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An Examination of Heterogeneous Beliefs with a Short-Sale Constraint in a Dynamic Economy*

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Abstract. We study the effects of a market-wide short-sale constraint in a dynamic economy with heterogeneous beliefs. Imposing the constraint reduces the stock price if the optimistic investors' intertemporal elasticity of substitution (IES) is less than one and increases the stock price if the optimist's IES is greater than one. In calibrated examples, the optimist's market price of risk falls and the interest rate rises when the constraint binds. Imposing the constraint leads to a higher stock volatility if the optimist's IES is less than one and a lower stock volatility if the IES is greater than one.

JEL Classification: D51, G11, G12, G14

1. Introduction

Equilibrium stock prices reflect investors' beliefs. If investors cannot short sell stock, then equilibrium stock prices may not reflect the beliefs of the most pessimistic investors. Stock prices may therefore be higher than otherwise in the presence of a short-sale constraint. This is indeed the case in the work of Lintner (1969), Miller (1977), and Jarrow (1980) who study one period models with heterogeneous beliefs and an exogenous interest rate. They show that in the presence of a short-sale constraint and heterogeneous beliefs, stock prices are higher than otherwise when pessimistic investors face binding short-sale constraints.

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In contrast to this previous work, we explore the asset pricing effects of a market-wide short-sale constraint in a dynamic general equilibrium economy. Our model is appropriate for situations in which investors disagree about the overall economy, but cannot necessarily trade on these beliefs. Clearing across all markets makes the role of the short-sale constraint more complex, since prices in all markets depend on the constraint. Indeed, we show that in a dynamic general equilibrium setting, the short-sale constraint can have effects on stock prices, stock volatility, and interest rates. Such a market-wide analysis is important both in US and foreign markets as market participants face several impediments to short selling and in some instances are even completely precluded from shorting.

In the USA, market structures are not favorable toward shorting through procedures and restrictions such as margin requirements, lending fees, and the "uptick rule" imposed by the Federal Reserve, the SEC, stock exchanges, and brokerage firms. Furthermore, some market participants are completely constrained from shorting—Almazan et al. (2004) show that approximately 69% of their sample of equity funds are prohibited from shorting. They also document that even when a fund is unconstrained, it rarely shorts. In a related work, Koski and Pontiff (1999) show that equity mutual funds do not appear to be substituting short-sale trades with derivative positions as only 21% of the funds in their sample use derivatives.

Additional evidence on the difficulty of shorting the market comes from the prices of index options. Put index options are priced high relative to their Black-Scholes-Merton values (see Bates (2003)). These high put prices may reflect restrictions on short selling as put writers face additional costs to cover their positions. Short-selling restrictions might also contribute to the current organization of options markets where only a small number of specialized option market-making firms predominately write options. Furthermore, Lamont and Stein (2004) show that market-wide short interest is low, arguing that mutual funds and other institutional traders do little shorting.

Internationally, short-sale restrictions can be even more severe as several foreign equity markets, especially in Southeast Asia and South America, have experimented with completely banning short sales (Bris et al. 2004). As of 2004, Charoenrook and Daouk (2005) document that fewer than half of the equity exchanges around the world allow short sales.

Given the intricacies of modeling all the market frictions faced when shorting, we study a dynamic continuous-time economy where a class of investors is completely precluded from short selling. Our economy is made up of optimistic power utility investors and pessimistic logarithmic utility investors who disagree about the expected growth rate of aggregate output.

Such an environment provides a setting where a short-sale constraint can bind and can be tractably characterized. All the investors learn about the expected growth rate by observing realized output growth. The stock price is a security paying out the aggregate output. In contrast to a single period model, the stock price can fall or rise when a short-sale constraint is imposed in an economy with heterogeneous beliefs.

Suppose that the pessimist short sells stock in an unconstrained economy, and a short-sale constraint is now imposed. For security markets to clear, prices must change so that the optimist reduces his stock demand. Two channels in the optimist's state prices can lead to a reduction in the optimist's stock demand. The market price of risk on the stock can drop, and the cost of borrowing at the riskless rate to purchase the stock can rise. Both channels occur in our model. For either channel, imposing the short-sale constraint reduces the expected return on the optimist's portfolio relative to the unconstrained economy. The reduction in the optimist's investment opportunity set changes his demands for current consumption and for savings.

The changes in current consumption and savings are characterized through classical income and substitution effects. Imposing the short-sale constraint reduces the expected return of the optimist's portfolio. The income effect is that the change in the expected return reduces the optimist's wealth, and so leads him to consume less today and save more for the future. The substitution effect is that the decrease in the expected return of the optimist's portfolio reduces the price of current consumption relative to future consumption and so leads him to consume more today and save less for the future. The net effect of a reduction in expected returns depends on the strength of the income and substitution effects. If the income effect is stronger than the substitution effect, then the reduction in expected returns leads the optimist to consume less today. If the substitution effect is stronger, then the reduction in expected returns leads the optimist to consume more today.

If the income effect dominates, the demand for savings rises and the demand for current consumption falls as a consequence of the constraint. Given that the aggregate supply of current consumption is fixed, the demand for current consumption must rise to clear the consumption market. In order for the demand for current consumption to rise, aggregate wealth must rise. Since aggregate wealth equals the stock market value, the stock's price must therefore rise when the short-sale constraint is imposed. By the same reasoning, if the opposing substitution effect dominates, the stock's price must fall when the short sale is imposed.

The relative strength of the income and substitution effects depends on the optimist's intertemporal elasticity of substitution (IES). For example, consider a simple partial equilibrium setting with a power utility investor who maximizes the expected utility

$$E\left[\int_{t=0}^{\infty} e^{-\rho t}\frac{c(t)^{1-\frac{1}{\mathrm{IES}}}-1}{1-\frac{1}{\mathrm{IES}}}dt\right]$$

where $c(t)$ is the consumption rate at time t, IES is the intertemporal elasticity of substitution, and ρ is the time discount parameter. The investor smooths intertemporal consumption over an infinite horizon by investing in a single risky asset. When the risky asset's price is a geometric Brownian motion with growth rate μ and volatility σ, consumption is

$$c(t) = W(t)\left[\mathrm{IES}\,\rho + (1-\mathrm{IES})\,\mu - \frac{1}{2}\left(\frac{1-\mathrm{IES}}{\mathrm{IES}}\right)\sigma^2\right]$$

where $W(t)$ is time t wealth. If the investor's opportunity set deteriorates through a fall in μ, his consumption either rises or falls. For an IES greater than one, the substitution effect dominates and consumption rises. For an IES less than one, the income effect dominates and consumption falls. This intuition generalizes to a non-constant coefficient price system as in our setting.

Once market clearing in the consumption market is imposed, the optimist's IES drives how the stock price changes when a short-sale constraint is imposed.[1] If the optimist's IES is greater than one, the substitution effect dominates and the stock price falls when the short-sale constraint is imposed. If the optimist's IES is less than one, the income effect dominates and the stock price rises when the short-sale constraint is imposed. If the optimist's IES is equal to one, neither effect dominates and the stock price does not change when the short-sale constraint is imposed.

In any equilibrium where the short-sale constraint binds with positive probability, the stock price is strictly higher than the pessimist's valuation for the asset. The result holds irrespective of the investors' IES, but the overvaluation does not necessarily mean that the stock price would fall if the short-sale constraint were removed from the economy.

We provide numerical examples of the asset pricing effects of the short-sale constraint. In all cases, imposing the constraint leads to a lower market price of risk for the optimist and a higher riskless rate. When the optimist's IES is greater than one, the market price of risk drops and the interest rate rises significantly when the short-sale constraint is imposed. The pessimist may

[1] See Epstein (1988) for a related analysis of how a representative agent's IES drives the level of equilibrium stock prices when the exogenous output's probability distribution becomes more favorable.

believe the stock to be overvalued by as much as 20%. Here, the stock price would rise if the short-sale constraint were removed from the economy.

When the optimist's IES is less than one, the market price of risk drops and the interest rate rises by much less when the short-constraint is imposed. The pessimist believes the stock to be overvalued by a much smaller amount. In this region, the stock price falls by a small amount when the short-sale constraint is removed from the economy.

Given our model is dynamic, we also study how constraints influence stock volatility, complementing the recent work by Bhamra and Uppal (2006). When the optimist's IES is greater than one, the stock's volatility drops once the short-sale constraint is imposed. When the optimist's IES is less than one, the opposite occurs—the stock's volatility increases when the short-sale constraint is imposed.

Our analysis of the market-wide impact of a short-sale constraint provides additional insights into the literature that studies the general equilibrium impact of constraints.[2] First, our work provides results analyzing how stock price levels are influenced by constraints in addition to return dynamics as commonly studied. Second, our short-sale constrained equilibrium leads to higher interest rates. This contrasts with other types of constraints that typically lead to lower interest rates such as limited participation (Basak and Cuoco, 1998), borrowing constraints (Kogan and Uppal, 2001), and liquidity constraints (Detemple and Serrat, 2003). Finally, our work demonstrates that short-sale constraints can lead to higher equity volatilities over a broad range of parameters whereas borrowing-constrained equilibria, such as Kogan and Uppal (2001) and Bhamra and Uppal (2006), typically lead to lower equity volatilities.

2. The Model

To study the market-wide influence of a short-sale constraint, our economic environment is a continuous-time pure exchange economy with heterogeneous investors. The economy is similar to the economies in Detemple and Murthy (1997) and Basak and Croitoru (2000) who study the effects of constraints on equilibrium quantities when logarithmic investors have heterogeneous beliefs about economic fundamentals.[3] As in this previous work, we use heterogeneous

[2] Examples include Heaton and Lucas (1996), Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000), Kogan and Uppal (2001), Detemple and Serrat (2003), and Bhamra and Uppal (2006).

[3] Work that studies heterogeneous beliefs without constraints includes the discrete-time model of Abel (1990) and the continuous-time models of Detemple and Murthy (1994), Zapatero (1998), Basak (2000), Gallmeyer (2002), and David (2008).

beliefs as a mechanism for a short-sale constraint to bind as no investor will short stock in a homogeneous beliefs setting. But all the investors in our model do not have logarithmic preferences and therefore the short-sale constraint will influence the stock's price. The economy has a finite horizon equal to $T < \infty$. All uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

The exogenous aggregate output $\delta(t)$ follows a geometric Brownian motion with dynamics

$$d\delta(t) = \delta(t)\left[\mu_\delta dt + \sigma_\delta dw(t)\right], \quad \delta(0) > 0, \tag{1}$$

where $w(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ with $\sigma_\delta > 0$.[4]

To incorporate heterogeneous beliefs about economic fundamentals, investors do not observe the expected growth rate of output μ_δ, but only observe the continuous record of realized output $\delta(t)$. All investors agree that the evolution of output is governed by Equation (1), but they do not know the true expected output growth. Heterogeneous beliefs are modeled with investor-specific priors about expected output growth as, for example, in Detemple and Murthy (1994) and Basak (2000). Since output is observed continuously, all investors can perfectly estimate its volatility σ_δ by computing the output process's quadratic variation.

The economy is populated by two classes of investors: optimists and pessimists. All investors are price-takers allowing each class to be summarized by a single representative investor. All investor-specific quantities are labeled with a superscript o for an optimist or a superscript p for a pessimist.

For investor $i = \{o, p\}$, uncertainty in the economy is represented by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}^\delta\}, \mathcal{P}^i\right)$. Investor-specific beliefs about expected output growth are

$$\mu^i_\delta(t) = E^i\left[\mu_\delta \left| \mathcal{F}^\delta_t \right.\right], \quad i = \{o, p\}. \tag{2}$$

The optimist has weakly higher expectations about output growth than the pessimist, $\mu^o_\delta(0) \geq \mu^p_\delta(0)$.

Given the two investors start with different priors, they will continue to disagree about the expected output growth over the life of the economy as they update their beliefs using the commonly observed output process. Although we assume that the investors are Bayesian, it is straightforward to allow for a

4 Our characterization of the equilibrium does not depend on the geometric Brownian motion output assumption. All we require is that the diffusion coefficient on output growth is $\mathcal{F}^\delta_t$-progressively measurable, where $\mathcal{F}^\delta_t$ is the augmented filtration generated by $\delta(t)$. Such a mild assumption allows for a variety of models for the expected output growth process, such as a mean-reverting process or a regime-switching process.

non-Bayesian updating rule. Following standard filtering theory as in Liptser and Shiryayev (1977), the investor-specific innovation processes are defined as

$$dw^i(t) = \frac{1}{\sigma_\delta}\left[\frac{d\delta(t)}{\delta(t)} - \mu^i_\delta(t)dt\right] = \frac{\mu_\delta - \mu^i_\delta(t)}{\sigma_\delta}dt + dw(t), \quad i = \{o,\ p\}. \tag{3}$$

From Equation (3), the innovation processes across investors are related by

$$dw^p(t) - dw^o(t) = \overline{\mu}(t)dt, \quad \text{where} \quad \overline{\mu}(t) \equiv \frac{\mu^o_\delta(t) - \mu^p_\delta(t)}{\sigma_\delta}. \tag{4}$$

The process $\overline{\mu}(t)$ measures investors' differences of opinion; we call it the disagreement process as in Basak (2000). The disagreement process depends on the initial priors and the path of realized aggregate output through the filtering problems of the two classes of investors. In contrast to a model with asymmetric information, the disagreement process does not depend on equilibrium asset prices or allocations.

