^\infty \\ \{p_2(D_1), V - S_2\}; & \text{for } D_1 > D_1^\infty \text{ if } p_2(D_1) \geq V - S_2 \end{cases}. \tag{52}$$

To see this, recall from Section 5 that $D_1 \leq D_1^0$ and $D_1 < D_1^\infty$ imply $A(D_1) \geq 0$ and $F(D_1) < 1$, respectively. From Figure 2, we already inferred $p_2 = \max\{p_2(D_1), V - S_2\}$ in this case (cf. (30)). For $D_1^0 \leq D_1 < D_1^\infty$, we have $A(D_1) \leq 0$ and $F(D_1) < 1$. From Figure 4, $p_2 = V - S_2$. The last line in (52) is inferred from the left panel of Figure 6 (where $A(D_1) < 0$ and $F(D_1) > 0$). p_2 is undefined if $D_1 > D_1^0$, $D_1 > D_1^\infty$, and $p_2(D_1) < V - S_2$ (see the right panel of Figure 6).

The different types of undervaluation equilibria are defined as in Section 6 (see Table I), except that p_2 now obeys (52). The conditions for the existence of the different types of undervaluation equilibria have to be adapted in order to account for the fact that D_1-values greater than D_1^∞ are now admissible.

PIE

A PIE exists if there is D_1^* such that p_2 is given by (52), $p_2 = \psi(D_1^*)$, and $\bar{D}_1 \geq F_1$.

FIE

A FIE exists if $p_2(F_1) \geq V - S_2$ and $p_2 \geq \psi(F_1)$. Notice that (52) allows for the possibility of a FIE with $p_2 = V - S_2 < p_2(F_1)$. As the derivation of (29) makes use of $D_1 < D_1^{\infty}$, the requirement $p_2(F_1) < V$ may not be satisfied, so that we have to check its validity in order to prove the existence of a FIE with $p_2 = p_2(F_1)$.

ZIE

A ZIE exists if $p_2 = p_2(0) \leq \psi(0)$ and $\bar{D}_1 \geq F_1$.

FIEV

A FIEV, with $p_1 = V - S_1 + F_1$ and $p_2 \in \{p_2(F_1), V - S_2\}$, exists if $\bar{D}_1 < F_1$, $EW(0) \leq EW(F_1)$, and $p_2 < V$. From (41) and (42), $EW(0) \leq EW(F_1)$ is satisfied iff $\psi(F_1) \leq t(F_1)$.

ZIEV

A ZIEV exists if $\psi(0) \geq p_2(0)$, $\bar{D}_1 < F_1$, and $\psi(0) \geq t(0)$ (so that from (41) and (42), $EW(0) \geq EW(F_1)$).

TILE

A TILE, with $p_1 = V - S_1 + D_1^{**}$ and $p_2 = V - S_2 + F_1 - D_1^{**}$, exists if there is D_1^{**} such that $\psi(D_1^{**}) = t(D_1^{**})$ (so that $EW(0) = EW(F_1)$) and $\bar{D}_1 < F_1$.

Existence

Theorem 3. *Given (49) and given (50) or (51), an equilibrium (viz., a PIE, ZIE, FIE, FIEV, ZIEV, or TILE) exists.*

Proof. For $D_1 < D_1^{\infty}$, (46) holds; for $D_1 > D_1^{\infty}$, the inequality signs are reversed. Equation (47) holds for all D_1. From (39) and (49), $\widetilde{D}_1 < F_1$. $\widetilde{D}_1 > 0$ or $\widetilde{D}_1 < 0$, depending on whether (44) or (45) holds, respectively. In the former case, using the definitions of D_1^{∞} in (22) and of $\widetilde{D}_1$ in (38), we find

$$\widetilde{D}_1 \overset{<}{\underset{>}{=}} D_1^{\infty} \Leftrightarrow a(F_1 + S_1 - S_2) \overset{<}{\underset{>}{=}} V - S_2 + F_1. \tag{53}$$

The case distinctions (49) and (50) or (51) imply $\widetilde{D}_1 < D_1^{\infty}$.

