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A Dynamic Model of Optimal Capital Structure*

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Abstract. This paper presents a continuous time model of a firm that can dynamically adjust both its capital structure and its investment choices. In the model we endogenize the investment choice as well as firm value, which are both determined by an exogenous price process that describes the firm's product market. Within the context of this model we explore cross-sectional as well as time-series variation in debt ratios. We pay particular attention to interactions between financial distress costs and debtholder/equityholder agency problems and examine how the ability to dynamically adjust the debt ratio affects the deviation of actual debt ratios from their targets. Regressions estimated on simulated data generated by our model are roughly consistent with actual regressions estimated in the empirical literature.

JEL Classification: G32, G33, G35

1. Introduction

The concept of a target debt ratio, which reflects the trade offs between the benefits and costs of debt financing, is quite familiar to most finance managers. For example, in a survey of CFOs, Graham and Harvey (2001) report that 37% of their respondents have a flexible target debt ratio, 34% have a somewhat tight target or range, and 10% have a strict target. This concept also plays a central role in many theories of optimal capital structure; however, there is substantial debate about the extent to which the idea of a target debt ratio is useful. For example, a recent paper by Fama and French (2002) suggests that firms move quite slowly towards their targets, and a number of papers suggest

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that earnings and stock price changes lead to capital structure changes that are only slowly reversed.[1]

There is a growing theoretical literature that considers transaction costs to explain the relatively slow movement towards target debt ratios.[2] These studies ignore debtholder/equityholder agency problems that reduce incentives to move towards the target as well as financial distress costs that can increase incentives to move towards the target.[3] These studies also ignore the possibility that economic shocks that move firms away from their target debt ratios may also cause their target debt ratios to change over time. If target debt ratios change over time, the estimates of the adjustment speed could be biased, which may further complicate the analysis of how quickly the observed capital structures move towards their targets.

To address these issues we develop and calibrate a dynamic capital structure model that allows us to quantify the benefits and costs associated with both movements towards and away from firms' target debt ratios. As we summarize in Table I, our model extends the existing dynamic capital structure literature along a number of dimensions. In particular, this is a fully dynamic, infinite horizon model where firm values are endogenously determined by continuous capital investments and financing choices. The market value of the firm in this model is determined by its earnings, which are themselves endogenously determined by investment choices as well as by exogenous price changes in the firm's product market.[4] The investment choice is also endogenously determined by the product price, the firm's existing capital, and the firm's leverage choice, which, of course, is also determined endogenously in the model. The capital structure choice is in turn determined by taxes and bankruptcy costs, as in the previous literature, and in addition, conflicts of interests between debtholders and equityholders and financial distress costs, which are modeled as cash flow losses incurred by the equityholders when the cash flows are small relative to interest payments.

Within the context of this model we examine the time-series as well as the cross-sectional variation in the debt ratio. In contrast to prior dynamic capital structure models, for example, Fischer et al. (1989a), and Leland (1998),

[1] See also Leary and Roberts (2004), Flannery and Rangan (2006), and Kayhan and Titman (2007).

[2] See Fischer et al. (1989a) and Strebulaev (2006).

[3] Leland (1998) and Moyen (2007) consider agency costs, but do not examine their effect on leverage adjustment towards target.

[4] Traditionally, a firm's earnings are modeled as an exogenously given process, irrespective of the characteristics of the firm. For example Kane et al. (1985), Fischer et al. (1989a,b) and Leland (1998) describe earnings as a constant fraction of the value of the firm's assets.

Table I. This table describes a set of capital structure models that use a contingent claims framework. The table summarizes the assumptions and/or features that have been employed in each of the models

Papers / Features of the models	Brennan and Schwartz (1984)	Fischer et al. (1989a)	Mello and Parsons (1992)	Mauer and Triantis (1994)	Leland (1998)	Mauer and Ott (2000)	Hennessy and Whited (2005)	Strebulaev (2006)	Model in this paper
Investment decisions are not fixed (dependent on financial decisions)	Y	N	Y	Y	N	Y	Y	N	Y
Financial decisions are not fixed (restructuring allowed)	Y	Y	N	Y	Y	N	Y	Y	Y
Value of assets is endogenous	Y	N	Y	Y	Y	Y	Y	N	Y
Depreciation of assets is modeled	Y	N	N	N	N	N	N	N	Y
The model is time invariant	N	Y	N	N	Y	N	Y	Y	Y
Firm is allowed to issue equity and reduce its debt	Y	Y*	N	Y	N	N	N	N	Y
Financial restructuring is costly	N	Y	N	Y	Y	N	Y	Y	Y
The model distinguishes between internal and external financing	N	N	N	N	N	N	Y	Y	Y
Financial distress costs are included	N	N	N	N	N	N	Y	Y	Y
Debt maturity is considered	N	N	N	N	Y	N	N	N	Y
The underinvestment problem is addressed	Y	N	Y	Y	N	Y	N	N	Y
The asset substitution problem is addressed	N	N	Y	N	Y	N	N	N	N
The firm maximizes its equity value (not firm's total value)	Y	N	Y	N	Y	Y	Y	Y	Y
Endogenous default choice	Y	N	Y	N	Y	Y	N	Y	Y
Bond covenants are considered	Y	N	N	N	N	N	N	N	Y
Closed form solution	N	Y	N	N	Y	N	N	N	N
The firm's investment flexibility is discussed	N	N	N	N	N	N	N	N	Y
Depreciation tax shield is incorporated	N	N	N	N	N	N	N	N	Y
Investment choice is a continuous decision (not a one-shot decision)	Y	N	N	N	N	N	Y	N	Y

* Within the context of Fischer et al. (1989) model, it is not optimal for the equityholders to raise equity to reduce debt.

the target debt ratio, determined by the various trade offs between the costs and benefits of debt financing, changes over time, depending on the firm's investment history and its product price. Hence, time-series changes in observed debt ratios are influenced by changes in the target debt ratio as well as by economic changes and corporate actions that move firms either away from or towards their targets.