Lemma 1. *Under each investor's beliefs, the aggregate output process is*

$$d\delta(t) = \delta(t)\left[\mu^i_\delta(t)\,dt + \sigma_\delta\,dw^i(t)\right], \quad \delta(0) > 0, \quad i \in \{o,\ p\}. \tag{5}$$

Each investor's beliefs about the expected growth of aggregate output follows the process

$$d\mu^i_\delta(t) = m^i(t)\,dt + \eta^i(t)\,dw^i(t), \quad i \in \{o,\ p\}, \tag{6}$$

where $m^i(t)$ and $\eta^i(t)$ are $\mathcal{F}^\delta_t$-measurable processes defined in Section 4, and $w^i(t)$ is a standard Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}^\delta_t\}, \mathcal{P}^i\right)$, for $i = \{o, p\}$.

Since Equation (5) has a strong solution, the investors' information and innovation filtrations coincide. Each investor acts as an investor with complete information who faces uncertainty in the economy generated by $w^i(t)$.

The investors continuously trade a stock and a locally riskless bond to share risk. The bond is in zero net supply with price $B(t)$. Posited bond price dynamics are

$$dB(t) = B(t)\,r(t)\,dt, \quad B(0) = 1, \tag{7}$$

with $r(t)$ the instantaneous riskless interest rate in the economy.

The stock is in fixed net supply of one share, paying a continuous dividend rate equal to aggregate output, $\delta(t)$. The stock is interpreted as the market portfolio. The posited gains process for the stock under the objective probability beliefs is

$$dS(t) + \delta(t)dt = S(t)\left[\mu(t)\,dt + \sigma(t)\,dw(t)\right], \quad S(T) = 0. \tag{8}$$

Under each investor's beliefs, the stock gains process is

$$dS(t) + \delta(t)dt = S(t)\left[\mu^i(t)\,dt + \sigma(t)\,dw^i(t)\right], \quad S(T) = 0, \quad i = \{o, p\}. \tag{9}$$

Here $\mu^i(t)$ is investor i's belief about the instantaneous expected stock return. All investors agree on the diffusion coefficient of the stock gains process. By agreeing on the traded stock price, the investor-specific expected stock returns are linked through the disagreement process,

$$\mu^o(t) - \mu^p(t) = \sigma(t)\,\overline{\mu}(t). \tag{10}$$

The endogenous price system $(r, \mu^o, \mu^p, \sigma)$ is determined in equilibrium.

All investors choose consumption and portfolio strategies to maximize lifetime utility. The optimist's portfolio strategy is unconstrained; he faces a complete financial market even though he has incomplete information about the economy (Riedel, 2001). The pessimist's portfolio strategy is constrained from shorting the stock; he faces an incomplete financial market.

Each investor's lifetime utility function is time and state separable and all investors have the same time-preference parameter ρ. The pessimist has logarithmic preferences implying a coefficient of relative risk aversion equal to one and an IES equal to one. The optimist has power preferences with a coefficient of relative risk aversion γ and an IES equal to $\frac{1}{\gamma}$.

Throughout our analysis, we do not restrict $\frac{1}{\gamma} \leq 1$ as typically assumed in the equity premium puzzle literature (Mehra and Prescott, 1985). First, given that a useful benchmark exists when the optimist's IES equals one, we want to explore the role of a short-sale constraint when the optimist's IES is on either side of the benchmark. Second, recent work by David (2008) argues that it is possible to fit the equity premium puzzle and the equilibrium short rate when $\frac{1}{\gamma} > 1$ for all investors in a setting with heterogeneous beliefs similar to our own.[5]

[5] David (2008) models heterogeneity in beliefs by assuming that the expected growth rate of dividends and output is driven by an unobserved finite state, continuous-time Markov chain. Investors learn about the current expected growth rates by observing the path of dividends and output. Heterogeneity in beliefs arises by the investors being endowed with different likelihood functions. We studied a similar formulation with a short-sale constraint in an earlier version of our paper. The asset pricing effects are similar to those reported in the current paper.

Assuming that the short-sale constrained pessimist has logarithmic preferences allows us to obtain a closed-form solution for the pessimist's consumption-portfolio problem for general no-arbitrage price systems. Assuming that the optimist has nonlogarithmic preferences leads to the possibility of both stock price and volatility effects from heterogeneous beliefs. Heterogeneous beliefs do not change stock prices if both investors have logarithmic preferences as shown by Zapatero (1998) without portfolio constraints, and as shown by Detemple and Murthy (1997) with collateral and short-sale constraints. To explore the robustness of our results when the pessimist has nonlogarithmic preferences, we also numerically studied discrete-time economies when both pessimists and optimists had the same IES. Qualitatively, our results were the same as those presented here. Details are available from the authors.

Turning to the consumption-portfolio problem of each investor, investor i's wealth $X^i(t)$ evolves as

$$\begin{aligned} dX^i(t) &= X^i(t)r(t)dt - c^i(t)dt \\ &\quad + \theta^i(t)S(t)\left[(\mu^i(t) - r(t))dt + \sigma(t)dw^i(t)\right], \\ X^i(0) &= x^i S(0) \end{aligned} \tag{11}$$

with x^i the initial stock endowment, $\theta^i(t)$ the shareholding, and $c^i(t)$ the consumption rate for investor i. Let $\alpha^i(t) = X^i(t) - \theta^i(t)S(t)$ denote investor i's bond holdings. Following Dybvig and Huang (1988), we impose a nonnegative wealth constraint, $X^i(t) \geq 0$, on the class of admissible trading strategies to rule out doubling strategies.

The optimist faces complete markets allowing us to apply standard martingale techniques to characterize the optimal consumption and portfolio for quite general price systems as in Cox and Huang (1989) and Karatzas et al. (1987). Summarizing the stock and bond price system, the optimist's state price density $\xi^o(t)$ has dynamics given by

$$d\xi^o(t) = -\xi^o(t)\left[r(t)dt + \kappa^o(t)dw^o(t)\right], \quad \xi^o(0) = 1, \tag{12}$$

with $\kappa^o(t) \equiv \frac{\mu^o(t) - r(t)}{\sigma(t)}$ the market price of risk, or instantaneous Sharpe ratio, under the optimist's beliefs. Proposition 2 verifies that $\kappa^o(t) > 0$ in equilibrium.

The optimist's consumption portfolio problem is

$$\max_{\{c^o(t)\}} E^o\left[\int_0^T e^{-\rho t}\frac{c^o(t)^{1-\gamma}-1}{1-\gamma}dt\right]$$

$$\text{subject to } E^o\left[\int_0^T \xi^o(t)c^o(t)dt\right] \leq x^o S(0). \tag{13}$$

The optimal consumption function is

$$\hat{c}^o(y^0\xi^0(t),t) = \left(\frac{e^{-\rho t}}{y^o\xi^o(t)}\right)^{\frac{1}{\gamma}},$$

$$\text{where } y^o = \left(\frac{E^o\left[\int_0^T e^{-\frac{\rho}{\gamma}t}\xi^o(t)^{\frac{\gamma-1}{\gamma}}dt\right]}{x^o S(0)}\right)^{\gamma} \tag{14}$$

is the Lagrange multiplier from the static budget constraint in (13).

The pessimist's consumption-portfolio problem is complicated by a short-sale constraint $\theta^p(t) \geq 0, \quad \forall t$. Given the pessimist faces an incomplete financial market, we use the approach developed by He and Pearson (1991) and Karatzas et al. (1991) to solve the pessimist's problem. The constrained pessimist's problem is embedded in a fictitious, unconstrained complete markets problem. With the appropriate choice of state prices in the fictitious market, the solution to the fictitious market problem is the solution to the original constrained problem. With logarithmic preferences for the constrained pessimist, we can analytically compute the fictitious state prices for general no-arbitrage price systems. Without logarithmic preferences such as power preferences with $\gamma \neq 1$, we cannot analytically compute the fictitious state prices without making strong restrictions on the underlying economy and endogenous price processes.

Given the optimist and pessimist agree on the interest rate, Equation (10) implies that the optimist's and pessimist's market prices of risk are linked through the disagreement process:

$$\kappa^o(t) - \kappa^p(t) = \overline{\mu}(t), \qquad \text{where } \kappa^p(t) \equiv \frac{\mu^p(t)-r(t)}{\sigma(t)}. \tag{15}$$

If the pessimist does not face a short-sale constraint, his optimal portfolio is the log-optimal portfolio. The log-optimal portfolio shorts the stock only when the pessimist's instantaneous market price of risk is negative. The short-selling constraint is therefore binding in the original lifetime consumption portfolio problem only when $\mu^p(t) < r(t)$. From Equation (10), $\mu^p(t) < r(t)$ when $\overline{\mu}(t) > \kappa^o(t)$ and $\sigma(t) > 0$. We only consider equilibria where $\sigma(t) > 0$—a positive shock to aggregate output increases stock prices, and the pessimist is short-sale constrained only when he believes the expected return on the stock is low enough. Since $\sigma(t)$ is endogenous, the condition puts a restriction on the parameters of the economy. We verify that $\sigma(t) > 0$ holds in our numerical examples.

The constrained pessimist's state-price density in the fictitious market is given by $\xi^p(t)$, and follows the process

$$d\xi^p(t) = -\xi^p(t)\left[r(t)dt + \max\left(\kappa^p(t), 0\right) dw^p(t)\right], \quad \xi^p(0) = 1. \tag{16}$$

The pessimist's consumption portfolio choice problem in the fictitious unconstrained economy is given by

$$\max_{\{c^p(t)\}} E^p\left[\int_0^T e^{-\rho t}\ln(c^p(t))dt\right]$$

$$\text{subject to } E^p\left[\int_0^T \xi^p(t)c^p(t)dt\right] \le x^p S(0). \tag{17}$$

The optimal consumption function is

$$\hat{c}^p(y^p\xi^p(t), t) = \frac{e^{-\rho t}}{y^p\xi^p(t)}, \quad \text{where} \quad y^p = \frac{1 - e^{-\rho T}}{\rho\ x^p S(0)} \tag{18}$$

is the Lagrange multiplier y^p from the static budget constraint in (17).

The optimal consumption function in Equation (18) and the dynamics of the pessimist's fictitious state-price density (16) imply that the pessimist's optimal consumption is locally deterministic when the constraint is binding. The optimal portfolio strategy is to fully invest in the bond when the market price of risk is negative and hold the log-optimal portfolio otherwise.

3. Equilibrium Characterization

We characterize a short-sale constraint at the market level by appealing to general equilibrium restrictions.

Definition 1. *Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations* $((\hat{c}^o, \hat{\alpha}^o, \hat{\theta}^o), (\hat{c}^p, \hat{\alpha}^p, \hat{\theta}^p))$ *and a price system*

$(r, \mu^o, \mu^p, \sigma)$ such that $(\hat{c}^i, \hat{\alpha}^i, \hat{\theta}^i)$ is an optimal solution to investor i's consumption-portfolio problem given his perceived price processes, security prices are consistent across investors, and all markets clear for $t \in [0, T]$:

$$\hat{c}^o(t) + \hat{c}^p(t) = \delta(t), \quad \hat{\theta}^o(t) + \hat{\theta}^p(t) = 1, \quad \hat{\alpha}^o(t) + \hat{\alpha}^p(t) = 0. \tag{19}$$

To determine the equilibrium price system, it suffices to find the two state-price density processes that clear the consumption market,

$$\hat{c}^o(y^o\xi^o(t), t) + \hat{c}^p(y^p\xi^p(t), t) = \delta(t). \tag{20}$$

Once the two state-price processes are determined from Equation (20), they in conjunction with the output process uniquely determine the equilibrium stock and bond prices. If the state price processes clear the consumption market, the associated optimal portfolio strategies also clear the stock and bond markets.

Rather than working directly with both state-price processes in the consumption market clearing condition above, it is convenient to define the stochastic weighting process $\lambda(t)$ given by

$$\lambda(t) = \frac{y^o\xi^o(t)}{y^p\xi^p(t)}, \tag{21}$$

where $\lambda(0) = \frac{y^o}{y^p}$, since $\xi^o(0) = \xi^p(0) = 1$. The stochastic weighting process $\lambda(t)$ summarizes the differences in the investor's opportunity sets from heterogeneous beliefs and the short-sale constraint.