The curves $p_2(D_1)$, $V - S_2 + F_1 - D_1$, and $t(D_1)$ are illustrated in Figure 7. Consider the left panel, which assumes (44), so that $\widetilde{D}_1 > 0$. There is $\widetilde{D}_1$ (0 <

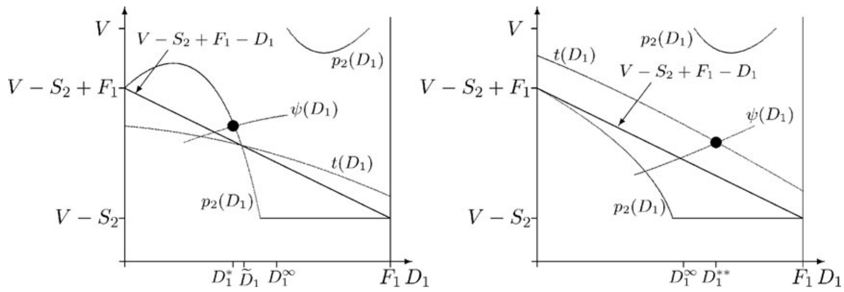

Figure 7. Existence of equilibrium when (49) and (50) or (51) are satisfied (left panel: (44) satisfied, right panel: (45) satisfied).

$\widetilde{D}_1 < D_1^\infty$) such that for $0 < D_1 < \widetilde{D}_1$, we have $p_2(D_1) > V - S_2 + F_1 - D_1 > t(D_1)$; for $\widetilde{D}_1 < D_1 < D_1^\infty$, we have $p_2(D_1) < V - S_2 + F_1 - D_1 < t(D_1)$; and for $D_1^\infty < D_1 < F_1$, we have $p_2(D_1) > t(D_1) > V - S_2 + F_1 - D_1 > V - S_2$. If (45) holds, so that $\widetilde{D}_1 < 0$, we have $p_2(D_1) < V - S_2 + F_1 - D_1 < t(D_1)$ for $D_1 < D_1^\infty$ and $V - S_2 < V - S_2 + F_1 - D_1 < t(D_1)$ for $D_1 > D_1^\infty$ (see the right panel of Figure 7).

As shown in the proof of Theorem 2, $\bar{D}_1 \geq F_1$ for a PIE if either $D_1^* < D_1^\infty$ and $D_1^* \leq \widetilde{D}_1$ or $D_1^* > D_1^\infty$ and $D_1^* \geq \widetilde{D}_1$. $\bar{D}_1 \geq F_1$ in a ZIE iff $0 \leq \widetilde{D}_1$, which is satisfied if (44) holds and is violated if (45) holds. $\bar{D}_1 < F_1$ in a ZIEV iff (45) is satisfied. $\bar{D}_1 < F_1$ in a TILE iff $D_1^{**} > \widetilde{D}_1$. The condition (48) for $\bar{D}_1 < F_1$ in a FIEV follows from (49).

Let q rise from zero to unity. Consider first the case in which (44) holds (see the left panel of Figure 7). For q such that $\psi(F_1) \leq V - S_2$, there is a FIE with $p_2 = V - S_2$. For $V - S_2 < \psi(F_1) \leq t(F_1)$, there is a FIEV with $p_2 = V - S_2$. For $\psi(F_1) > t(F_1)$ and $\psi(\widetilde{D}_1) < V - S_2 + F_1 - \widetilde{D}_1$, there is a TILE. The requirement $D_1^{**} > \widetilde{D}_1$ is satisfied. For q such that $\psi(\widetilde{D}_1) \geq V - S_2 + F_1 - \widetilde{D}_1$ and $\psi(0) < V - S_2 + F_1$, there is a PIE (see the left panel of Figure 7). The requirement $D_1^* \leq \widetilde{D}_1$ is satisfied. For $\psi(0) \geq V - S_2 + F_1$, there is a ZIE. This case arises iff, $S_2 \leq S_1 + F_1$. $\bar{D}_1 \geq F_1$ is implied by (44).