Our model implies that conflicts of interest between debtholders and equityholders and financial distress costs have a first-order effect not only on the level of the target debt ratio but also on how debt ratios evolve over time. Consistent with the existing literature, we find that debtholder/equityholder conflicts and financial distress costs lead firms to initially choose more conservative capital structures. We also find that debtholder/equityholder conflicts reduce the tendency of firms to move towards their target debt ratios, while financial distress costs increase the tendency of firms to move towards their targets. In addition, there exists an interaction effect between financial distress costs and these conflicts of interest, which that has not been previously discussed. Specifically, the interests of debtholders and equityholders tend to be more in line for firms with substantial financial distress costs. Intuitively, shareholders that are exposed to greater cash flow losses due to financial distress are more willing to reduce their debt ratios when they are over-levered even though a debt reduction benefits debtholders at the expense of the shareholders.

To quantify the dynamic implications of our model, we use parameters that are chosen to roughly match empirical observations for firms in the gold mining industry. Using these initial parameters, we first calculate comparative statics that allow us to examine how changes in the parameters influence the initial target capital structure choice. In addition to considering financial distress costs and the debtholder/equityholder conflict, we consider comparative statics with respect to firm-specific characteristics, such as profitability, the expected growth rate of the product price, the rate at which a firm's capital depreciates, and market characteristics such as the transaction costs associated with issuing and repurchasing debt and equity.

In order to explore the forces that cause firms to move towards and away from their target ratios, we use our model to create a panel of simulated data, which includes model-generated debt ratios, cash flows, and investment choices.[5] Using this model-generated data, we regress changes in the debt ratio on variables that have been examined in the past empirical literature. In particular, we examine the effect of past stock returns, past profits, investment

[5] There are several corporate finance papers that use regressions on simulated data. See for example Alti (2003), Hennessy and Whited (2005), and Strebulaev (2006).

expenditures, and deviations from the target debt ratio. We find that the estimated speed at which firms move towards their target debt ratios tends to be relatively slow. However, the adjustment speed depends on firm characteristics. In particular, the adjustment speeds are somewhat faster for firms with higher financial distress costs and for firms that are managed to maximize firm values rather than equity values. If one assumes that smaller firms are more subject to financial distress costs and are more subject to bank monitoring, then these results are consistent with the Flannery and Rangan (2006) estimates that suggest that adjustment speeds are faster for smaller firms. Our model also implies that the capital structure changes of firms with higher financial distress costs and those managed to maximize total firm value are less sensitive to earnings changes, which is consistent with the Fama and French (2002) findings that changes in earnings have greater influence on changes in the capital structures of large firms.

The remainder of the paper is organized as follows. The next section develops the theoretical model of the firm. Section 3 reports model calibration and extensive numerical results for different types of firms, and the last section concludes the paper. Stochastic control problems of the firm valuation are formulated in Appendix A, and the technical details of the numerical algorithm are presented in Appendix B.

2. Description of the Model

2.1 TIME LINE AND SUMMARY OF THE MODEL

The model developed in this paper endogenously determines the firm's optimal investment and financing strategies as functions of an exogenous state variable that determines the price of the firm's product. To briefly summarize the model we present a timeline that specifies the firm's investment and financing decisions:

<u>At time 0</u>

- The entrepreneur raises capital to finance the firm by issuing equity and debt. The optimal initial mix of debt and equity is the ratio that maximizes the total value of debt plus equity.

<u>Each subsequent time period</u>

- The firm realizes cash flows that are determined by (i) the current price of the product it sells, (ii) its current capacity that is determined by its past investment choices and the rate at which its fixed assets depreciate, and (iii) whether or not the firm is in financial distress.

◦ The firm can

- default as soon as its equity value equals zero, in which case the equityholders get nothing and debtholders recover the value of the unlevered firm less bankruptcy costs or,
- increase, decrease, or keep its current debt level constant
- meet debt payments (which are tax deductible) and choose the amount to invest where funds for investment can be taken from:
  - internally generated cash flows
  - proceeds from newly issued debt or
  - raising additional equity
- pay out any residual cash flow (after taxes) as dividends to the equityholders.

After the initial time period, the firm's decisions with respect to investment and capital structure choices reflect either

1. the objectives of the shareholders, and thus maximize the value of their equity stake or
2. the objectives of all claimholders, and thus maximize the total value of the firm's debt and equity.[6]

In the case where the firm acts in the equityholders' interests, it has incentives to underinvest and to deviate from the optimal financing strategy. Since debt is priced to account for the effect of these incentives, the costs of debt financing can be reduced if equityholders can write contracts that commit them to follow value-maximizing investment and financing strategies. The first objective can thus be viewed as the case where such contracts are precluded, and the second represents the case where the contracts can be costlessly written. The difference between the values in these two cases is defined as the agency costs. In reality, given observed debt covenants, reputational concerns, and the ways in which managers are compensated, we expect that actual managerial strategies lie somewhere between these two extreme cases.[7]

[6] To keep our discussion reasonably focused, we ignore a number of incentive issues. For example, we ignore the asset substitution problem identified by Jensen and Meckling (1976) and managerial agency problem also discussed by Jensen and Meckling (1976) along with Jensen (1986), Hart (1993), and Hart and Moore (1995). However, since our model can be solved for any reasonable objective function, it can be extended in ways that allow us to consider these possibilities.