Using the stochastic weighting process, the consumption market clearing condition is

$$\hat{c}^o(y^o\xi^o(t), t) + \hat{c}^p(y^o\xi^o(t)/\lambda(t), t) = \delta(t). \tag{22}$$

Market clearing can be used to uniquely solve for the optimist's state-price density in terms of aggregate output, $\delta(t)$, and the stochastic weighting process, $\lambda(t)$,

$$y^o\xi^o(t) = e^{-\rho t}U_c(\delta(t), \lambda(t)), \tag{23}$$

where $U_c(\delta(t), \lambda(t))$ is a well-defined function and can be interpreted as the marginal utility of a state-dependent representative agent[6] defined by

[6] Such a notion of aggregation was introduced by Cuoco and He (1994) and has been used in a variety of models with incompleteness. See for example Basak and Cuoco (1998); Basak (2000); Detemple and Serrat (2003) and Basak and Gallmeyer (2003). Properties of $U_c(\delta(t), \lambda(t))$ are given in Karatzas et al. (1990) for example.

$$U(c,\lambda) \equiv \max_{c^o+c^p=c} \frac{(c^o)^{1-\gamma}-1}{1-\gamma} + \lambda \ln(c^p). \tag{24}$$

Using the dynamics of the state-price densities in Equations (12) and (16), the dynamics of the stochastic weighting process under the optimist's beliefs are

$$d\lambda(t) = -\lambda(t)\sigma_\lambda(t)dw^o(t), \quad \text{with} \quad \sigma_\lambda(t) \equiv \min\left(\overline{\mu}(t), \kappa^o(t)\right). \tag{25}$$

Given $\kappa^o(t) > 0$ and the optimist has a higher expected growth rate than the pessimist $\overline{\mu}(t) > 0$, then by (25), the weighting process's diffusion coefficient is negative—a positive output shock $dw^o(t)$ leads to a decrease in $\lambda(t)$. From the representative agent construction in Equation (24), a decrease in $\lambda(t)$ through a positive shock to aggregate output reduces the pessimist's consumption share relative to the optimist's consumption share.

In an unconstrained economy, the weighting process's diffusion coefficient is given by the disagreement process $\overline{\mu}(t)$. The short-sale constraint binds when $\overline{\mu}(t) \geq \kappa^o(t)$; a binding short-sale constraint reduces the sensitivity of the optimist's consumption to output shocks relative to an unconstrained economy. Imposing the short-sale constraint reduces the optimist's ability to consume more in good states and less in bad states—the short-sale constraint reduces the optimist's ability to trade on his beliefs.

Proposition 1 provides a construction of the equilibrium.

Proposition 1. *Assume that an equilibrium exists with $\sigma(t) > 0$. In equilibrium, the state-price density for the optimist and the state-price density in the fictitious market for the pessimist are*

$$\xi^o(t) = e^{-\rho t}\frac{U_c(\delta(t),\lambda(t))}{U_c(\delta(0),\lambda(0))}, \quad \xi^p(t) = \frac{\lambda(0)}{\lambda(t)}\xi^o(t). \tag{26}$$

The stock price $S(t)$ is computed from the optimist's marginal valuation,

$$S(t) = E^o\left[\int_t^T e^{-\rho s}\frac{U_c(\delta(s),\lambda(s))}{U_c(\delta(t),\lambda(t))}\delta(s)ds \,\middle|\, \mathcal{F}_t^\delta\right]. \tag{27}$$

The equilibrium consumption allocations are

$$\hat{c}^o(\delta(t),\lambda(t)) = \left(\frac{1}{U_c(\delta(t),\lambda(t))}\right)^{\frac{1}{\gamma}}, \quad \hat{c}^p(\delta(t),\lambda(t)) = \frac{\lambda(t)}{U_c(\delta(t),\lambda(t))}. \tag{28}$$

The stochastic weighting process $\lambda(t)$ has dynamics given by (25) where $\lambda(0)$ solves

$$\hat{c}^p(\delta(0), \lambda(0)) = \frac{\rho}{1 - e^{-\rho T}} x^p S(0). \tag{29}$$

Conversely, if there exists processes $\xi^o(t)$, $\xi^p(t)$, $\lambda(t)$, and $S(t)$ satisfying equations (26)–(29), with $\sigma(t) > 0$, the associated optimal consumption-portfolio policies clear all markets.

The equilibrium consists of two regions depending on whether the short-sale constraint is binding. In the first region, the disagreement process is large enough so that the pessimist desires to short the stock. Here, the short-sale constraint binds; the dynamics of the weighting process depend only on the optimist's market price of risk, similar to the limited participation equilibrium in Basak and Cuoco (1998). In the second region, the disagreement process is small enough so that the pessimist desires a long position in the stock. Here, the short-sale constraint does not bind; the dynamics of the weighting process depend only on the disagreement process $\overline{\mu}(t)$. This is also the only region that would occur in an economy with no short-sale constraint but heterogeneous beliefs. The next proposition reports the equilibrium price of risk and the instantaneous interest rate across the two regions of the equilibrium.

Proposition 2. *The market prices of risk for the optimist and pessimist are*

$$\kappa^o(t) = \begin{cases} A^o(t)\delta(t)\sigma_\delta, & \text{if the constraint binds,} \\ A(t)\delta(t)\sigma_\delta + \frac{A(t)}{A^p(t)}\overline{\mu}(t), & \text{otherwise,} \end{cases}$$

$$\kappa^p(t) = \begin{cases} 0, & \text{if the constraint binds,} \\ A(t)\delta(t)\sigma_\delta - \frac{A(t)}{A^o(t)}\overline{\mu}(t), & \text{otherwise,} \end{cases} \tag{30}$$

where $A^i(t)$, $i = \{o, p\}$ is the absolute risk aversion coefficient for each investor and $A(t)$ is the absolute risk aversion coefficient for the state-dependent representative agent,

$$A^o(t) \equiv \frac{\gamma}{\hat{c}^o(t)}, \quad A^p(t) \equiv \frac{1}{\hat{c}^p(t)}, \quad A(t) \equiv \frac{1}{\frac{1}{A^o(t)} + \frac{1}{A^p(t)}}$$
$$= \frac{1}{\delta(t) + \hat{c}^o(t)\left(\frac{1-\gamma}{\gamma}\right)}. \tag{31}$$

The instantaneous interest rate is

$$r(t) = \begin{cases} \rho + A(t)\delta(t)\mu^o_\delta(t) - \frac{1}{2}A(t)B^o(t)\delta(t)^2\sigma^2_\delta, \ \textit{if the constraint binds,} \\ \rho + A(t)\delta(t)\left(\frac{A(t)}{A^o(t)}\mu^o_\delta(t) + \frac{A(t)}{A^p(t)}\mu^p_\delta(t)\right) - \frac{1}{2}A(t)B(t)\delta(t)^2\sigma^2_\delta \\ \quad -\frac{A(t)^2}{A^o(t)A^p(t)}\left(\frac{1}{2}\frac{B^o(t)+B^p(t)}{A^o(t)+A^p(t)} - 1\right)\overline{\mu}(t)^2 \\ \quad -\frac{A(t)^3}{A^o(t)A^p(t)}\left(\frac{B^o(t)}{A^o(t)} - \frac{B^p(t)}{A^p(t)}\right)\delta(t)\sigma_\delta\overline{\mu}(t), \quad \textit{otherwise,} \end{cases} \tag{32}$$

where $B^i(t),\ i = \{o, p\}$ is the absolute prudence coefficient for each agent and $B(t)$ is the absolute prudence coefficient of the state-dependent representative agent,

$$B^o(t) \equiv \frac{\gamma+1}{\hat{c}^o(t)},\ B^p(t) \equiv \frac{2}{\hat{c}^p(t)},\ B(t) \equiv \left(\frac{A(t)}{A^o(t)}\right)^2 B^o(t) + \left(\frac{A(t)}{A^p(t)}\right)^2 B^p(t). \tag{33}$$

For the same time t consumption allocation, the optimist's market price of risk is lower in a constrained than in an unconstrained economy, while the instantaneous interest rate is higher in a constrained than in an unconstrained economy for all $\gamma < \overline{\gamma}$ where $\overline{\gamma} \geq 1$.

The market prices of risk in our model are consistent with the one period models with heterogeneous beliefs and short-sale constraints in Lintner (1969), Miller (1977), and Jarrow (1980). The optimist's market price of risk only depends on the optimist's beliefs when the constraint is binding, and on both investors' beliefs when the constraint is not binding. Conditional on the same consumption allocation at time t,[7] the optimist's market price of risk is lower in a constrained versus an unconstrained economy.

Unlike in one period models where the interest rate is exogenous, the interest rate in our model is endogenous. The first three terms of the interest rate (32) in each region are of the same form as in an unconstrained common belief economy. But the interest rate now depends on the optimist's beliefs when the constraint is binding and on both investors' beliefs when the constraint is not binding. The extra two terms in the unconstrained region's interest rate capture the effects of heterogeneous beliefs and heterogeneous preferences on

[7] Given the short-sale constraint influences intertemporal consumption trade-offs, identical consumption allocations at time t across the constrained and unconstrained economies do not imply that the two economies have identical endowments. Our numerical work in Section 4 considers identical endowments across the two economies.

the interest rate arising from the risk shifting induced by the disagreement process. These two terms are both negative when the optimist is less risk averse than the pessimist. Conditional on the same consumption allocation at time t, the interest rate is higher in a constrained versus an unconstrained economy as long as the optimist's risk aversion is not too much higher than the pessimist's risk aversion.

In contrast to a one period model, the stock's volatility is endogenous in our setting and is needed to calculate the stock's risk premium $\mu^i(t) - r(t)$ under either investor's belief. Once the stock's volatility is known, its risk premium can be computed by using the market prices of risk given by (30). We use the Clark-Ocone formula to compute the stock's volatility.[8]

Proposition 3. *The stock's volatility is given by*

$$\sigma(t) = A(t)\delta(t)\sigma_\delta + \frac{E^o\left[\int_t^T \xi^o(s)(1 - A(s)\delta(s))\mathcal{D}_t\delta(s)ds \middle| \mathcal{F}_t^\delta\right]}{E^o\left[\int_t^T \xi^o(s)\delta(s)ds \middle| \mathcal{F}_t^\delta\right]} \tag{34}$$

$$+\frac{A^o(t)}{A^o(t)+A^p(t)}\sigma_\lambda(t) + \frac{E^o\left[\int_t^T \xi^o(s)\frac{A^o(s)}{A^o(s)+A^p(s)}\frac{\delta(s)}{\lambda(s)}\mathcal{D}_t\lambda(s)ds \middle| \mathcal{F}_t^\delta\right]}{E^o\left[\int_t^T \xi^o(s)\delta(s)ds \middle| \mathcal{F}_t^\delta\right]}.$$

where $\mathcal{D}_t\delta(s)$ is the Malliavin derivative for $\delta(s)$, and $\mathcal{D}_t\lambda(s)$ is the Malliavin derivative for $\lambda(s)$. Both Malliavin derivatives are defined in the proof of Proposition 3.

Proposition 3 provides a decomposition of how the stock's volatility reacts to an output shock through changes in both $\delta(t)$ and $\lambda(t)$. The terms in the first row of Equation (34) measure the effect of a shock to output on the stock price holding $\lambda(t)$ constant. The first term is the effect on today's state price density. The second term is the effect on the variability of future cash flows from a shock to today's output, holding the future path of $\lambda(t)$ fixed. The Malliavin derivative $\mathcal{D}_t\delta(s)$ measures how perturbing $w^o(t)$ affects the instantaneous change of $\delta(s)$ for $s \geq t$.

The terms in the second line of Equation (34) capture the effect of the shock through $\lambda(t)$ on the stock price, holding output fixed. The first term in the second line of Equation (34) is the effect of the shock through $\lambda(t)$ on today's state-price density. The second term in the second line of Equation (34) is the contribution to the stock's volatility due to future shifts in $\lambda(s)$. The Malliavin

[8] The Clark-Ocone formula was previously used by Ocone and Karatzas (1991) and Detemple et al. (2003) to compute trading strategies in an optimal investment problem and by Fourniéet al. (1999) to compute hedging parameters in option pricing applications.

derivative $\mathcal{D}_t\lambda(s)$ measures how perturbing $w^o(t)$ affects the instantaneous change of $\lambda(s)$ for $s \geq t$. When beliefs are identical and the short-sale constraint does not bind, the weighting process is constant and both the terms in the second line are zero.