Next, suppose (45) holds, so that $\widetilde{D}_1 < 0$ (see the right panel of Figure 7). If $\psi(F_1) \leq V - S_2$, there is a FIE with $p_2 = V - S_2$. For $V - S_2 < \psi(F_1) \leq t(F_1)$, there is a FIEV with $p_2 = V - S_2$. If $\psi(F_1) > t(F_1)$ and $\psi(0) < t(0)$, there is a TILE (see the right panel of Figure 7). As $\widetilde{D}_1 < 0$, $D_1^{**} > \widetilde{D}_1$ is satisfied. For $\psi(0) \geq t(0)$, there is a ZIEV. $\bar{D}_1 < F_1$ is implied by (45). The definitions of ψ and t in (35) and (41), respectively, imply that this case arises as q approaches unity. ■

Theorem 3 extends the existence result for undervaluation equilibria in Theorems 1 and 2 to parameter combinations which violate SV's condition (1). Remarkably, an equilibrium exists, even though the function $p_2(D_1)$, which relates aggregate period-1 investment, D_1, to the period-2 price level, p_2, is discontinuous and not even well defined for all $D_1 \in [0, F_1]$. Recall SV's conjecture that if (1) is violated, the only equilibrium period-2 price is $V - S_2$. The proof of Theorem 3 shows that, contrary to this conjecture, any of the six types of undervaluation equilibrium possibly emerges.

We proceed to show that, moreover, an early recovery equilibrium (ERE), in which the asset price returns to its fundamental value even though the noise trader shock worsens (i.e., $p_2 = V$) may occur.

ERE

In an ERE, the arbitrageurs are fully invested in period 1 (i.e., $D_1 = F_1$) because the asset price rises from $p_1 = V - S_1 + F_1$ ($<V$) to $p_2 = V$ with certainty. The crucial requirement for $(p_1, p_2, D_1, D_2, F_2, W) \in \mathbb{R}^6_+$ to be an ERE is that period-2 funds under management are sufficient to counterbalance the period-2 noise trader shock: $F_2 \geq S_2$ ($=D_2$). SV (p. 39) push this possibility aside with their assumption $F_2 < S_2$, but it is a genuine possibility. Using (4) and (20), $F_2 \geq S_2$ can be written as

$$A(F_1) + F(F_1)V \geq V. \tag{54}$$

This condition tends to be satisfied when a is large (so that $F(F_1)$ is large). This is because pronounced PBA means that the increase in the asset price to its fundamental value leads to an inflow of funds that is sufficiently large so as to enable the arbitrageurs to compensate for worsening noise trader sentiment.

Theorem 4. *Given (49) and given (50) or (51), an ERE exists iff*

$$a \geq \frac{S_2 - F_1}{S_1 - F_1}\left(1 + \frac{V - S_1}{F_1}\right). \tag{55}$$

Proof. The fact that $a > 1 + (V - S_1)/F_1$ implies $A(F_1) < 0$ and $F(F_1) > 1$. From the left panel of Figure 6, we infer that (54) is equivalent to $p_2(F_1) \leq V$. Using (25), this condition can be rewritten as (55). ■

Example 5. *Let* $V = 1$, $F_1 = 0.2$, $S_1 = 0.3$, $S_2 = 0.6$. *As* $S_2 = 0.6 > 0.5 = S_1 + F_1$, *(50) holds. (49) holds if* $a > 4.5$. *In order for an ERE to emerge, a has to be four times as large: (55) requires* $a > 18$.

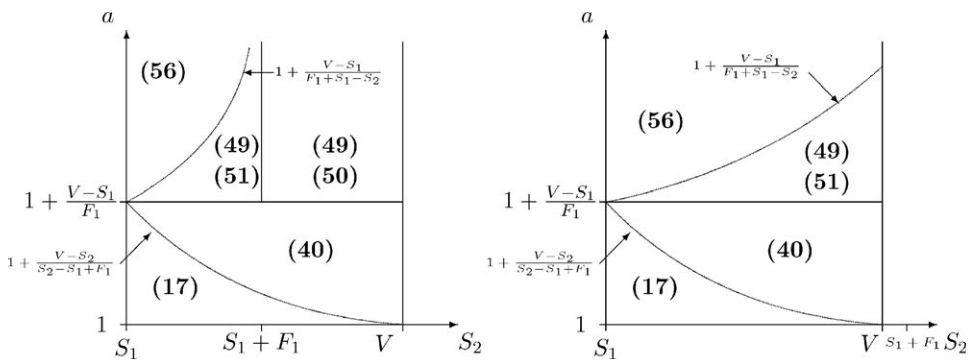

Figure 8. Case distinctions: numbers refer to Equation numbers in the running text (left panel: $S_1 + F_1 < V$, right panel: $S_1 + F_1 \geq V$).