[7] In addition to a dynamic setting where the debt level can be adjusted over time, we have also considered a static setting where the firm initially sets its capital structure and cannot change

2.2 DIMENSIONALITY CONSIDERATIONS

The model presented in this study requires the solution of a three-dimensional stochastic optimal control problem. The model assumes that there is one exogenous state variable, the price. However, at each point in time, the investment and financing decisions are made as a function of the exogenous price as well as the firm's current capital structure and capacity, which are endogenous variables. Therefore, to solve the model numerically, the algorithm must account for three dimensions, the exogenous price and the two endogenous variables that are determined from past decisions. Because of a need to limit the dimensionality of the model we are forced to make various modeling compromises. First, the debt must all have the same priority in the event of default. Therefore, to capture the idea that existing debt cannot be expropriated by the issuance of new debt, we assume that the firm repurchases all existing debt at its face value before issuing additional debt. Second, we cannot allow the firm to change the maturity structure of its debt over time. Third, since tracking the firm's cash holdings would require an additional dimension, we assume that the firm holds no cash, which implies that it pays out all its residual cash flows as dividends.[8] While these assumptions place some unrealistic limitations on the firm, the restrictions placed on the firm in this model are weaker than the restrictions imposed in existing dynamic models.

2.3 THE FIRM'S INCOME AND INVESTMENT

The firm we examine produces and sells a product (commodity) whose unit market price, p, continuously evolves through time in the manner described by the following stochastic process:

$$\frac{\mathrm{d}p}{p} = (r - \alpha)\mathrm{d}t + \sigma_p \mathrm{d}W_p, \tag{1}$$

where W_p is a Wiener process under the risk-neutral measure Q, σ_p is the instantaneous volatility coefficient, r is the risk free rate, which is assumed to

the amount of its debt (coupon size and face value) as time progresses. Since our focus is on the more realistic dynamic setting, we have excluded the static case which is available upon request.

[8] Since the firm can hold no cash, the firm may invest more and take on less debt to avoid the transaction costs associated with issuing equity to meet temporary financial short falls. Evaluating how the ability to hold cash influences capital structure and investment choices is clearly of interest, but is beyond the scope of this model. Allowing cash holdings to vary over time complicates the model not only because it increases the dimensionality of the problem but also because it adds an additional decision problem to the firm.

be constant, and α ($\alpha \geq 0$) is the convenience yield. There are fixed production costs b ($b \geq 0$) which are assumed to be constant.

The firm's instantaneous earnings before interest, production costs, taxes, and depreciation are assumed to equal the product $p \times c$, where c is the firm's output level, which, to save on notation, equals its capacity level. The capacity of the firm is described by a strictly concave and increasing function $c(\cdot)$ of the value of the firm's fixed (tangible) assets A. A can be viewed as the *book value* of the firm. The capacity function corresponds to a typical production function with diminishing marginal returns. We assume that $c(0) = 0$ and $c(A) \to 1$ as $A \to \infty$ and normalize the capacity to be between 0 and 1 (maximum capacity). The change in the value of the firm's fixed (tangible) assets A is given by:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\gamma A + i, \ i \geq 0, \tag{2}$$

where γ, the depreciation rate, is assumed to be constant, and i, the instantaneous investment rate (or maintenance rate), is a continuous choice variable of the firm. Thus, the fixed (tangible) assets of the firm depreciate with time and can be increased with investment. The firm's capacity also decreases with the depreciation of its fixed assets and increases with investments.[9] If the investment rate i is higher than the depreciation rate γA, the firm expands its capacity.[10] We assume that the firm cannot sell its assets, that is, $i \geq 0$.

The firm's instantaneous net cash flow before taxes is the difference between its earnings after costs and its investment rate $p \cdot c(A) - b - i$, where b is the firm's production costs. We assume that the firm operates at its full capacity $c(A)$ and that it cannot change the product or its production technology, that is, the depreciation rate of its assets, its production costs, or the capacity function. We assume that the firm has the option to permanently shut down its operations if the spot price drops sufficiently below its production costs b, which implies that the value of the firm is always positive.

Without market imperfections such as costs of raising equity, distress costs, and taxes, a "bang-bang" investment strategy is optimal. In other words, if the firm is currently below its optimal capacity level or if the product price increases, it will invest at an infinite rate, allowing it to instantly move to its optimal capacity, which is a concave function of product prices. If the firm has excess capacity, it will not invest until its capacity declines to the optimal level due to depreciation of its assets. Thus, without imperfections the investment

9 We assume that the tax depreciation rate equals the rate of physical depreciation.

10 This setup implies that generally when the price increases, the firm will expand its capacity, that is, $i > \gamma A$.

rates are either infinite or zero. As we show later, with nontrivial costs of raising capital, distress costs, and taxes, the optimal investment strategy is not generally "bang-bang". In most cases, a firm that is below its optimal capacity will invest all of its free cash, but will not raise sufficient external capital to move to its optimal capacity. Moreover, as we show later, the firm's investment strategy will depend on its objectives as well as the level of its outstanding debt.

2.4 TAXES

Although we ignore personal taxes, we assume that the firm's cash flow after debt payments is taxed continuously at a constant corporate rate τ. We also assume that there are no loss offset or carryforward provisions. The firm's instantaneous tax obligation equals

$$TAX = \tau \times \max[0,\ p \cdot c(A) - b - \gamma A - d] \geq 0, \tag{3}$$

where d is the periodic debt payments. Notice that we assume that the periodic debt payments (coupon) d and depreciation γA are fully tax deductible and that the principal payments do not serve as a tax shield. The firm uses its income, $p \cdot c(A)$, to pay costs, meet its debt obligations, pay taxes, and to invest, with any residual being paid out as a dividend. If there is insufficient cash flow to meet debt and tax obligations and to fund investment needs, the firm can raise capital by issuing additional debt or equity. The conditions under which outside capital can be raised, and the costs associated with raising it, will be described later.

2.5 THE DEBT STRUCTURE

As we mentioned previously, the firm chooses an initial capital structure that maximizes firm value, but can subsequently recapitalize at any time, which incurs transaction costs. As we will describe in more detail below, the firm has the option to maintain its current debt level, or alternatively, can instantaneously increase or decrease its debt.