Given that the expectations in Equation (34) cannot be computed in closed-form, it is difficult to establish the effect of the short-sale constraint on the stock's volatility. But the expectations in Equation (34) can be computed through a Monte Carlo simulation. One case that can be computed in closed-form is when the optimist also has logarithmic preferences. This case provides a convenient benchmark in our numerical analysis in the next Section as it provides the knife-edge case for many of the asset pricing effects.

Corollary 1. *Assume that the optimist has logarithmic preferences ($\gamma = 1$). The stochastic weighting process can be expressed as a ratio of the relative wealths across the two investors $\lambda(t) = \frac{X^p(t)}{X^o(t)}$.*
The equilibrium stock price and dynamics in either an unconstrained or a constrained economy are

$$S(t) = \frac{e^{-\rho t} - e^{-\rho T}}{\rho}\delta(t), \quad \mu^i(t) = \mu^i_\delta(t), \quad \sigma(t) = \sigma_\delta. \tag{35}$$

The market prices of risk for the optimist and pessimist and the interest rate are

$$\kappa^o(t) = \begin{cases} (1+\lambda(t))\sigma_\delta, \\ \sigma_\delta + \frac{\lambda(t)}{1+\lambda(t)}\overline{\mu}(t), \end{cases} \quad \kappa^p(t) = \begin{cases} 0, & \textit{if the constraint binds,} \\ \sigma_\delta - \frac{1}{1+\lambda(t)}\overline{\mu}(t), & \textit{otherwise,} \end{cases} \tag{36}$$

$$r(t) = \begin{cases} \rho + \mu^o_\delta(t) - (1+\lambda(t))\sigma^2_\delta, & \textit{if the constraint binds,} \\ \rho + \left(\frac{1}{1+\lambda(t)}\mu^o_\delta(t) + \frac{\lambda(t)}{1+\lambda(t)}\mu^p_\delta(t)\right) - \sigma^2_\delta, & \textit{otherwise.} \end{cases} \tag{37}$$

For the same endowment, the market price of risk is lower in the constrained than in the unconstrained economy, and the interest rate is higher in the constrained than in the unconstrained economy.

When the optimist has logarithmic preferences, the stock price is not influenced by a short-sale constraint on the pessimist—its level is the same across otherwise identical unconstrained or constrained economies. When the constraint binds, the stock price does not change as the income and substitution effects exactly offset each other. Imposing the short-sale constraint does however lead to the optimist facing a lower market price of risk and the interest rate rising.

In contrast to Basak and Cuoco's limited participation model, the interest rate is higher in the constrained than the unconstrained economy. This is driven by how beliefs enter the equilibrium interest rate. With the constraint imposed, only the optimist's higher beliefs drive the interest rate as compared to a lower wealth-weighted average of the investors' beliefs in the unconstrained case.

The next proposition summarizes how the stock price is influenced by a short-sale constraint as a function of the optimist's IES.[9] For an IES close to one, imposing the constraint reduces the stock price if the optimist's IES is greater than one and increases the stock price if the optimist's IES is less than one.

Proposition 4. *Consider two economies with identical beliefs, preferences, and endowments. One economy is short-sale constrained and one economy is unconstrained.*

- *If the constraint binds with positive probability and the optimist's IES is greater than one ($\gamma < 1$), then the initial stock price in the short-sale constrained economy is lower than the initial stock price in the unconstrained economy for γ close to one.*
- *If the constraint binds with positive probability and the optimist's IES is smaller than one ($\gamma > 1$), then the initial stock price in the constrained economy is higher than the initial stock price in the unconstrained economy for γ close to one.*

The intuition behind how the stock price is influenced by the short-sale constraint is driven by the relative strength of the optimist's income and substitution effects for current consumption relative to savings. Consider an unconstrained economy where a pessimist short sells. If a short-sale constraint is now imposed, security prices must adjust to reduce the optimist's stock demand. Such an adjustment occurs through a fall in the optimist's price of risk and an increase in the interest rate. Both changes reduce the expected return on the optimist's portfolio. Reducing the expected return on the optimist's portfolio changes his demand for financial assets relative to current consumption which is driven by income and substitution effects.

The income effect increases the demand for financial assets relative to consumption, while the substitution effect decreases the demand for financial assets relative to consumption. The income effect dominates when the optimist's IES is less than one. Since the supply of aggregate consumption is

[9] The proof of the Proposition is based on perturbing the marginal utility of the representative agent around the case when the optimist has logarithmic preferences. See for example Judd (1999), and Kogan and Uppal (2001).

fixed, the demand for current consumption must rise to clear the consumption market. The demand for current consumption can only rise if the value of aggregate wealth rises. Given that aggregate wealth equals the stock market, the stock price must rise when the short-sale constraint is imposed. The substitution effect dominates when the optimist's IES is greater than one. By the same reasoning, the stock price must therefore fall when the short-sale constraint is imposed and the optimist's IES is greater than one.

Although the initial stock price can either rise or fall when the constraint is imposed, the pessimist always finds the stock overpriced if the constraint ever binds. The pessimist's marginal valuation for the stock at time t is the price that the pessimist would be willing to pay for a small quantity of the stock at time t if he were required to hold the stock forever. If the pessimist did not face a short-sale constraint, the equilibrium price of the stock would equal the pessimist's marginal valuation for the stock. The pessimist's marginal valuation can be computed with the fictitious state-price density process used to solve the pessimist's consumption-portfolio problem. The pessimist's marginal valuation for the stock has the following dynamics:

$$dS^p(t) + \delta(t)dt = S^p(t)\left[\max\left(\mu^p(t), r(t)\right)dt + \sigma(t)dw^p(t)\right], S^p(T) = 0 \tag{38}$$

and solution:

$$S^p(t) = E^p\left[\int_t^T e^{-\rho s}\frac{u_c^p(\hat{c}^p(\delta(s), \lambda(s)))}{u_c^p(\hat{c}^p(\delta(t), \lambda(t)))}\delta(s)ds \,\middle|\, \mathcal{F}_t^\delta\right], \tag{39}$$

with $u_c^p(\delta(s), \lambda(s))$ the pessimist's marginal utility.

Proposition 5. *The equilibrium stock price and marginal valuation processes satisfy*

$$\begin{aligned} &S(t) > S^p(t), \textit{ if the constraint binds with positive probability for some } t' \geq t, \\ &S(t) = S^p(t), \textit{ else}. \end{aligned} \tag{40}$$

If the constraint is not binding for the pessimist at time t, the pessimist holds the log-optimal portfolio, which is a long position in the stock. If there is a positive probability that the short-sale constraint will bind at some time in the future, the stock price is higher than the pessimist's marginal valuation for the stock. The difference between the stock price and the pessimist's marginal valuation can be interpreted as a speculative premium from the perspective of the pessimist. Harrison and Kreps (1978), Morris (1996), and Scheinkman and Xiong (2003) define the speculative premium from the perspective of the most optimistic investor which is zero in our setting since the identity of the optimist and the pessimist does not change through time. Following Detemple and Murthy (1997), we allow for a more general definition of the speculative premium which is investor-specific. The speculative premium arises because the pessimist expects to be able to resell the stock to the optimist at a favorable price in the future.

4. Numerical Results

We provide numerical examples of the effects of imposing a short-sale constraint on asset prices when the optimist's IES is no longer close to one. Proposition 1 provides a method to construct the equilibrium numerically. From Equation (25), the weighting process dynamics do not depend on the stock price dynamics. Conditional on an initial guess for $\lambda(0)$, we simulate the weighting and output processes to solve for the stock's price and volatility through a Monte Carlo simulation of Equations (27) and (34). We iterate to find a $\lambda(0)$ satisfying the pessimist's budget constraint given his endowment. The Malliavin derivatives needed to compute the stock's volatility from Equation (34) are provided in Appendix B. Technical details of the simulations are given in Appendix C.

We assume each investor has a different Gaussian prior about the initial drift of the output process, $\mu_\delta^i \sim \mathcal{N}(\mu_\delta^i(0), \nu^i(0)),\ i \in \{o, p\}$, where $\mu_\delta^i(0)$ is investor i's belief about the output's expected growth rate and $\nu^i(0)$ is investor i's belief variance. An investor's optimal filtering problem is given by a standard Kalman-Bucy filter. Investor i's prediction of the expected output growth $\mu_\delta^i(t)$ is

$$d\mu_\delta^i(t) = \frac{\nu^i(t)}{\sigma_\delta} dw^i(t), \quad \nu^i(t) = \frac{\nu^i(0)\sigma_\delta^2}{\nu^i(0)t + \sigma_\delta^2}, \quad i \in \{o, p\}. \tag{41}$$

As t grows large, investor i learns the true expected output growth rate unless he has dogmatic beliefs defined by $\nu^i(0) = 0$. In this case, investor i is so confident in his prior that he completely ignores any information from the output process implying $\mu_\delta^i(t) = \mu_\delta^i(0)$ for all t. Details about the optimal filtering problem can be found in Liptser and Shiryayev (1977).

Under the optimist's beliefs, the disagreement process $\overline{\mu}(t)$ has dynamics:

$$d\overline{\mu}(t) = -\frac{\nu^p(t)}{\sigma_\delta}\overline{\mu}(t)dt + \frac{\nu^o(t) - \nu^p(t)}{\sigma_\delta}dw^o(t). \tag{42}$$

If both investors have strictly positive belief variances, the disagreement process converges to zero as t grows large. If both investors have the same belief variances, the disagreement process is deterministic, and one investor is always strictly more optimistic than the other. In particular, when both investors have dogmatic priors ($\nu^o(0) = \nu^p(0) = 0$), the disagreement process is a constant.

In all our numerical examples, the time-preference parameter ρ for both investors equals 0.03 and the optimist's IES $\frac{1}{\gamma}$ is varied between 0.5 and 2.0. The initial endowment of each investor type is 50% of the market portfolio with no initial bond position. The economy has a 20 year horizon. All plots that follow represent equilibrium quantities as a function of the optimist's IES. Solid lines (–) represent quantities for unconstrained economies, while dashed–dotted lines (–·) represent quantities for constrained economies unless otherwise labeled in the plot.

4.1 BASELINE CASE

In our baseline parameterization, the true expected output growth rate $\mu_\delta = 4\%$ with an instantaneous volatility $\sigma_\delta = 5\%$. To parameterize the optimist's beliefs, we appeal to Gordon (2000) who reports that the average growth rate in GNP in the USA from 1995 to 1999 was 4.90%. We therefore set the optimist's initial expected growth rate $\mu_\delta^o(0)$ equal to 5%. The pessimist's beliefs are correct, so $\mu_\delta^p(0) = 4\%$. Both investor types have dogmatic beliefs implying $\nu^o(0) = \nu^p(0) = 0$. Here, the disagreement process is constant: $\overline{\mu}(t) = 0.20\%$ for all t. Figure 1 plots characteristics of the equilibrium stock price and the short-sale constraint, while Figure 2 plots additional properties of the unconstrained and constrained economies.

The behavior of the equilibrium stock prices in both economies is consistent with Proposition 4 as seen in the top plot of Figure 1. When the optimist's IES is less than one, the stock price is lower in the unconstrained than in the constrained economy; whereas when the optimist's IES is greater than one, the stock price is higher in the unconstrained than in the constrained economy. As the optimist's IES rises across both economies, the representative agent's risk aversion decreases, leading to an increase in the equilibrium stock price.