8. Completing the Partition of the Parameter Space: Sunspots

The final case we have to consider is:

$$S_2 < S_1 + F_1 \quad \text{and} \quad a > 1 + \frac{V - S_1}{F_1 + S_1 - S_2}. \tag{56}$$

The Dependence of p_2 on D_1

This implies $D_1^\infty < D_1^0 < F_1$. That is, $p_2(D_1)$ diverges to plus infinity when D_1 approaches D_1^∞ from below and to minus infinity when D_1 approaches D_1^∞ from above. Figure 8 illustrates that this case completes the partition of the parameter space.[19]

From (19) and Figures 2, 4, and 6, we infer the equilibrium period-2 price level:

$$p_2 \in \begin{cases} \max\{p_2(D_1), V - S_2\}; & \text{for } 0 \leq D_1 < D_1^\infty \\ \{p_2(D_1), V - S_2\}; & \text{for } D_1 > D_1^0 \text{ if } p_2(D_1) \geq V - S_2 \end{cases}.$$

p_2 is undefined for the remaining D_1-values. The conditions for the emergence of the different types of undervaluation equilibria are the same as in Section 7.

Theorem 5. *Given (56), an equilibrium (viz., a PIE, FIE, FIEV, or TILE) exists.*

Proof. For $D_1 < D_1^\infty$, (46) holds; for $D_1 > D_1^\infty$, the inequality signs are reversed. Equation (47) holds for all D_1. Furthermore,

$$p_2(D_1) \underset{>}{\overset{<}{=}} t(D_1) \Leftrightarrow D_1 \underset{>}{\overset{<}{=}} \widetilde{D}_1$$

[19] Subject to the caveat in footnote 14.

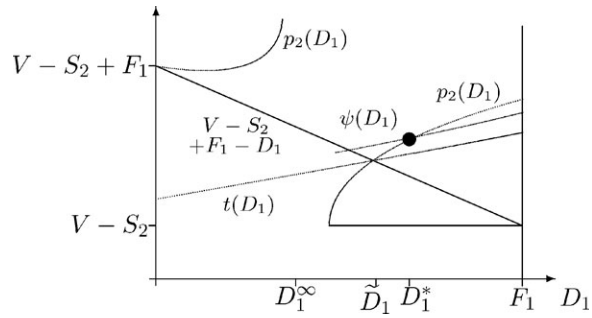

Figure 9. Existence of equilibrium when (56) is satisfied.

for $D_1 > D_1^\infty$. From (28) and (56), $p_2(F_1) > V - S_2$. Condition (55) is necessary and sufficient for $p_2(F_1) \leq V$. From (53) and (56), $\tilde{D}_1 > D_1^\infty$. From (39) and (56), $\tilde{D}_1 < F_1$. This implies that there is $\tilde{D}_1$ ($D_1^\infty < \tilde{D}_1 < F_1$) such that $p_2(D_1) < t(D_1) < V - S_2 + F_1 - D_1$ for $D_1^\infty < D_1 < \tilde{D}_1$ and $p_2(D_1) > t(D_1) > V - S_2 + F_1 - D_1$ for $\tilde{D}_1 < D_1 < F_1$.

The conditions for the existence of the different types of equilibria are the same as in the proof of Theorem 3. $\bar{D}_1 \geq F_1$ for a PIE with $D_1^* > D_1^\infty$ requires $D_1^* \geq \tilde{D}_1$. $\bar{D}_1 < F_1$ in a TILE iff $D_1^{**} > \tilde{D}_1$. The validity of the condition (48) for a FIEV is implied by (56).

Consider Figure 9. Let q rise from zero to unity. Notice that for $q = 1$, we have $\psi(D_1) = V - S_1 + D_1$. For $D_1 > D_1^\infty$, it then follows that

$$p_2(D_1) \overset{<}{\underset{>}{=}} \psi(D_1) \Leftrightarrow D_1 \overset{<}{\underset{>}{=}} S_1 + F_1 - S_2 \tag{57}$$

and, in particular, that $p_2(F_1) > \psi(F_1)$. For q such that $\psi(F_1) \leq V - S_2$, there is a FIE with $p_2 = V - S_2$. For $V - S_2 < \psi(F_1) \leq t(F_1)$, there is a FIEV with $p_2 = V - S_2$. For q large enough such that $(p_2(F_1) >)\ \psi(F_1) > t(F_1)$, $\psi(D_1)$ intersects either $t(D_1)$ or $p_2(D_1)$ in the interval $(\tilde{D}_1, F_1)$ (this follows from the observation that it must enter the area formed by these two curves for $D_1 > \tilde{D}_1$ and the vertical line at F_1). If it intersects $t(D_1)$, there is a TILE. The condition $D_1^{**} > \tilde{D}_1$ is satisfied. If it intersects $p_2(D_1)$, there is a PIE. The condition $D_1^* \geq \tilde{D}_1$ is satisfied. The latter case (i.e., a PIE) arises when q approaches unity. This follows from the observation that