Debt Maturity Structure

In order to incorporate the maturity structure of the debt, we follow Leland (1998) and assume that the firm issues perpetual callable coupon debt. The debt structure obligates the firm to continuously retire its current debt, by repurchasing the debt at its face value, at a constant predetermined rate w. The parameter w, which we will call the debt retirement rate w, indirectly introduces the maturity structure of the debt: when w is higher the debt matures faster. Ignoring default or restructuring, the average duration (maturity) of the debt is equal to $\frac{1}{w}$. Since the parameter w is exogenous, the average duration of the debt is exogenous and does not change over time.[11]

For simplicity, we assume the face value of the debt is equal to the value of perpetual debt with periodic payment d discounted at the risk-free rate, that is, $F = \frac{d}{r}$, where r is the risk-free rate, which is constant over time. This implies that, since the debt is generally risky, in general it is sold at a discount relative to its face value.[12] If the firm is not restructuring its debt level, it has to maintain a fixed level of debt by continuously reissuing debt at the same rate w as it matures. For example, if $w = 0.05$, the firm has to annually retire about 5% of its current debt at its face value, and reissue the retired 5% portion of its debt where the total coupon and the face value remains the same at the level of d and F, respectively.[13]

The firm has to reissue its maturing debt at the current market price $w \cdot D(p, A, d)$, where $D = D(p, A, d)$ is the market value of the total debt outstanding. Although the face value of the outstanding debt remains constant in this case, the transaction can yield a net cash flow that arises because the face value of the maturing debt does not necessarily equal the market value of the new debt that replaces it. The above discussion implies that the net instantaneous cash flow from the firm to the debtholders can be described by the following expression:

$$d + wF - w \cdot D(p, A, d), \tag{4}$$

where the first term is the coupon, and the last two terms determine refunding expenses; wF is the cash flow that must be paid for the fraction of the debt retired and $w \cdot D$ is the proceeds of the newly issued debt. If the new debt is risky, net refunding expenses $wF - w \cdot D(p, A, d)$ are positive because $F - D(p, A, d) > 0$. Thus, the total dividend flow to the shareholders equals

$$\begin{aligned} &p \cdot c(A) - b - i - d - wF + w \cdot D(p, A, d) - \tau \\ &\quad \times \max[0, p \cdot c(A) - b - \gamma A - d]. \end{aligned} \tag{5}$$

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[11] An interesting extension of the model would be to allow the firm to change the duration of its debt overtime. This is a nontrivial extension that would increase the dimensionality of the problem.

[12] The reason that we do not assume that the debt is sold at par is that it would require an extra numerical iteration to find the par coupon rate every time the firm reissues its debt. Since the par rate is different at each state, this will increase the dimensionality of the problem.

[13] In Dangl and Zechner (2006), who extend the maturity structure introduced in Leland (1998), the firm can endogenously choose the rate at which the firm gradually retires its debt. They show that the optimal retirement rate depends on debt maturity structure and the firm' leverage.

It should be noted that we are assuming that when the firm keeps the face value of its debt constant it is not subject to transaction costs. In contrast, as we discuss in the next subsection, changes in the debt ratio generate transaction costs, which keep firms from constantly changing their debt ratio as economic conditions change.

Restructuring Debt

If the firm wishes to either increase or decrease its debt level, the transaction generates transaction costs. The model requires the firm to simultaneously repurchase all of its outstanding debt at its face value and issue the desired amount of new debt at market value. This assumption, which preserves the rights of the current debtholders if the firm increases its debt level,[14] makes the transaction especially costly when the debt is risky and its face value exceeds its market value. We follow Fischer et al. (1989a) and assume that when the firm adjusts its debt level, it has to pay transaction costs that are proportional to the face value of the new debt. Specifically, when the firm changes its debt level by replacing old debt that has a face value of $F = \frac{d}{r}$, with new debt that has a coupon $\widehat{d}$ and face value $\widehat{F} = \frac{\widehat{d}}{r}$, the firm has to pay transaction costs of

$$TC_{debt} = C_{Debt} \times \widehat{F}, \tag{6}$$

where C_{Debt} is a constant parameter. When the firm increases its debt, (i.e., $\widehat{F} > F$), it uses the net proceeds of the debt issue $D(p, A, \widehat{d}) - F$ to either repurchase equity or to invest, where $D(p, A, \widehat{d})$ is market value of new debt. When the firm reduces its debt (i.e., $\widehat{F} < F$), the firm has to raise equity to cover the difference between the face value of the debt being called and the proceeds of the newly issued debt $F - D(p, A, \widehat{d}) > 0$. As we discuss in the next section, raising equity generates additional transaction costs. Depending on the firm's objectives, the firm chooses to recapitalize only if the net benefit of recapitalization (net increase in equity value or in total firm value) exceeds transaction costs.[15]

[14] If a firm that acts in the interest of its equityholders is allowed to issue new debt, without repurchasing the existing debt, it will continuously increase its debt ratio over time thereby expropriating current debtholders. Our assumptions with respect to debt restructuring are similar to those in Fischer et al. (1989a).

[15] When the firm does recapitalize, it moves to a debt ratio that is close but not exactly the same as what we will refer to as the target debt ratio, which is the debt ratio that maximizes the total market value of debt and equity. The difference between the post-restructuring debt ratio and the target debt ratio reflects the transaction costs associated with restructuring. To understand this, consider the case of a firm that is underlevered and restructures by increasing its outstanding debt and repurchasing shares. With transaction costs, such a firm will issue

Our model of the firm's ability to change its capital structure extends the existing literature in a couple of ways. First, we account for the fact that the target debt ratio changes over time with changes in p and A. In addition, we account for differences in the recapitalization strategies of the value- and the equity-maximizing firms. The value-maximizing firm will always move towards its target if the net increase in the firm's total value after the change in capital structure exceeds the transaction costs. However, because a decrease in leverage transfers wealth to the firm's existing debtholders, an equity-maximizing firm has less incentive to reduce its debt when it is doing poorly.