The magnitude of the difference in stock prices across the unconstrained and constrained economies is captured by the solid line (–) in the middle plot of

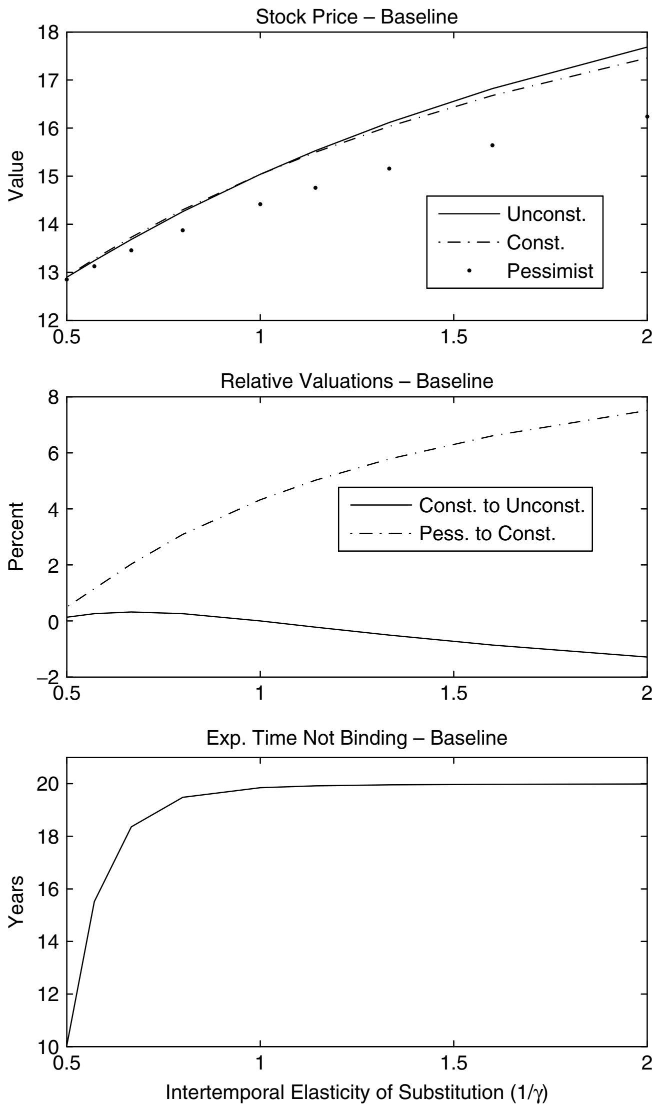

Figure 1. Valuations with the Baseline Parameters.
Parameters: $x^o = x^p = 0.5$, $\delta(0) = 1.0$, $\mu_\delta = 0.04$, $\sigma_\delta = 0.05$, $\mu_\delta^o(0) = 0.05$, $\nu^o(0) = 0$, $\mu_\delta^p(0) = 0.04$, $\nu^p(0) = 0$, $\rho = 0.03$, and $T - t = 20$. The largest 95% Monte Carlo confidence intervals for the unconstrained stock price, the constrained stock price, and the pessimist's valuation occur at $\frac{1}{\gamma} = 2.0$ and are respectively: $\pm 5.62 \times 10^{-3}$, $\pm 2.15 \times 10^{-3}$, and $\pm 5.94 \times 10^{-3}$. The largest 95% Monte Carlo confidence intervals for the expected time until the constraint is not binding occurs at $\frac{1}{\gamma} = 0.5$ and is $\pm 7.33 \times 10^{-2}$.

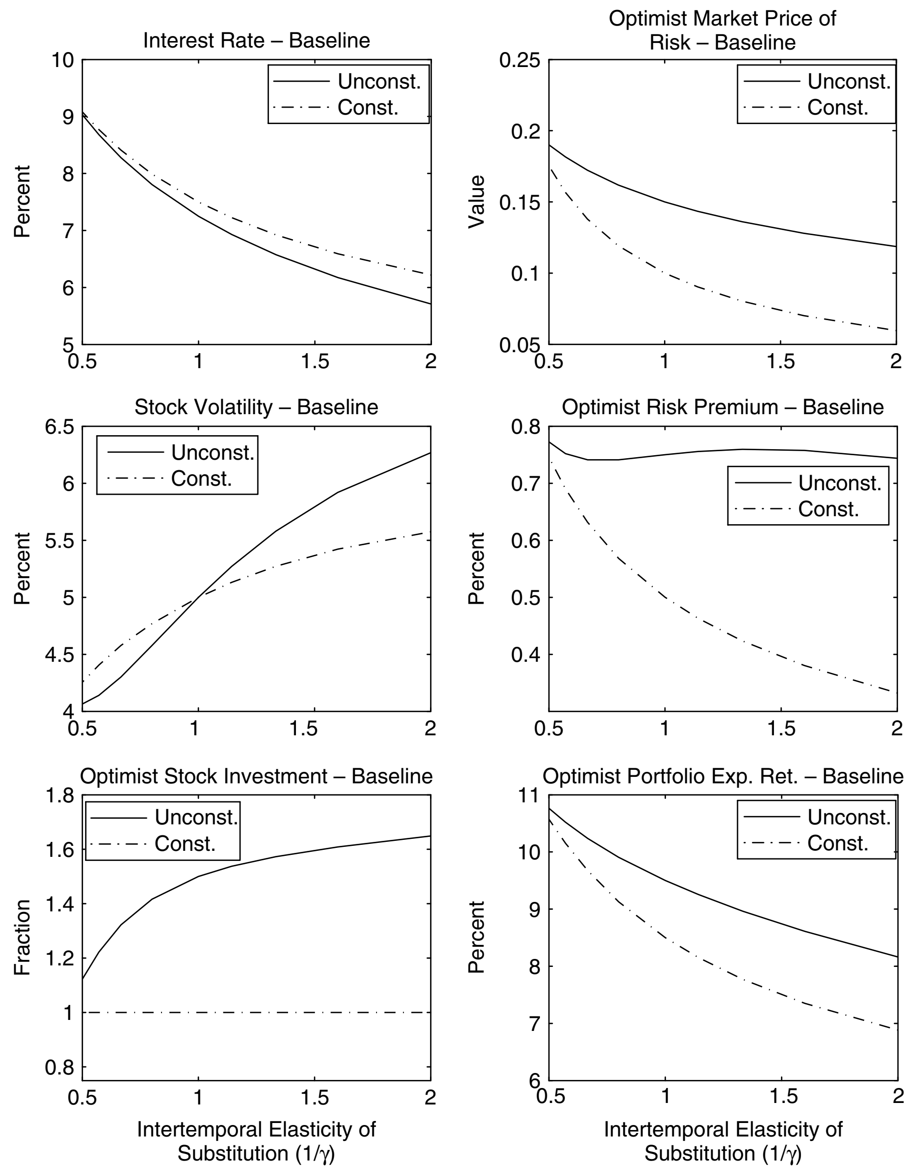

Figure 2. Equilibrium Properties with the Baseline Parameters.
Parameters: $x^o = x^p = 0.5$, $\delta(0) = 1.0$, $\mu_\delta = 0.04$, $\sigma_\delta = 0.05$, $\mu_\delta^o(0) = 0.05$, $\nu^o(0) = 0$, $\mu_\delta^p(0) = 0.04$, $\nu^p(0) = 0$, $\rho = 0.03$, and $T - t = 20$. The largest 95% Monte Carlo confidence intervals for the unconstrained volatility, the constrained volatility, the unconstrained optimist's risk premium, the constrained optimist's risk premium, and the unconstrained optimist's stock investment occur at $\frac{1}{\gamma} = 2.0$ and are respectively: $\pm 4.12 \times 10^{-5}$, $\pm 5.12 \times 10^{-5}$, $\pm 4.90 \times 10^{-6}$, $\pm 8.96 \times 10^{-6}$, and $\pm 2.87 \times 10^{-3}$.

Figure 1 which plots the constrained stock price relative to the unconstrained stock price, expressed as a percentage. When the optimist's IES is 0.5, the constrained price is 0.14% higher than the unconstrained price. The percentage overvaluation is maximized when the optimist's IES is approximately 0.67. Here the constrained price is 0.32% higher than the unconstrained price. The percentage overvaluation is decreasing for elasticities greater than 0.67. At an elasticity of 2.0, the constrained price is 1.29% lower than the unconstrained price.

While the magnitude of the difference in stock prices across the unconstrained and constrained economies is relatively small, the constrained pessimist's stock valuation can deviate significantly from the market-clearing stock price. In Figure 1, the constrained pessimist's marginal valuation of the stock is given by the dotted line (··) in the top plot and expressed as a percentage relative to the constrained economy's stock price by the dashed–dotted line ($- \cdot -$) in the middle plot. The pessimist always views the stock as overvalued in equilibrium, and the percentage overvaluation is increasing in the optimist's IES. At an optimist IES of 2.0, the pessimist views the stock as overvalued by 7.5%.

The bottom plot of Figure 1 reports the expected time the constraint binds. The pessimist's perceived overvaluation of the stock is increasing in the optimist's IES because the short-sale constraint binds more frequently as the IES increases. From Equation (38), the pessimist's marginal stock valuation grows at the riskless rate when the constraint binds and at the pessimist's expected stock return when not binding. The first order effect of increasing the time that the constraint binds is that the expected growth rate of the pessimist's marginal valuation increases relative to the expected growth rate of the equilibrium stock price. Since the constraint binds more often when the optimist's IES increases, the percentage overvaluation also increases.

Figure 2 illustrates additional properties. The short-sale constraint's influence on the interest rate and the optimist's market price of risk is highlighted in the top two plots of Figure 2. Suppose that the pessimist short sells stock in an unconstrained economy. For markets to clear when a short-sale constraint is imposed, security prices must change to reduce the optimist's stock demand. Two changes can lead to a reduction in the optimist's stock demand–his market price of risk can drop, and the cost of borrowing at the riskless rate to purchase the stock can rise. From the figure, both changes occur. The initial interest rate increases and the initial optimist's price of risk decreases in the constrained economy relative to the unconstrained economy.

The middle left plot in Figure 2 reports the initial stock volatility across the two economies. The volatilities are computed through Monte Carlo simulations of Equation (34) for both economies. In both economies, an increase in the optimist's IES leads to an increase in stock volatility.

This volatility pattern is driven by the volatility's sensitivity to the stochastic weighting process. In both the unconstrained and constrained economies, stock returns depend on innovations to the output and stochastic weighting processes. Holding the level of the stochastic weighting process fixed, stock returns are positively correlated to output innovations. Numerically, this correlation is relatively insensitive to the optimist's IES.

Innovations in the stochastic weighting process are perfectly negatively correlated to innovations in output—a positive shock to $\delta(t)$ leads to a decrease in $\lambda(t)$. For an optimist's IES less than one, the optimist is more risk averse than the pessimist. A decrease in the weighting processes makes the representative agent more risk averse and so reduces the stock price. The negative correlation between innovations to output and the weighting process thereby diminishes the effect of a positive shock to output on stock returns—stock returns respond less to output shocks when the optimist's IES is less than one. For an optimist's IES greater than one, the optimist is less risk averse than the pessimist. Now a decrease in the weighting processes makes the representative agent less risk averse and so increases the stock price. The negative correlation between innovations to output and the weighting process thereby amplifies the effect of a positive shock to output on stock returns—stock returns respond more to output shocks when the optimist's IES is greater than one. The overall result is that increasing the optimist's IES leads to an increase in stock volatility in both economies.

The effect of the short-sale constraint on the stock's volatility is driven by the investors' differences in preferences. Imposing the constraint reduces the magnitude of the instantaneous covariance between innovations to output and the weighting process which diminishes the effect of the weighting process on the stock's volatility. As a consequence, in comparison to the unconstrained stock price, the constrained stock price becomes less volatile when the IES is less than one and more volatile when the IES is greater than one.

Our result that the stock's volatility can either rise or fall when the short-sale constraint is relaxed complements the analytical stock volatility results in Bhamra and Uppal (2006). They study an economy where two agents are borrowing constrained and can only invest in a stock. They show that the stock volatility can rise when the borrowing constraint is relaxed if the precautionary demand for savings is not too high.[10] They illustrate that this precautionary

[10] The condition in Bhamra and Uppal (2006) is on the representative agent's precautionary demand for savings and is given in Proposition 2 of their paper. Additionally, the two investors cannot have the same IES coefficient.

savings condition is satisfied for a broad range of parameter values. Using the same output process parameters as in our numerical examples ($\mu_\delta = 0.04$, $\sigma_\delta = 0.05$), their condition would hold for a logarithmic investor and an investor with an IES coefficient greater than $\frac{1}{17}$. Our numerical volatility result highlights that a short-sale constraint can lead to a different effect. When the optimist's IES is less than one, stock volatility actually falls when our short-sale constraint is removed in contrast to an increase in volatility when removing the borrowing constraint in Bhamra and Uppal (2006).

The middle right plot in Figure 2 reports the optimist's instantaneous risk premium in the two economies. Increasing the optimist's IES decreases the market price of risk and increases stock volatility. Since the risk premium is the product of the optimist's market price of risk and the stock volatility, the risk premium is nonmonotonically related to the optimist's IES. Consistent with a one period model, imposing the short-sale constraint reduces the risk premium.

The optimist's initial stock holdings across both economies are reported in the bottom left plot in Figure 2. The stock holding is measured as a fraction of the aggregate stock supply. Increasing the IES decreases the optimist's risk aversion and so increases the optimist's stock demand in the unconstrained economy. For a low IES, slightly less than 0.5, the optimist's stock holding is 1 implying that the short-sale constraint would not bind if it was imposed. In the constrained economy, the optimist holds all of the stock for all levels of the elasticity we report; the constraint is always binding.

The bottom right plot in Figure 2 reports the expected return of the optimist's portfolio across the two economies. In the unconstrained economy, the stock risk premium is almost flat with respect to the optimist's IES, and in the constrained economy, the stock's risk premium is decreasing in the optimist's IES. In both economies, the optimists borrow at the instantaneous riskless rate to purchase stock. As the optimist's IES increases, the instantaneous riskless rate rises and so the expected return on the optimist's portfolio is decreasing in his IES. At all levels of the optimist's IES, imposing the constraint increases the interest rate, reduces the optimist's risk premium, and reduces the optimist's stock holdings. Imposing the constraint therefore reduces the expected return on the optimist's portfolio.