$$S_1 + F_1 - S_2 > D_1^\infty \Leftrightarrow a > 1 + \frac{V - S_1}{S_1 + F_1 - S_2},$$

which, together with (56) and (57) implies that for $q = 1$, there is a (unique) intersection of $\psi(D_1)$ and $p_2(D_1)$ on the interval (D_1^∞, F_1). ■

As in Section 7, the fact that $p_2(D_1)$ is not continuous and not even well defined for all D_1 does not lead to non-existence of an equilibrium. Contrary to the previous cases, the relation between the equilibrium investment level D_1 and q is not monotonic in this case. The assertion of Theorem 4 (existence of an ERE) is valid when (56) holds; the proof goes through without any modification. So an ERE coexists alongside with an undervaluation equilibrium if (55) is satisfied. As Example 2 illustrates, it may even happen that three equilibria exist:

Example 6. *Consider the parameters introduced in Example 2:* $S_2 = 0.85 < 1.5 = S_1 + F_1$ *and* $a = 2 > 1.3077 = 1 + (V - S_1)/(F_1 + S_1 - S_2)$ *implies that (56) holds. In Section 4, we have demonstrated the existence of a FIE, a PIE, and an ERE.*

Here is another example with three equilibria:

Example 7. *Let* $V = 1$, $F_1 = 0.2$, $a = 8$, $S_1 = 0.4$, $S_2 = 0.5$, *and* $q = 0.45$. *As* $S_2 = 0.5 < 0.6 = S_1 + F_1$ *and* $a = 8 > 7 = 1 + (V - S_1)/(F_1 + S_1 - S_2)$, *this example also satisfies (56). There is a FIE with* $p_1 = 0.8$, $p_2 = p_2(F_1) = 0.9$ $(>0.6429 = \psi(F_1))$. *There is a FIEV with* $p_1 = 0.8$, $p_2 = V - S_2 = 0.5$. *And there is an ERE with* $p_1 = 0.8$, $p_2 = 1$, *and* $F_2 = 0.6$ $(>0.5 = S_2)$.

Sunspot Equilibria

The multiplicity of equilibria naturally gives rise to sunspot equilibria: any extrinsic random variable which coordinates beliefs in period 1 potentially governs the selection of one out of multiple equilibria $(p_1, p_2, D_1, D_2, F_2, W) \in \mathbb{R}^6_+$. Interestingly, as already pointed out by means of Example 2 in Section 4, there is also scope for sunspot equilibria related to the realization of an extrinsic random variable in period 2. In Example 7, period-2 sunspots possibly govern equilibrium selection among all three equilibria.

Example 8. *Consider the parameters introduced in Example 7 and a random variable with three possible outcomes,* $\{1, 2, 3\}$, *which materialize with probabilities* π_i, *where* $0 \leq \pi_i < 1$ *for all* $i \in \{1, 2, 3\}$. *Let* $p_{21} = 0.9$, $p_{22} = 0.5$, *and* $p_{23} = 1$. *These are the equilibrium period-2 price levels,* p_2, *in the FIE, FIEV, and ERE, respectively. Furthermore, let* $p_1 = 0.8$. *Suppose (conditional on worsening noise trader expectations) arbitrageurs expect the period-2 price* p_{2i} *if* i *is realized. Analogously to (32), expected wealth as of period 1 is*

$$EW(D_1) = \sum_{i=1}^{3} \pi_i \left(0.2 \left[0.2 + 8D_1 \left(\frac{1}{0.8} - 1\right)\right]\right.$$

$$+0.8\frac{1}{p_{2i}}\max\left\{0.2+8D_1\left(\frac{p_{2i}}{0.8}-1\right),0\right\}\bigg).$$

As $D_1 = 0.2$ *(i.e., full investment) maximizes expected wealth for each* $i \in \{1, 2, 3\}$*, it also maximizes* $EW(D_1)$*. In period 2, arbitrageurs choose* $D_2 = 0.4$, $D_2 = 0$*, or* $D_2 = 0.5$*, depending on which* $i \in \{1, 2, 3\}$ *is realized.*