2.6 FINANCIAL SHORTFALL AND THE COSTS OF EQUITY ISSUES

When the firm's internally generated cash flow cannot cover its periodic coupon payments, refunding expenses, and its investment expenditures, it experiences what we call a financial shortfall, which requires the firm to issue equity. Our model assumes that there are no fixed costs that arise from an equity issue, so that small shortfalls are funded by an equity issue at a cost proportional to the value of the equity being issued.[16] The firm will not fund a small shortfall with a new debt issue because the costs associated with raising debt are proportional to the face value of the new debt. Specifically, if the firm's cash flow is negative, that is,

$$p \cdot c(A) - b - i - d - wF + w \cdot D(p, A, d) < 0 \tag{7}$$

the firm has to issue equity, which is subject to transaction costs, that is,

$$TC_{Equity} = C_{Equity} \times \max[0, \ - p \cdot c(A) + b + i + d + wF - w \cdot D], \tag{8}$$

where C_{Equity} is the constant parameter of the proportional cost of an equity issue.

The cost of issuing equity is also incurred when the firm decreases its debt. Specifically, when the firm decreases its debt from d to some $\widehat{d}$, $\widehat{d} < d$, the firm incurs a transaction cost of issuing equity that is proportional to the amount of equity needed, which is the difference between the face value of the

slightly less debt and repurchase slightly fewer shares than it would without transaction costs. In other words, it moves towards its target debt ratio, but not completely to its target debt ratio.

[16] It would be more realistic to assume that the firm issues equity in lumpsum amounts and uses proceeds to build up cash in order to use it whenever the firm is in financial shortfall. This would allow us to incorporate a fixed cost of issuing equity. However, such an assumption would increase the dimensionality of the model.

old debt, F, that the firm repurchases and the proceeds of newly issued debt $C_{Equity} \times (F - D(p, A, \widehat{d}))$.

2.7 FINANCIAL DISTRESS COSTS

Firms are financially distressed when their cash flows are low relative to their debt obligations. In the event of financial distress, firms suffer a reduction in their cash flows, owing, for example, difficulties that they face in dealing with customers, employees and strategic partners. We account for the effect of financial distress with two parameters: (i) the trigger point, that is, the parameter that determines when financial distress arises, (ii) the percentage decline in cash flows in the event of distress. Specifically, our numerical calculations assume that distress is triggered when the interest coverage ratio, $\frac{p \cdot c(A) - b}{d}$, falls below a certain threshold s, which is a parameter that we vary in our comparative statics. The continuous reduction in cash flows due to distress is proportional to the amount by which the firm's income falls below this threshold, $(s \cdot d - p \cdot c(A) + b)$. Specifically, the distress costs equal

$$DC = C_{Distress} \times max[0,\ s \cdot d - p \cdot c(A) + b], \tag{9}$$

where $C_{Distress}$ is the constant parameter for proportional distress costs.[17]

2.8 DEFAULT

The firm will endogenously default when the shortfall exceeds the present value of the firm's future dividends, after issuing shares to cover the shortfall. When this is the case, the firm cannot raise equity to cover the shortfall and the value of its equity is zero. We assume that in the event of default, the equityholders get nothing and the debtholders recover the liquidation value of the firm $E^U(p, A)$ minus default costs, $C_{default}$, which are proportional to $E^U(p, A)$, that is, at default the debt value satisfies $D(p, A) = (1 - C_{default}) \cdot E^U(p, A)$.[18] For simplicity we assume that the liquidation value of the firm equals the present

[17] Financial distress, described here, does not create any permanent damage to the firm's assets and thus has only a temporary affect on the future productivity of the firm. Alternatively, we can assume that financial distress causes the assets of the firm (e.g., the firm's reputation or its organizational capital) to depreciate at a faster rate, which would have a more permanent effect of the firm's productivity.

[18] In reality, if default and agency costs are sufficiently large, the debtholders and equityholders may have incentives to renegotiate the debt prior to the default. This is an interesting issue but it is beyond the focus of our analysis. See Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), and Christensen et al. (2002), for models that consider strategic debt service and renegotiations.

value of the firm's cash flows, assuming that the firm will always be all-equity financed.[19]

3. Valuation of Equity and Debt

The market value of the equity $E = E(p, A, d)$, and debt $D = D(p, A, d)$, are determined by the product price p, the value of the firm's fixed (tangible) assets A, and the level of the periodic coupon payment d. These values can be determined by solving stochastic control problems with free boundary conditions, where the control variables are the investment rate $i = i(p, A, d)$ and the debt restructuring strategy as well as default strategy.

The following subsections describes these valuation problems for the case where the firm follows the equity-maximizing strategy. The case where the firm follows a value-maximizing strategy, and the related case where the firm is all-equity financed, is provided in Appendix A. In Appendix B we describe the numerical algorithm that we use for solving these stochastic control problems.

3.1 VALUATION OF THE EQUITY FOR THE EQUITY-MAXIMIZING FIRM

In each state (p, A, d), the firm makes its investment choice $i = i(p, A, d)$, as well as its recapitalization and default choices. These choices maximize the market value of the firm's equity, which is the present value of cash flow to the equityholders. The solution involves free boundary conditions that divide the state space (p, A, d) into the three regions that characterize the firm's choices: the *no recapitalization* region, the *recapitalization* region, and the *default* region.[20]

In the *no recapitalization* region, it is not optimal for the firm to restructure its debt. In this region, the equity value $E(p, A, d)$ equals the instantaneous cash flow t, $CF(i)$, plus the expected value of the equity at time $t + \Delta t$ calculated under the risk neutral measure Q. The following expression provides the maximization problem over all investment choices i:

[19] An alternative assumption would be to assume that, after the default costs are paid, the debtholders that take over the firm optimally recapitalize the firm and continue to operate it. Unfortunately, this assumption would create additional complexity in the numerical algorithm since the value of the firm would depend on the financing strategy after the default.