4.2 INTRODUCING LEARNING

In our baseline case, both the pessimist and optimist have dogmatic beliefs; they do not learn from observing realized output. We now allow for learning by increasing the common prior variance to 1%: $\nu^o(0) = \nu^p(0) = 0.01$. All other parameters are kept at the baseline values. Here, the disagreement process starts at $\overline{\mu}(0) = 0.20$ and declines deterministically to approximately 0.15 at the end of the economy. Figure 3 compares equilibrium quantities across economies with the baseline and the learning parameters. In the figure, the left-hand side reports baseline parameter quantities, while the right-hand side reports learning parameter quantities.

The equilibrium stock prices and constrained pessimist's valuations reported in the top plots are roughly similar across the baseline and learning parameterizations—introducing learning has a small effect on the level of the equilibrium prices. The relative valuations, given in the middle plots, are slightly smaller with learning. For example, the maximum overvaluation according to the pessimist falls to 6% from 7.5% under the baseline parameters when $\frac{1}{\gamma} = 2.0$. The reduction is driven by the change in dynamics of the disagreement process. Since the disagreement process now decreases through time, the pessimist's short-sale constraint tends to bind less severely with learning, as can be seen from the bottom plots that report the expected time until the constraint stops binding.

4.3 INCREASING DISAGREEMENT

Our next example studies the impact on equilibrium quantities of increasing the spread in the investors' beliefs. We increase the optimist's prior beliefs about output growth from 5% to 6%: $\mu^o(0) = 6\%$. All other parameters are kept at the baseline values. Relative to the baseline parameters, the disagreement process now doubles to $\overline{\mu}(0) = 0.4\%$. Since beliefs are dogmatic, the disagreement process is constant. Figure 4 summarizes the differences in equilibrium quantities relative to the baseline parameters as a function of the optimist's IES. The layout of the figure is identical to Figure 3 where introducing learning was studied. The only difference now is that the right-hand side of the figure plots the equilibrium quantities for the increased disagreement parameters.

By increasing the disagreement across investors, the income and substitution effects are amplified, leading to larger unconstrained stock price movements relative to the baseline parameters as seen in the top plots of Figure 4. When the optimist's IES is less than one, the income effect of higher expected output growth is stronger than the substitution effect. In this case, increasing the optimist's prior about the expected output growth further reduces the unconstrained economy's stock price. When the optimist's IES is greater than one, by the same reasoning, increasing the optimist's prior about the expected output growth increases the unconstrained economy's stock price.

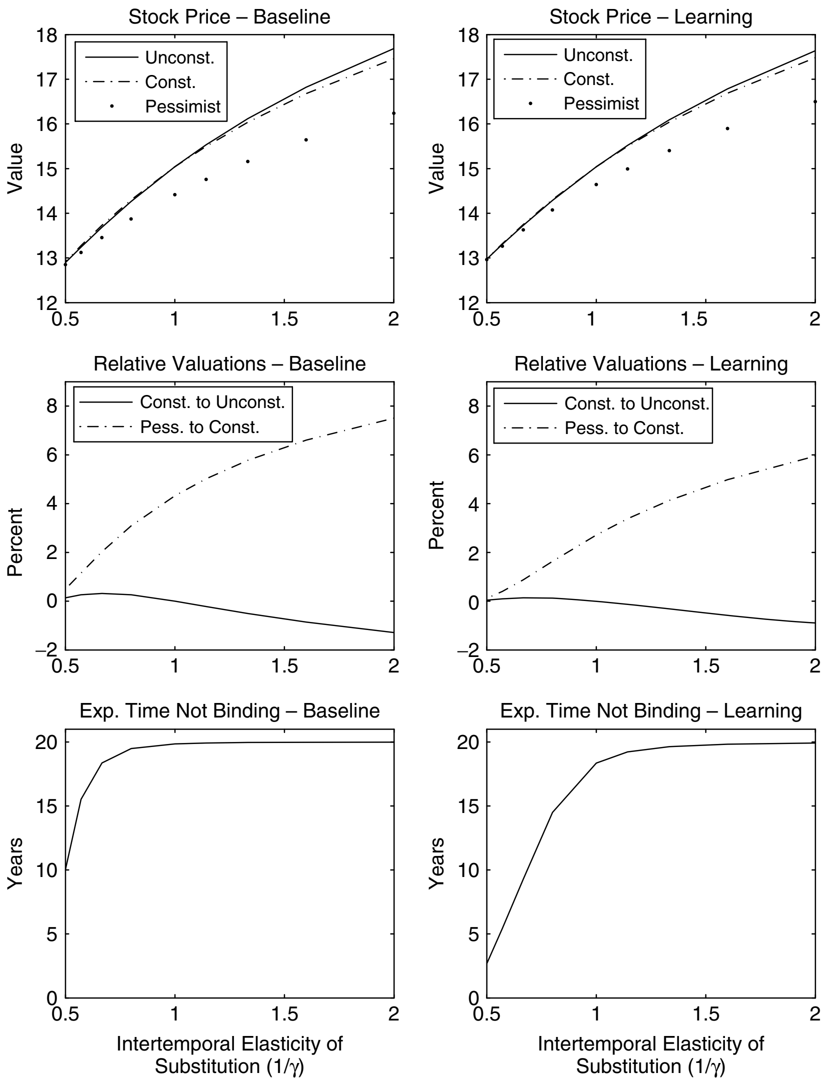

Figure 3. Introducing Learning.
Parameters: $x^o = x^p = 0.5$, $\delta(0) = 1.0$, $\mu_\delta = 0.04$, $\sigma_\delta = 0.05$, $\mu_\delta^o(0) = 0.05$, $\nu^o(0) = 0.01$, $\mu_\delta^p(0) = 0.04$, $\nu^p(0) = 0.01$, $\rho = 0.03$, and $T - t = 20$. The right panel's largest 95% Monte Carlo confidence intervals for the unconstrained stock price, the constrained stock price, and the pessimist's valuation occur at $\frac{1}{\gamma} = 2.0$ and are respectively: $\pm 7.27 \times 10^{-3}$, $\pm 4.29 \times 10^{-3}$, and $\pm 6.23 \times 10^{-3}$. The largest 95% Monte Carlo confidence intervals for the expected time until the constraint is not binding in the right panel occurs at $\frac{1}{\gamma} = 0.5$ and is $\pm 5.11 \times 10^{-2}$.

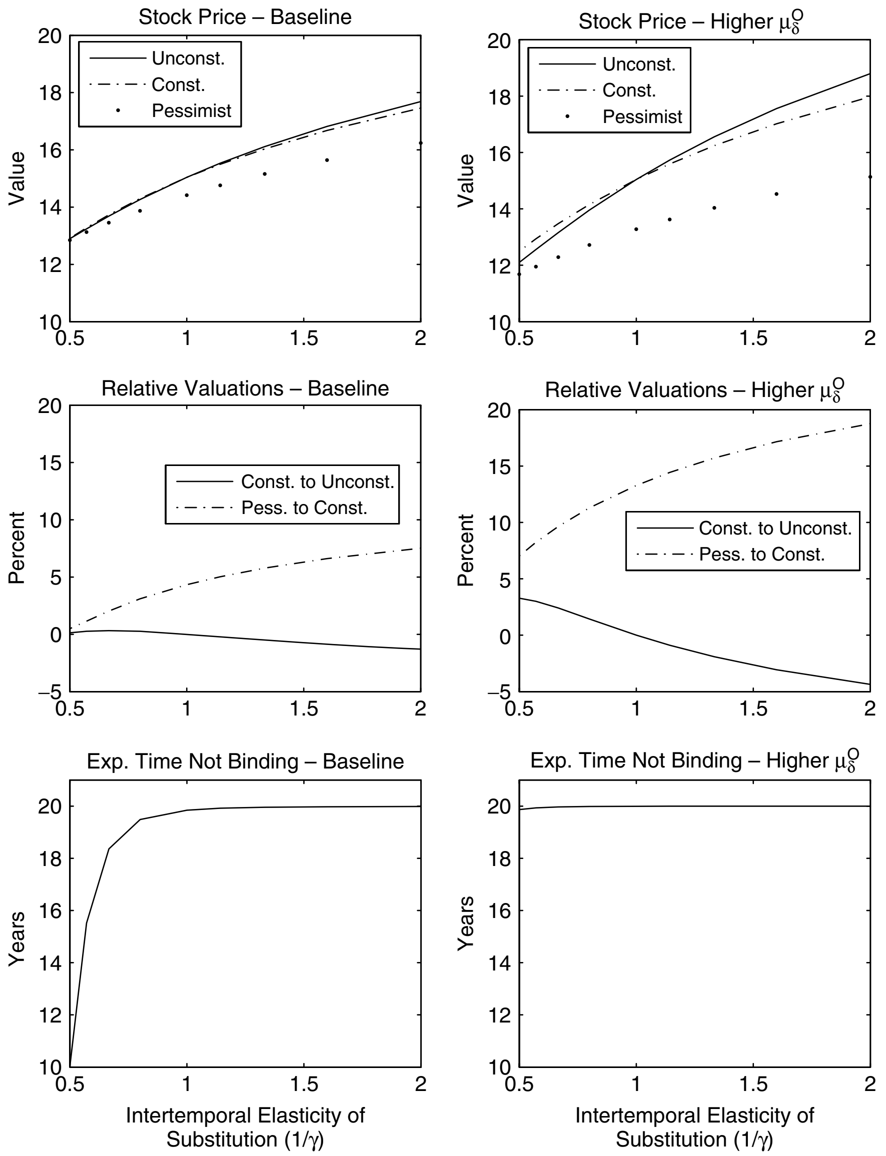

Figure 4. Increasing the Optimist's Beliefs.
Parameters: $x^o = x^p = 0.5$, $\delta(0) = 1.0$, $\mu_\delta = 0.04$, $\sigma_\delta = 0.05$, $\mu_\delta^o(0) = 0.06$, $\nu^o(0) = 0$, $\mu_\delta^p(0) = 0.04$, $\nu^p(0) = 0$, $\rho = 0.03$, and $T - t = 20$. The right panel's largest 95% Monte Carlo confidence intervals for the unconstrained stock price, the constrained stock price, the pessimist's valuation, and the expected time until the constraint is not binding occur at $\frac{1}{\gamma} = 0.5$ and are respectively: $\pm 3.69 \times 10^{-3}$, $\pm 4.80 \times 10^{-3}$, $\pm 1.02 \times 10^{-3}$, and $\pm 9.57 \times 10^{-2}$.

The increase in the optimist's expected output growth has a similar amplifying effect on the constrained stock prices as can be best seen from the solid lines in the relative valuation plots in the middle of Figure 4. For all levels of the IES, an increase in the optimist's prior about the expected output growth increases the optimist's demand for the stock, implying that prices must change by more for the optimist to optimally hold all shares in the constrained economy. Imposing the constraint now has a more pronounced effect on the stock price when the optimist's prior increases—in the low IES case, the stock price rises by more as a consequence of the imposing the constraint, and in the high IES case, the stock price falls by more as a consequence of imposing the constraint.

The size of the pessimist's overvaluation, given in the middle plots of Figure 4, depends on how long the constraint binds as well as the spread between the pessimist's expected stock return and the riskless interest rate. Increasing the optimist's expected output growth increases both of these quantities. The bottom plot in the figure shows that the expected time until the constraint stops binding increases to the economy's ending date ($T = 20$) for all levels of the optimist's IES. In conjunction with a larger spread between the constrained pessimist's expected stock return and the equilibrium riskless interest rate, the pessimist's perceived overvaluation increases significantly relative to the baseline parameters. At an optimist's IES of 2.0, the pessimist's overvaluation is maximized at 18.8% for the increased disagreement parameters versus 7.5% for the baseline parameters.

5. Conclusions

We study the effects of a short-sale constraint on prices and allocations in a dynamic economy with optimistic and pessimistic investors. If the optimist's IES is less than one, then imposing the short-sale constraint increases the stock price. If the optimist's IES is greater than one, then imposing the short-sale constraint decreases the stock price. In all cases, the presence of the constraint implies that the pessimist's marginal valuation of the stock is lower than the equilibrium price. The pessimist finds the stock most overvalued when the optimist's IES is greater than one—exactly when imposing the constraint reduces the equilibrium stock price.

Consistent with a static economy, we also find numerically that the optimist's market price of risk always falls when the short-sale constraint is imposed. The existence of the short-sale constraint also leads to higher instantaneous interest rates. Like the stock price results, the behavior of the stock volatility when the short-sale constraint is relaxed is driven by the optimist's IES. Relaxing the constraint leads to a lower stock volatility when the optimist's IES is less than one and to a higher volatility when the optimist's IES is greater than one.