Examples 2 and 8 prove:

Theorem 6. *There are parameters which satisfy (56) such that a sunspot equilibrium equilibrium exists in which the period-2 price depends on extrinsic uncertainty resolved in period 2.*

Comparative statics

SV (pp. 44-47) characterize the model's comparative statics under condition (1) in depth. For instance, they show that, given (1), an increase in the period-2 noise trader shock, S_2, leads to lower period-2 prices, p_2, in a FIE (SV, Proposition 2, p. 44). From their conjecture that $p_2 = V - S_2$ is the only equilibrium period-2 price if (1) is violated, it follows that S_2 negatively affects p_2 in this case as well: $dp_2/dS_2 = -1$. However, we have seen that equilibria with full investment and with $p_2 \neq V - S_2$ may exist when (1) is violated. The following theorem shows that dp_2/dS_2 may be positive in such an equilibrium.

Theorem 7. *If (1) holds,* $dp_2/dS_2 < 0$ *in a FIE. If (55) holds,* $dp_2/dS_2 > 0$ *in a FIE with* $p_2 = p_2(F_1)$*.*

Proof. From (22) and (25),

$$\frac{\partial p_2(D_1)}{\partial S_2} = -\frac{V - S_1 + D_1}{V - S_1 - (a-1)D_1} \lessgtr 0 \Leftrightarrow D_1 \lessgtr D_1^\infty.$$

If (1) holds, $D_1 \leq F_1 < D_1^\infty$, and from Figures 3 and 5, it can be inferred that $dp_2/dS_2 < 0$ in a FIE (irrespective of whether $p_2 > V - S_2$ or $p_2 = V - S_2$, in which case[20] $dp_2/dS_2 < -1$). By contrast, (55) implies that (1) is violated, so that $\partial p_2(F_1)/\partial S_2 > 0$. It follows that $dp_2/dS_2 > 0$ in a FIE. Example 7 shows that (55) holds and a FIE exists. ■

9. No-Default Constraint

Theorems 1-3 and 5 jointly prove existence of equilibrium in the SV model. What renders the analysis so difficult is the potential non-concavity of the expected wealth

[20] Cf. SV (Proposition 4, p. 46).

function, which makes the consideration of equilibria with V-shaped expected wealth functions necessary. As illustrated in Example 1 in Section 4, it is much easier to come up with an existence result if one is willing to make the additional assumption that arbitrageurs plan their investments such that they avoid default risk. The analogous assumption was introduced to the banking literature by Bernanke and Gertler (1987, p. 96).[21]

As in Section 4, assume the contractual rate of return paid to the investors is zero, independently of whether they withdraw their money in period 2 or 3.[22] As explained in Section 3, the change in period-2 funds unter control due to withdrawals is $(a-1)D_1(p_2/p_1-1)$. The absence of default risk requires that final wealth after a worsening noise trader shock, $(V/p_2)F_2$, does not fall short of outstanding claims, $F_1+(a-1)D_1(p_2/p_1-1)$. If $p_2 \geq p_1$, then arbitrageurs never realize losses, so default cannot occur. If $p_2 < p_1$, using (4), the no-default condition can be expressed as

$$D_1 \leq \frac{F_1}{a\left(1-\frac{p_2}{p_1}\right)\left(1+\frac{\frac{1}{a}}{\frac{V}{p_2}-1}\right)}. \tag{58}$$

Assume that if $p_2 < p_1$, arbitrageurs obey the additional constraint (58) to the maximization problem in (6). By virtue of (33), the right-hand side of (58) is less than $\bar{D}_1$. Hence, this rules out non-concavity of expected wealth.

In the presence of the no-default constraint (58), we have to define a new type of undervaluation equilibrium, in which expected wealth is increasing in D_1 ($<F_1$), but the arbitrageurs refrain from raising D_1 because (58) is binding. We call this an equilibrium with no default (END). An END exists if there is D_1 ($0 < D_1 < F_1$) such that (58) holds with equality and $p_2(D_1) \geq \psi(D_1)$. The following two theorems state that either an SV-type equilibrium or an END exists. We start with the simple special case treated in Section 5:

Theorem 8. *Given (17) and given that arbitrageurs obey (58), an equilibrium (a PIE, FIE, ZIE, or END) exists.*

Proof. From (31), we have $p_2 = p_2(D_1)$. Let p_1 and p_2 be given by (2) and (25) in (58). Using $D_1 < D_1^\infty$ it follows that

$$\frac{p_2}{p_1} \mathrel{\substack{<\\=\\>}} 1 \Leftrightarrow D_1 \mathrel{\substack{>\\=\\<}} S_1 + F_1 - S_2. \tag{59}$$

[21] See also Arnold (2002, Section 6.2).