[20] For brevity, we omit the discussion of smooth pasting conditions.

$$
\begin{aligned}
E(p_t, A_t, d_t) = \max_{i \geq 0} \{ & CF(i) \cdot \Delta t \\
& + e^{-r\Delta t} \mathbb{E}_Q [E(p_{t+\Delta t},\ A_t(1 - \gamma \Delta t) + i \cdot \Delta t,\ d_t)] \Big\}, \\
CF(i) = {} & p_t \cdot c(A_t) - b - i - d_t - wF + w \cdot D - TAX(p_t, A_t, d_t) \\
& - TC_{Equity}(p_t, A_t, d_t) - DC(p_t, A_t, d_t)
\end{aligned} \tag{10}
$$

where $\mathbb{E}_Q$ is an expectation operator, $TAX(p_t, A_t, d_t) = \tau \times \max[0, p_t \cdot c(A_t) - b - d_t - \gamma A_t]$, and $TC_{Equity}(p_t, A_t, d_t) = C_{Equity} \times \max[0, -p_t \cdot c(A) + b + d_t + i + wF - w \cdot D(p, A, d)]$, $DC(p_t, A_t, d_t) = C_{Distress} \times \max[0,\ s \cdot d_t - p_t \cdot c(A_t) + b]$ describe the instantaneous taxes, costs of issuing equity, and distress costs introduced in Equations (3), (8) and (9). The term $-wF + w \cdot D$ corresponds to the net refunding expenses specified in (4).

Using standard arbitrage arguments, the value of the equity $E(p, A, d)$ in the *no recapitalization* region is given by the solution to the following stochastic control problem:

$$
\begin{aligned}
\max_{i \geq 0} \Big[& \frac{1}{2} \sigma_p^2 p^2 E_{pp} + (r - \alpha) p E_p + (-\gamma A + i) E_A - rE \\
& + p \cdot c(A) - b - i - d - wF + w \cdot D - TAX(p, A, d) \\
& - TC_{Equity}(p, A, d) - DC(p, A, d) \Big] = 0,
\end{aligned} \tag{11}
$$

where subscripts denote partial derivatives. Taking derivatives with respect to i in (11), one can determine the optimal investment rate i.[21] Note, that, since there is no horizon in the problem, the value of the equity is independent of time, that is, its partial derivative with respect to time t is zero, $E_t(p, A, d) = 0$.

With a sufficient increase or decrease in the product price the firm can enter either the *default* region or the *recapitalization* region. In the *recapitalization* region, the firm optimally increases or decreases its debt level. The firm recapitalizes only if net increase in equity value exceeds transaction costs. When the firm recapitalizes, it replaces its existing debt, which has a periodic payment d and a face value of $F = \frac{d}{r}$, with new debt, which has a periodic payment $\widehat{d}$ and a face value of $\widehat{F} = \frac{\widehat{d}}{r}$, where the choice of the new debt $\widehat{d}$ maximizes the value of the firm's existing equity. With this transaction, the firm instantly transits from the states (p, A, d) of the *recapitalization* region to the new states $(p, A, \widehat{d})$ in the *no recapitalization* region. We denote by

[21] The firm optimally chooses not to invest that is, $i = 0$, if $E_A \leq 1$. The firm invests up to all its internally generated cash and pays no dividends, i.e., $i = p \cdot c(A) - b - d - wF + w \cdot D$, if $1 < E_A \leq 1 + C_{Equity}$; and the firm instantly moves to a greater capacity $i = \infty$, if $E_A > 1 + C_{Equity}$.

(p^+, A^+, d^+) the states where the firm optimally increases its debt level from d^+ (the face value $F^+ = \frac{d^+}{r}$) to some optimal $\widehat{d}$.

When this is the case, the value of the equity must satisfy the following free boundary (value matching) condition where the maximum is taken over all debt choices $\widehat{d} > d^+$:

$$E(p^+, A^+, d^+) = \max_{\widehat{d} \text{ s.t. } \widehat{d} > d^+} [E(p^+, A^+, \widehat{d}) + D(p^+, A^+, \widehat{d}) - F^+ - C_{Debt} \times \widehat{F}], \tag{12}$$

$$\text{such that } D(p^+, A^+, \widehat{d}) - F^+ > 0, \quad E(p^+, A^+, \widehat{d}) > 0$$

where the last term is the proportional transaction costs that are introduced in (6). The amount $D(p^+, A^+, \widehat{d}) - F^+$ is the net debt proceeds of new debt issuance which is paid to current shareholders for the portion of their shares that is repurchased. The inequality $E(p^+, A^+, \widehat{d}) > 0$ rules out the possibility that the recapitalization leads to an immediate default.[22]

Similarly, we denote by (p^-, A^-, d^-) the states where the firm optimally decreases its debt from d^- to some $\widehat{d}$, $\widehat{d} < d^-$. When this is the case, the value of equity must satisfy the following free boundary condition:

$$E(p^-, A^-, d^-) = \max_{\widehat{d} \text{ s.t. } \widehat{d} < d^-} [E(p^-, A^-, \widehat{d}) - (F^- - D(p^-, A^-, \widehat{d})) - C_{Equity} \times (F^- - D(p^-, A^-, \widehat{d})) - C_{Debt} \times \widehat{F}], \tag{13}$$

$$\text{such that } F^- - D(p^-, A^-, \widehat{d}) > 0, \quad E(p^-, A^-, d^-) > 0,$$