We study the effects of heterogeneity and a short-sale constraint on the valuation of the market as a whole. An interesting extension of our analysis is to study a short-sale constraint at the sector or stock level where more empirical evidence is known about the effects of short-sale constraints. For example, Diether et al. (2002) provide evidence that expected stock returns are higher for stocks with high dispersion of analysts' beliefs, interpreting the result as evidence of binding short-sale constraints.

In Jarrow (1980), a short-sale constraint increases the value of the constrained stock and also the value of the stock market as a whole. In this case, the substitution and income effects that drive our model would continue to hold. The value of the stock market as a whole would rise only if the income effect is stronger than the substitution effect on current consumption demand. In a dynamic setting, it would be interesting to understand under what conditions the constrained stock price would rise relative to the price of the other stocks.

Another avenue to study is the role of constraints on the survival of investors with different beliefs as in the unconstrained models of Kogan et al. (2006), Berrada (2004), and Yan (2006).

Appendix A: Proofs

Proof of Lemma 1. The result follows from Theorem 8.1 of Liptser and Shiryayev (1977). ■

Proof of Proposition 1. The proof follows Karatzas et al. (1990) with the appropriate modifications taken to accommodate for investors facing different state prices. From clearing in the consumption good market, Equations (26) and (28) follow from the representative agent construction. The stock price (27) follows by substituting (26) into the stock price valued using the state-price density process for the optimistic investor. The valuation equation exists from the existence of a state-price density. To compute the dynamics of the weighting process, apply Itô's lemma to $\lambda(t) = \frac{y^p \xi^o(t)}{y^p \xi^p(t)}$ using the dynamics of the state-price densities in the constrained and unconstrained regions.

To show the converse, assume that there exists ξ^o, ξ^p, λ, and S satisfying (26) and (27). Clearing in the consumption goods market (20) follows from the solutions to the investors' consumption-portfolio problems with (26) substituted. To show clearing in the bond market, note that $\xi^o(t)(X^o(t) + X^p(t))$ is a martingale since the pessimist's optimal wealth process is a tradeable strategy, and the optimist and the pessimist must agree upon the prices of all tradeable strategies.

$$X^o(t) + X^p(t) = \frac{1}{\xi^o(t)} E^o \left[\int_t^T \xi^o(s)(\hat{c}^o(s) + \hat{c}^p(s))ds \middle| \mathcal{F}_t^\delta \right].$$

By substituting clearing in the consumption goods market, clearing in the bond market results. Finally, to show clearing in the stock market, compare the diffusion dynamics of $\xi^o(t)S(t)$ with $\xi^o(t)(X^o(t) + X^p(t))$ using clearing in both the goods market and the bond market. Clearing in the stock market then results. ■

Proof of Proposition 2. Applying Itô's lemma to each agent's first order conditions,

$$e^{-\rho t} u_c^i(\hat{c}^i(t)) = y^i \xi^i(t),$$

and matching diffusion terms,

$$\sigma_{c^i}(t) = \frac{\kappa^i(t)}{A^i(t)}, \quad i \in \{o, p\},$$

where $\hat{c}^i(t)$ satisfies

$$d\hat{c}^i(t) = \mu_{c^i}(t)dt + \sigma_{c^i}(t)dw^i(t).$$

Market clearing in the consumption goods market implies

$$\sigma_{c^o}(t) + \sigma_{c^p}(t) = \delta(t)\sigma_\delta(t)$$

and

$$1/A(t) = 1/A^o(t) + 1/A^p(t).$$

Note that $\sigma_{c^p}(t) = 0$ when the short-sale constraint binds. This implies Equation (30).

Denoting the optimist's market price of risk in the unconstrained and constrained economies as $\kappa^{o,u}$ and $\kappa^{o,c}$ respectively, the difference between the two is given by

$$\kappa^{o,u}(t) - \kappa^{o,c}(t) = \left(A(t) - A^o(t)\right)\delta(t)\sigma_\delta + \frac{A(t)}{A^p(t)}\overline{\mu}(t).$$

Given the short-sale constraint binds when $\overline{\mu}(t) \geq \kappa^{o,c}(t)$, this implies

$$\kappa^{o,u}(t) - \kappa^{o,c}(t) \geq \left(A(t) - A^o(t)\right)\delta(t)\sigma_\delta + \frac{A(t)A^o(t)}{A^p(t)}\delta(t)\sigma_\delta$$
$$= \delta(t)\sigma_\delta\left(A(t) - A^o(t) + \frac{A(t)A^o(t)}{A^p(t)}\right).$$

By using the definition of $A(t)$ from (31), the term in parenthesis above equals zero, implying $\kappa^{o,u}(t) - \kappa^{o,c}(t) \geq 0$ when the short-sale constraint binds.

The interest rate follows from applying Itô's lemma to the state-price density for each agent, changing to the true probability measure, matching deterministic terms, and rearranging. Using (32), it is straightforward to show that for $\gamma < 1$, the constrained interest rate $r^c(t)$ is higher than unconstrained interest rate $r^u(t)$ for the same time t consumption allocation. When $\gamma = 1$,

$$r^c(t) - r^u(t) = A(t)\delta(t)\sigma_\delta\left(\frac{A(t)}{A^p(t)}\overline{\mu} + \frac{1}{2}\delta(t)\sigma_\delta\left(B(t) - B^o(t)\right)\right)$$
$$= A(t)\delta(t)\sigma_\delta\left(\sigma_\delta\left(1 - \frac{\delta(t)}{c^o(t)}\right) + \frac{\delta(t) - c^o(t)}{\delta(t)}\overline{\mu}(t)\right).$$

Given the short-sale constraint binds when $\overline{\mu}(t) \geq \kappa^{o,c}(t)$, we can substitute $\kappa^{o,c}(t)$ for $\overline{\mu}(t)$ in the above equality leading to the inequality $r^c(t) - r^u(t) \geq 0$. If $\overline{\mu}(t) > \kappa^{o,c}(t)$, then $r^c(t) - r^u(t) > 0$. By continuity of the interest rate expressions in γ, there exists a $\overline{\gamma} > 1$ where $r^c(t) - r^u(t) \geq 0$. ■

We use the following lemma to prove Proposition 3.

Lemma 2. *(Proposition 1.3.5 (Clark-Ocone) - Nualart (1995))*
Let $F \in D^{1,2}$, where the space $D^{1,2}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$||F||_{1,2} = \left[E(|F|^2) + E(||\mathcal{D}F||^2_{L^2(T)})\right]^{1/2}.$$

Suppose that w is a one-dimensional Brownian motion. Then

$$F = E(F) + \int_0^T E(\mathcal{D}_s F|\mathcal{F}_s)dw(s),$$

Taking conditional expectations,

$$E(F|\mathcal{F}_t) = E(F) + \int_0^t E(\mathcal{D}_s F|\mathcal{F}_s)dw(s).$$

Proof of Lemma 2. See page 42 of Nualart (1995). ■

Proof of Proposition 3. From Proposition 1, the stock is priced with respect to the optimist:

$$S(t) = \frac{1}{U_c(\delta(t), \lambda(t))} E^o \left[\int_t^T e^{-\rho s} U_c(\delta(s), \lambda(s)) \delta(s) ds \middle| \mathcal{F}_t^\delta \right].$$

Define the $\mathcal{L}^2(\mathcal{P}^o)$ martingale $M(t)$

$$M(t) = E^o \left[\int_0^T e^{-\rho s} U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right].$$

By the martingale representation theorem, there exists a process ϕ such that $E^o \left[\int_0^T |\phi(t)|^2 dt \right] < \infty$ where $M(t) = M(0) + \int_0^T \phi(t) dw^o(t)$.

The value of the stock can be rewritten as

$$S(t) = \frac{1}{U_c(\delta(t), \lambda(t))} \left[M(t) - \int_0^t e^{-\rho s} U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \quad \text{(A1)}$$

By applying Itô's lemma to (A1) and matching diffusion coefficients with the dynamics of S given by (8),

$$\sigma(t) = -\frac{U_{cc}(\delta(t), \lambda(t))}{U_c(\delta(t), \lambda(t))} \delta(t) \sigma_\delta(t) + \frac{U_{c\lambda}(\delta(t), \lambda(t)) \lambda(t)}{U_c(\delta(t), \lambda(t))} \sigma_\lambda(t) + \frac{\phi(t)}{E^o \left[\int_t^T e^{-\rho s} U_c(\delta(s), \lambda(s)) \delta(s) ds | \mathcal{F}_t^\delta \right]}. \quad \text{(A2)}$$

Equation (A2) gives a representation of the stock volatility in terms of the representative agent's utility function; the processes δ and λ; and the process ϕ from the martingale representation theorem.

To complete the characterization of the stock's volatility, we need to relate the dynamics of the martingale $M(t)$ to the dynamics of the processes δ and λ. From Lemma 2 above, the dynamics of $M(t)$ are

$$\begin{aligned} \phi(t) = {} & E^o \left[\int_t^T e^{-\rho s} U_{c\lambda}(\delta(s), \lambda(s)) \delta(s) \mathcal{D}_t \lambda(s) ds \middle| \mathcal{F}_t^\delta \right] \\ & + E^o \left[\int_t^T e^{-\rho s} \left(U_{cc}(\delta(s), \lambda(s)) \delta(s) + U_c(\delta(s), \lambda(s)) \right) \mathcal{D}_t \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \end{aligned}$$

The Malliavin derivatives $\mathcal{D}_t \lambda(s)$ and $\mathcal{D}_t \delta(s)$ are

$$\mathcal{D}_t\delta(s) = \delta(s)\left[\int_t^s \mathcal{D}_t\mu^o(z)dz + \sigma_\delta\right],$$

$$\mathcal{D}_t\lambda(s) = \lambda(s)\left[\int_t^s \sigma_\lambda(z)\mathcal{D}_t\sigma_\lambda(z)dz - \int_t^s \mathcal{D}_t\sigma_\lambda(z)dw^o(z) - \overline{\mu}(t)\right].$$

The results are computed by using the chain rule for Malliavin derivatives. See Nualart (1995) for a full exposition.

From the representative agent construction and implicit differentiation,

$$U_{c\lambda}(\delta(s), \lambda(s)) = \frac{U_c(\delta(s), \lambda(s))}{\lambda(s)}\frac{A(s)}{A^p(s)} = \frac{U_c(\delta(s), \lambda(s))}{\lambda(s)}\frac{A^o(s)}{A^o(s) + A^p(s)}.$$

Substituting gives the result. ■

Proof of Corollary 1. When the optimist has logarithmic preferences, the marginal utility of the representative agent is given by

$$U_c(\delta(t), \lambda(t)) = \frac{1 + \lambda(t)}{\delta(t)}, \tag{A3}$$

which can be computed by substituting into (22). Substituting (A3) into (27) solves for the stock price in closed-form.

Expressing the weighting process as $\lambda(t) = \frac{W^p(t)}{W^o(t)}$ can be found by using the stock price expression, market clearing $S(t) = W^o(t) + W^p(t)$, and $\xi^o(t)\, W^o(t) = E_t^o\left[\int_t^T \xi^o(s)c^o(s)ds\right]$.

Substituting the definitions of the absolute risk aversion and absolute prudence coefficients into (30) and (32) gives the market price of risk and interest rate expressions. The comparisons of the market prices of risk and the interest rates across the constrained and unconstrained economies follow immediately from Proposition 2.

We use the following lemma to prove Proposition 4. ■

Theorem 1. [*Mean Comparison Theorem Adapted from Hajek (1985)] Consider the dynamics of the stochastic weighting process* λ *in an equilibrium where the short-sale constraint can possibly bind,*

$$\lambda(t) = \lambda(0) - \int_0^t \lambda(s)\, \min(\overline{\mu}(s), \kappa^o(s))\, dw^o(s).$$

Under the assumption that $\overline{\mu}(s)$ *is bounded,* λ *is a martingale under the optimist's beliefs. Let* λ^u *be the unique solution to the stochastic differential equation*

$$\lambda^u(t) = \lambda(0) - \int_0^t \lambda^u(s)\,\overline{\mu}(s)\,dw^o(s),$$

where $\min(\overline{\mu}(s), \kappa^o(s)) \leq \overline{\mu}(s)$. *The process* λ^u *denotes the stochastic weighting process in an equilibrium with no short-sale constraint where the initial level of the process is the same as the constrained economy.*
Then for any convex function Φ *and any* $t \geq 0$,

$$E\,[\Phi(\lambda(t))] \leq E\,[\Phi(\lambda^u(t))]\,. \tag{A4}$$

Proof of Theorem 1. The proof is adapted from Hajek (1985). Define a nondecreasing process ϕ by

$$\phi(t) = \int_0^t \frac{(\lambda(s)\min(\overline{\mu}(s), \kappa^o(s)))^2}{(\lambda(s)\overline{\mu}(s))^2}ds$$

with the convention that the integrand is one when $\min(\overline{\mu}(s), \kappa^o(s)) = \overline{\mu}(s) = 0$. Note that $\phi(t) \leq t$. We can also assume without loss of generality that for some constants a and $b > 0$, $\phi(t) \geq a + bt, \quad \forall t$. If this is not true, simply modify the process λ by choosing its diffusion to equal the diffusion of λ^u for all u larger than the given t in (A4).