[22] An independent motivation for ruling out default is that the actual rate of return is less than zero if default occurs when noise trader expectations worsen.

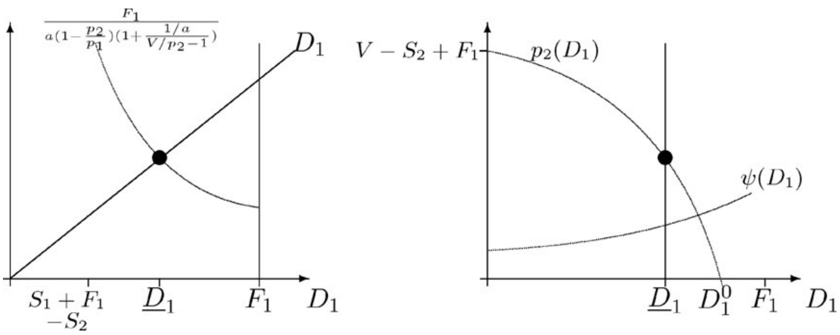

Figure 10. Existence of equilibrium when arbitrageurs obey (58) and (17) holds.

If $S_2 > S_1 + F_1$, then $p_2/p_1 < 1$ for all $D_1 \geq 0$. If $S_2 < S_1 + F_1$, then $p_2/p_1 > 1$ for $0 \leq D_1 < S_1 + F_1 - S_2$ ($<F_1$) and $p_2/p_1 < 1$ for $D_1 > S_1 + F_1 - S_2$. If (58) holds for all $D_1 \in [0, F_1]$, then there is no default risk, irrespective of how much the arbitrageurs invest. Otherwise there is D_1 ($0 < D_1 < F_1$) such that (58) holds with equality. Let $\underline{D}_1$ denote the smallest such D_1-value (the case with $S_2 < S_1 + F_1$ is illustrated in the left panel of Figure 10). Then, if each arbitrageur invests $\underline{D}_1$, default risk would arise if D_1 were raised.

If (58) holds for all $D_1 \in [0, F_1]$, the arguments put forward in the final paragraph in the proof of Theorem 1 prove the existence of a FIE, PIE, or NIE. In the opposite case, let q rise from zero to unity. $p_2(D_1)$ and $\underline{D}_1$ are independent of q. For q such that $\psi(\underline{D}_1) \leq p_2(\underline{D}_1)$, there is an END (see the right panel of Figure 10). All arbitrageurs choose $D_1 = \underline{D}_1$, the period-2 price is $p_2 = p_2(\underline{D}_1)$. The fact that $p_2 = p_2(\underline{D}_1) > \psi(\underline{D}_1)$ means that expected wealth is increasing in the arbitrageur's investment, but he refrains from raising D_1 beyond $\underline{D}_1$ because that would imply the emergence of default risk. If $\psi(\underline{D}_1) > p_2(\underline{D}_1)$ and $\psi(0) < p_2(0)$, there is a PIE. For q such that $\psi(0) \geq p_2(0)$, there is a ZIE. From $D_1 < \underline{D}_1$ and the definition of $\underline{D}_1$, there is no default risk. ■

The assertion of Theorem 8 readily generalizes to all other cases:

Theorem 9. *Given that arbitrageurs obey (58), an equilibrium (a PIE, FIE, ZIE, or END) exists.*

Proof. Suppose (40) holds. As shown in Section 6, there is a unique positive D_1 such that $p_2(D_1) = V - S_2$. $p_2(D_1) = V - S_2$ implies $F_2 = 0$ and, hence, $D_1 = \bar{D}_1$, so that (58) is violated. This implies that there is a smaller D_1-value for which (58) holds with equality. By the same reasoning as in the proof of Theorem 8, there is an END, PIE, or ZIE. The same argument also applies when (49) and either (50) or (51) hold.