[22] We also check whether it is optimal for the firm to instantly invest part of its net proceeds from new debt issuance. Therefore, the following boundary condition is also incorporated:

$$E(p^+, A^+, d^+) = \max_{\widehat{d} > d^+,\ \Delta A \geq 0} [E(p^+, A^+ + \Delta A, \widehat{d}) + D(p^+, A^+, \widehat{d}) - F^+ - \Delta A - C_{Debt} \widehat{F}],$$

$$\text{such that } \quad \widehat{d} > d^+, \ 0 < \Delta A \leq D(p^+, A^+, \widehat{d}) - F^+, \quad E(p^+, A^+, \widehat{d}) > 0$$

where $F^+ = \frac{d^+}{r}$, and the amount $D(p^+, A^+, \widehat{d}) - F^+$ is the net debt proceeds of new debt issuance and the amount ΔA is the portion of net debt proceeds that are instantly invested in the firm's assets. The amount $D(p^+, A^+, \widehat{d}) - F^+ - \Delta A$ is paid to current shareholders for the portion of their shares that is repurchased. In most cases, the optimal choice is $\Delta A = 0$; however, in the states where both the firm's initial capacity and leverage are significantly below optimal, the firm can choose to instantly invest using debt proceeds, $\Delta A > 0$.

where $F^- = \frac{d^-}{r}$, $F^- - D(p^-, A^-, \widehat{d})$ is the amount of new equity required to repurchase part of the existing debt, and the last terms are the transaction costs of issuing equity and transaction costs of recapitalization. Note also that in the states of the *no recapitalization* region there is a strict inequality (>) for any $\widehat{d}$ in both (12) and (13), implying that it is not optimal for the firm to increase or reduce its current debt.

We also need to impose the free boundary condition which ensures that the equity value is greater or equal to zero. In the states of the default region denoted as (p^d, A^d, d^d), the equity value becomes zero and the firm defaults.

$$E(p^d, A^d, d^d) = 0.$$

3.2 VALUATION OF THE DEBT FOR THE EQUITY-MAXIMIZING FIRM

This subsection describes how the debt of the equity-maximizing firm is valued. The debt entitles its holders to receive a continuous coupon payment d and the net refunding payments $wF - w \cdot D$ until the firm either defaults or is recapitalized. In the *no recapitalization* region, the value of the debt $D(p, A, d)$ depends on the equityholders' investment and debt reissuance decisions, and satisfies:

$$\begin{aligned}&\frac{1}{2}\sigma_p^2 p^2 D_{pp} + (r-\alpha)pD_p + (-\gamma A + i^*)D_A \\ &\quad -rD + d + wF - w \cdot D = 0,\end{aligned} \tag{14}$$

where i^* is the equityholders' choice of the investment strategy which is the solutions to the optimal control problem in (11). The term $d + wF - w \cdot D$ is the instantaneous payment to the debtholders.

In the states of the *recapitalization* region, that is, the states in either (p^+, A^+, d^+) or (p^-, A^-, d^-), the firm increases or decreases its debt by repurchasing its current debt at its face value $F = \frac{d}{r}$, and the debtholders receive the face value F for their debt, implying that debt value in this region satisfies

$$\begin{aligned}D(p, A, d) = F = \frac{d}{r},\ &\text{for either } (p, A, d) \\ &\subseteq (p^+, A^+, d^+) \text{ or } (p, A, d) \subseteq (p^-, A^-, d^-).\end{aligned}$$

Note that by construction, when the firm reaches the states of the *recapitalization* region, the firm's debt is repurchased at its face value $F = \frac{d}{r}$, which reflects the present value of its coupons discounted at the risk-free rate. However, the new debt is issued at a market price that reflects the fact that the debt may default.

In default, the debtholders recover the value of the unlevered firm minus default costs $(1 - C_{default})E^U$. Thus, in the *default* region (p^d, A^d, d^d), the debt value satisfies

$$D(p^d, A^d, d^d) = (1 - C_{default}) \cdot E^U(p^d, A^d), \quad \text{if} \quad E(p^d, A^d, d^d) = 0.$$

4. Numerical Results

4.1 OVERVIEW OF THE NUMERICAL ALGORITHM

The numerical algorithm used to solve for the values of the equity and debt in (11) and (14) is based on the finite-difference method augmented by a "policy iteration". Values are determined numerically using dynamic programming on the discretized grid of the state space (p, A, d) with a discrete time step Δt. At each node on the grid the partial derivatives are computed according to Euler's method. We start the procedure at the terminal date, which is sufficiently far from $t = 0$ by approximating ("guessing") the values for debt and equity. By running backward recursion long enough and taking into account the investment and financing decisions, the values for E and D on each node of the grid converge to the steady-state "true" values since the errors of the initial approximation at the terminal date are "smoothed" away because of the effect of discounting. In Appendix B we describe the numerical algorithm in more detail.

4.2 BASE CASE PARAMETERS AND VARIABLES OF THE MODEL

The base case parameter values that we use in our numerical analysis are displayed in Table II. These parameters are chosen to roughly match empirical observations for selected firms in the gold mining industry. Gold mining firms provide a natural setting for generating initial parameters for our model for the following reasons: First, the only exogenous source of uncertainty in our model comes from the commodity prices and arguably, the main uncertainty that gold-mining firms face is uncertainty about gold prices. Second, gold price data is easily available. Third, the gold-mining firms' production costs and other operating and financial ratios are available and relatively easy to calculate. Moreover, various gold-related financial instruments (e.g., gold futures) are widely available and are relatively liquid, which would further justify our arbitrage-free valuation approach. Finally, a number of corporate finance articles, for example, Tufano (1996), Brown et al. (2001) and Fehle and Tsyplakov (2005), have examined gold firms so we have some familiarity with this industry.

Table II. Base case parameters and variables value

Parameters and variables	Values
p, the current market price of gold	\$360 per ounce
σ_p volatility of the gold price	10%
α, gold convenience yield	2%
R, risk-free rate	3% per year
$c(A) = 1 - e^{-0.002A}$, capacity of the firm with assets valued at A	80%
A, value of the firm's fixed assets	\$819
γ, depreciation rate	10% per year
w, rate of continuous redemption of the debt	0.05/year
$(1/w)$, average debt maturity	20 years
τ, corporate tax	35%
C_{Equity}, proportional transaction costs for issuing equity	5%
C_{Debt}, proportional transaction costs for issuing debt	2%
$C_{Distress}$, proportional costs of financial distress	50%
s, distress triggering interest coverage ratio	1

Table III provides data on a sample of pure-play gold mining firms for which Compustat data is available. The table reports several financial ratios that provide guidance in our choice of base case parameter values. It should be noted that it is impossible to perfectly match all of the observed ratios, since a number of the ratios are determined endogenously in our model as functions of the other parameters. In addition, since we do not want to give the impression that we can precisely calibrate our model, we use round numbers.