For each $s \geq 0$ define the stopping time $\tau(s) = \inf\{t : \phi(t) > s\}$. Then

$$s \leq \tau(s) \leq \frac{s-a}{b}. \tag{A5}$$

Define a random process z by

$$z(s) = \lambda(\tau(s)), \tag{A6}$$

then since λ and ϕ have the same intervals of constancy,

$$z(\phi(t)) = \lambda(t). \tag{A7}$$

Applying Lebesgue's formula for the transformation from Lebesgue to Stieltjes integrals,

$$z(s)^2 - \int_0^s (z(t)\overline{\mu}(t))^2 dt = z(s)^2 - \int_0^{\tau(s)} (z_{\phi(t)}\overline{\mu}(\phi(t)))^2 d\phi(t) \tag{A8}$$

$$= \lambda(\tau(s)) - \int_0^{\tau(s)} (\lambda(t)\min(\overline{\mu}(t), \kappa^o(t)))^2 dt. \tag{A9}$$

By the optional sampling theorem, $z(t)$ and $z(t)^2 - \int_0^t (z(s)\overline{\mu}(s))^2 dt$ are continuous martingales. Thus z is the unique weak solution to the equation defining λ^u. Thus $z \sim \lambda^u$. If Φ is a convex function, then $\Phi(z(s))$ is a submartingale. Therefore, since $\phi(t) \leq t$, the optional sampling theorem implies that $E\Phi(z_t) \geq E\Phi(z_{\phi(t)})$. We can then conclude that $E\Phi(\lambda^u(t)) \geq E\Phi(\lambda(t))$. ■

Proof of Proposition 4. The stock price under the optimist's beliefs is given by

$$S(t) = E^o\left[\int_t^T e^{-\rho s}\frac{U_c(\delta(s),\lambda(s))}{U_c(\delta(t),\lambda(t))}\delta(s)ds\,\middle|\,\mathcal{F}_t^\delta\right] \tag{A10}$$

and each investor's consumption allocation in terms of the representative agent's utility is given by

$$\hat{c}^o(\delta(t),\lambda(t)) = \left(\frac{1}{U_c(\delta(t),\lambda(t))}\right)^{\frac{1}{\gamma}}, \quad \hat{c}^p(\delta(t),\lambda(t)) = \frac{\lambda(t)}{U_c(\delta(t),\lambda(t))}.$$

Using market clearing in the consumption good, the marginal utility of the representative agent can be linearized around the optimist's relative risk aversion for $\gamma = 1$ leading to

$$U_c(\delta(t),\lambda(t)) \approx \frac{1+\lambda(t)}{\delta(t)} + \frac{\ln\left(\frac{1+\lambda(t)}{\delta(t)}\right)}{\delta(t)}(\gamma-1)\,. \tag{A11}$$

Substitute (A11) into (A10) to approximate the stock price around the optimist's relative risk aversion of $\gamma = 1$:

$$S(t) \approx \frac{\delta(t)\times E^o\left[\int_t^T e^{-\rho s}\left\{1+\lambda(s)+(\gamma-1)\ln\left(\frac{1+\lambda(s)}{\delta(s)}\right)\right\}ds\,\middle|\,\mathcal{F}_t^\delta\right]}{1+\lambda(t)+(\gamma-1)\ln\left(\frac{1+\lambda(t)}{\delta(t)}\right)}. \tag{A12}$$

Differentiating with respect to γ yields

$$\begin{aligned}\frac{\partial S(t)}{\partial\gamma} \approx\ & \frac{\delta(t)\times E^o\left[\int_t^T e^{-\rho s}\ln\left(\frac{1+\lambda(s)}{\delta(s)}\right)ds\,\middle|\,\mathcal{F}_t^\delta\right]}{1+\lambda(t)+(\gamma-1)\ln\left(\frac{1+\lambda(t)}{\delta(t)}\right)} \\ & -\frac{\delta(t)\times\ln\left(\frac{1+\lambda(t)}{\delta(t)}\right)\times E^o\left[\int_t^T e^{-\rho s}\left\{\begin{matrix}1+\lambda(s)+(\gamma-1)\\ \ln\left(\frac{1+\lambda(s)}{\delta(s)}\right)\end{matrix}\right\}ds\,\middle|\,\mathcal{F}_t^\delta\right]}{\left(1+\lambda(t)+(\gamma-1)\ln\left(\frac{1+\lambda(t)}{\delta(t)}\right)\right)^2}.\end{aligned} \tag{A13}$$

At $\gamma = 1$, the expression for the partial derivative is exact. Evaluating at $\gamma = 1$ and using that $\lambda(t)$ is a martingale, we have

$$\left.\frac{\partial S(t)}{\partial \gamma}\right|_{\gamma=1} = \frac{\delta(t)}{1+\lambda(t)}\left(E^o\left[\int_t^T e^{-\rho s}\ln\left(\frac{1+\lambda(s)}{\delta(s)}\right)ds\,\middle|\,\mathcal{F}_t^\delta\right]\right.$$
$$\left. - \ln\left(\frac{1+\lambda(t)}{\delta(t)}\right)\times\int_t^T e^{-\rho s}ds\right).$$

Consider the case when $\gamma = 1$. When both investors have logarithmic preferences, the initial equilibrium stock price is $S(0) = \frac{1-e^{-\rho T}}{\rho}\delta(0)$ in both the unconstrained and constrained equilibria from Corollary 1. From (29), this implies that the initial level of the weighting process $\lambda(0)$ is the same across the two economies for the same initial endowments.

Denoting the weighting process across the unconstrained and constrained economies as $\lambda^u(t)$ and $\lambda^c(t)$ respectively, we can compare $\frac{\partial S(0)}{\partial \gamma}$ across the two economies:

$$\left.\frac{\partial S^c(0)}{\partial \gamma}\right|_{\gamma=1} - \left.\frac{\partial S^u(0)}{\partial \gamma}\right|_{\gamma=1}$$
$$= \frac{\delta(0)}{1+\lambda(0)}E^o\left[\int_0^T e^{-\rho s}\left[\ln\left(1+\lambda^c(s)\right) - \ln\left(1+\lambda^u(s)\right)\right]ds\,\middle|\,\mathcal{F}_0^\delta\right]$$
$$= \frac{\delta(0)}{1+\lambda(0)}\int_0^T e^{-\rho s}E^o\left[\ln\left(1+\lambda^c(s)\right) - \ln\left(1+\lambda^u(s)\right)\middle|\,\mathcal{F}_0^\delta\right]ds.$$

Since $\ln(1+\lambda)$ is a concave function in λ, we know from Theorem 1 that

$$E^o\left[\ln\left(1+\lambda^c(s)\right) \mid \mathcal{F}_0^\delta\right] \geq E^o\left[\ln\left(1+\lambda^u(s)\right) \mid \mathcal{F}_0^\delta\right]$$

which implies

$$\left.\frac{\partial S^c(0)}{\partial \gamma}\right|_{\gamma=1} - \left.\frac{\partial S^u(0)}{\partial \gamma}\right|_{\gamma=1} \geq 0.$$

Given $S^c(0) = S^u(0)$ when $\gamma = 1$, we know locally around $\gamma = 1$ that

$$S^c(0) - S^u(0) \text{ is } \begin{cases} > 0 & \text{if } \gamma > 1, \\ < 0 & \text{if } \gamma < 1. \end{cases}$$

■

Proof of Proposition 5. under the pessimist's beliefs given by (9) and the pessimist's marginal valuation for the stock given by (38) follow backward stochastic differential equations with $S(T) = 0$. The comparison theorem presented in El Karoui et al. (1997) is directly applicable leading to $S(t) \geq S^p(t)$ almost surely for any time t. ■

Appendix B: The Malliavin Derivatives

When the exogenous output process follows a geometric Brownian motion, the Malliavin derivatives $\mathcal{D}_t\delta(s)$ and $\mathcal{D}_t\lambda(s)$ are

$$\mathcal{D}_t\delta(s) = \delta(s)\left[\sigma_\delta + \frac{\nu^o(t)}{\sigma_\delta}(s-t)\right] > 0,$$

$$\mathcal{D}_t\lambda(s) = -\lambda(s)\left[\sigma_\lambda(t) + \int_t^s \sigma_\lambda(z)\mathcal{D}_t\sigma_\lambda(z)dz + \int_t^s \mathcal{D}_t\sigma_\lambda(z)dw^o(z)\right],$$

where $\mathcal{D}_t\sigma_\lambda(z)$ is

$$\mathcal{D}_t\sigma_\lambda(z) = \begin{cases} \mathcal{D}_t\overline{\mu}(z) & \text{if } \sigma_\lambda(z) = \overline{\mu}(z), \\ \frac{\partial\kappa_c^0(z)}{\partial\delta(z)}\mathcal{D}_t\delta(z) + \frac{\partial\kappa_c^0(z)}{\partial\lambda(z)}\mathcal{D}_t\lambda(z) & \text{if } \sigma_\lambda(z) = \kappa_c^o(z). \end{cases}$$

Appendix C: Construction of Numerical Examples

To construct the numerical examples presented in Figures 1 through 4 requires numerically evaluating the stock price (27) and the stock volatility (34) in both the unconstrained and the constrained economies as well as the pessimist's stock valuation (39) in the constrained economy. We compute these quantities by Monte Carlo simulation. Specifically, the equilibrium stock price (27) and stock volatility (34) can be written as conditional expectations under the optimist's beliefs that are driven by the stochastic processes $\delta(s)$, $\lambda(s)$, $\mu_\delta^o(s)$, $\mu_\delta^p(s)$, $\mathcal{D}_t\lambda(s)$, and $\mathcal{D}_t\delta(s)$. The pessimist's valuation (39) can also be computed under the optimist's beliefs after a change of the probability measure.

Since the output and weighting processes are strictly positive processes, we express them in exponential form:

$$\delta(s) = \delta(0)\exp\left(\int_0^s (\mu_\delta^o(t) - \frac{1}{2}\sigma_\delta^2)dt + \int_0^s \sigma_\delta dw^o(t)\right),$$

$$\lambda(s) = \lambda(0)\exp\left(-\int_0^s \frac{1}{2}\sigma_\lambda^2(t)dt - \int_0^s \sigma_\lambda(t)dw^o(t)\right).$$

The stochastic processes are then simulated forward along with the processes $\mu_\delta^o(s)$, $\mu_\delta^p(s)$, $\mathcal{D}_t\lambda(s)$, and $\mathcal{D}_t\delta(s)$ using a Milstein scheme[11] with a time-step of 0.005 years.

The pathwise integrals in the stock price (27), the stock volatility (34), and the pessimist's valuation (39) are calculated using the trapezoid rule with

[11] For the processes $\delta(s)$ and $\mu_\delta^o(s)$ the Milstein scheme collapses to an Euler scheme.

30,000 paths used to calculate each expectation. To minimize the Monte Carlo standard errors of the expectations in (27), (34), and (39), we use control variate techniques given the expectations in the stock price and stock volatility are known in closed-form when the optimist has logarithmic preferences in both the unconstrained and the constrained economies. For the pessimist's valuation, we use the constrained economy logarithmic optimist's stock valuation as a control variate. The optimal weights for the control variates to minimize the Monte Carlo standard errors are determined by the regression approach outlined for example in Boyle et al. (1998).

Figures 1 through 4 are constructed by computing unconstrained and constrained economies with a different IES for the optimist $\frac{1}{\gamma}$ where $\gamma =$ 0.5, 0.625, 0.75, 0.875, 1.0, 1.25, 1.5, 1.75, and 2.0. When $\gamma = 1.0$, the stock price and the stock volatility are known analytically and are free of simulation error. For the other cases when $\gamma \neq 1.0$, 95% confidence intervals are computed for all quantities subject to Monte Carlo simulation error—the stock price, the stock volatility, the pessimist's valuation, the optimist's risk premium through the stock volatility, the optimist's stock investment through the stock volatility, and the expected time until the short-sale constraint are not binding. Given the stock volatility is a nonlinear function of three different simulated expectations, its 95% confidence interval is computed using the delta method. All other quantities are known analytically.

When comparing simulated unconstrained quantities to constrained quantities when $\gamma \neq 1.0$, no 95% confidence interval for an unconstrained quantity overlaps with a constrained quantity implying that the figures presented accurately depict the relationship between the unconstrained and constrained economies. The largest 95% confidence interval for each simulated quantity is recorded in each figure's caption.

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