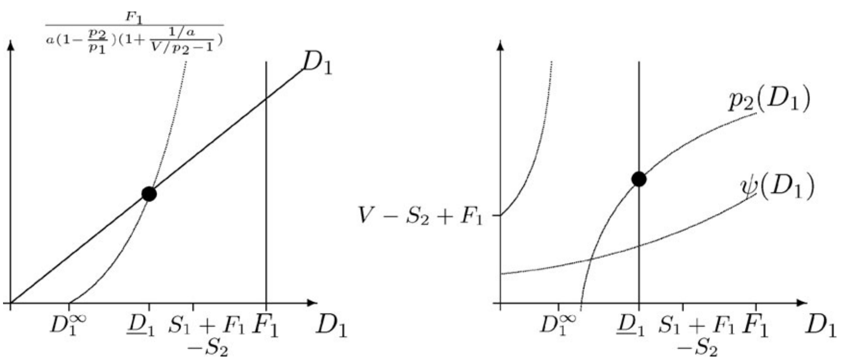

Figure 11. Existence of equilibrium when arbitrageurs obey (58) and (56) holds.

Finally, suppose (56) holds. As shown in the proof of Theorem 5, $S_1 + F_1 - S_2 > D_1^\infty$. For $D_1 > D_1^\infty$, the inequality signs in (59) are reversed. Hence, $p_2 < p_1$ iff $D_1^\infty < D_1 < S_1 + F_1 - S_2$. The right-hand side of (58) rises from zero to infinity as D_1 rises from D_1^∞ to $S_1 + F_1 - S_2$ (see the left panel of Figure 11). This follows from the observation that $p_2/p_1 \to -\infty$ as $D_1 \to D_1^\infty$ from above (so that $1 - p_2/p_1 \to \infty$) and $p_2/p_1 \to 1$ as $D_1 \to S_1 + F_1 - S_2$ (so that $1 - p_2/p_1 \to 0$). So there exists D_1 ($D_1^\infty < D_1 < S_1 + F_1 - S_2$) such that (58) holds with equality. Define $\underline{D}_1$ as the greatest such D_1-value. Notice that there is no default risk for $\underline{D}_1 < D_1 \leq F_1$ because $p_2 > p_1$ for $D_1 > S_1 + F_1 - S_2$.

Evaluating $p_2(D_1)$ at $D_1 = S_1 + F_1 - S_2$ yields $V - S_2 < p_2(S_1 + F_1 - S_2) < V$. For q such that $\psi(\underline{D}_1) \leq p_2(D_1)$, there is an END with $p_2 = p_2(D_1)$ ($<V$, see the right panel of Figure 11). As (58) holds with equality, the arbitrageurs do not raise D_1, even though expected wealth would rise. For q such that $\psi(\underline{D}_1) > p_2(\underline{D}_1)$, $p_2(D_1)$ and $\psi(D_1)$ intersect in the interval $(\underline{D}_1, F_1)$ (because, as shown in Section 8, $p_2(F_1) > \psi(F_1)$). Since there is no default risk for $\underline{D}_1 < D_1 \leq F_1$, a PIE exists. ■

By ruling out non-concave expected wealth, the no-default assumption ensures that it suffices to consider SV-type equilibria to prove existence, which simplifies the existence proof considerably. However, as already demonstrated by Example 2 in Section 4, the no-default assumption does not rule out multiplicity, sunspots, or early recovery if (1) is violated.

Example 9. *Consider the parameters introduced in Example 2. From (58),* $\underline{D}_1 = 0.4760$. *An END exists if* $q < 0.1597$.

10. Conclusion

The apparent simplicity of the SV model is deceptive. There are several possible types of equilibria that have to be distinguished. In equilibrium, arbitrageurs may be fully invested or not invest at all. It may happen that the only equilibrium is of the all-or-nothing type: some arbitrageurs choose full and others zero investment, while all intermediate investment levels are unprofitable. If the investors react sensitively to the arbitrageurs' past performance, multiplicity and sunspot equilibria arise. The initial undervaluation may continue or disappear. One way to rule out many difficulties in the model analysis and some of its surprising implications is to add a no-default constraint to the arbitrageurs' expected wealth maximization problem.

The most promising route for future research, however, is to model the investors' signal extraction problem explicitly. It would be interesting to see whether the allocation problems identified in this paper continue to arise.

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