In the 1986 to 2002 period, the daily COMEX gold closing prices obtained from Bloomberg fluctuates between \$242 and \$447 per oz, with an average price of around \$360/oz and a monthly volatility of 10.4%. Based on this, we set the initial gold price at $p = \$360/\text{oz}$ and the volatility of the spot price σ at 10%. The 12-month lease rate for gold, which is used as a proxy for the convenience yield of gold, averaged 2.04% (as reported from Bloomberg for Feb–1995 to Jan–2000), so the convenience yield α is set at 2%.[23] Since the interest rate is set to 3%, this means that gold prices are growing approximately 1% per year in the risk-neutral measure, and since production costs are assumed to be fixed, earnings will grow substantially faster. Hence, in our base case, gold companies are growth firms with very high price/earnings ratios. We will also consider cases where gold prices are expected to be stable.

The capacity function is chosen to be $c(A) = 1 - e^{-\beta A}$, where we set $\beta = 0.002$ and the initial level of assets A=\$819. These parameter values were chosen to guarantee that the initial capacity takes the value of the efficient

[23] Schwartz (1997) reports similar numbers in his calibration approach for spot and futures prices as well as for the average convenience yield.

Table III. This table reports year 2002 COMPUSTAT data for 20 pure-play gold mining firms. All data is in $millions

Company name	Market leverage (%)	Sales/ assets (%)	Depreciation/ assets (%)	Investment/ sales (%)	Depreciation, $M	Assets, $M	Investments, $M	Market value, $M
Agnico-eagle	10.4	18.2	2.2	60.0	13	594	65	1,387
American Barrick	8.3	37.4	9.9	11.6	519	5,261	228	9,133
Campbell resources	64.4	11.2	2.7	29.0	2	83	3	45
Coeur d'alene	21.4	49.7	7.8	12.0	14	173	10	312
Hecla mining	1.0	66.0	14.1	10.6	23	160	11	448
Newmont gold	12.6	26.2	5.0	11.3	506	10,155	300	13,487
Placer Dome	6.4	30.3	4.7	10.5	187	3,985	127	5,386
Kinross mining	3.3	43.6	14.3	8.7	85	598	23	1,058
Canyon resources	0.1	46.9	21.8	6.1	8	37	1	23
Meridian gold	0.0	19.0	3.2	28.1	23	703	38	1,745
Glamis gold	0.0	17.0	3.8	30.6	18	475	25	1,429
Cambior	10.4	72.4	10.3	8.6	29	279	17	257
Freeport McMoRan copper & gold	43.9	45.6	6.2	9.8	260	4,192	188	4,470
Goldcorp	0.0	40.5	4.7	14.5	22	458	27	2,320
Bema gold	5.0	17.8	6.1	44.7	12	204	16	362
Richmont mines	0.0	114.8	14.6	2.6	4	29	1	60
Wheaton river	0.0	22.8	2.1	15.0	3	152	5	181
Miramar mining	0.0	26.1	3.1	34.2	4	129	11	158
Crystallex mining	10.7	25.8	7.1	96.9	8	111	28	155
MK gold company	67.0	6.9	0.0	72.4	0	70	4	48
Sample Mean	13.2	36.9	7.2	25.9	87.0	1392.4	56.4	2,123.2
Sample St Dev	20.7	25.3	5.5	25.1				

steady state level, that is, the level at which the initial optimal investment rate of the value-maximizing firm equals the depreciation rate, $i = \gamma A$. The base case capacity level c equals 80%, which means that the firm can expand its capacity by up to 20%. The initial capacity level of $c = 80\%$ implies that the firm initially produces 0.8 ounces of gold per year. The depreciation rate γ is set at 10%, which approximates the observed ratio of (Annual Depreciation)/(Assets) for gold-mining firms. As one can see in Table III, the sample average ratio of (Annual Depreciation)/(Assets) is 7.2% with a standard deviation of 5.5%.

The base case total production costs are set at \$240/oz, which is consistent with data reported in Tufano (1996), who documents that the average (median) production cost is between \$239 and \$243/oz (\$235–\$239/oz) with a standard deviation across firms of \$58/oz. This implies an initial sales to cost ratio for the firm of $\frac{sales}{\cos t} = \frac{price/\mathrm{oz}}{\cos t/\mathrm{oz}} = \frac{\$360/\mathrm{oz}}{\$240/\mathrm{oz}} = 1.5$. In the framework of the model, the total production costs is the sum of two terms: $i + b$ where $i = \gamma A = 81.9$ is an initial investment rate, and b is fixed production costs. Given the initial production volume of $c(A) = 80\%$ ounces, we choose the value of the parameter of fixed production costs b such that the model-generated price to cost ratio $\frac{p \cdot c(A)}{b+\gamma A}$ is approximately 1.5. Therefore, for the base case parameter values, fixed production costs are set at $b = \$110/\mathrm{oz}$ so that the ratio of sales/costs, $\frac{p \cdot c(A)}{\gamma A+b} = \frac{360 \cdot 0.8}{81.9+110} = 1.5$, which matches the average observed ratio. Also, given these parameters, the initial investment ratio generated by the model approximately matches the observed annual investment ratios of the gold-mining firms. For example, as reported in Table III, the observed average ratio of (Annual Investment)/(Annual Sales) is 25.9% and the standard deviation of 25.1%. Given our base case parameters, the corresponding ratio of the investment rate to sales in the model is $\frac{i}{p \cdot c(A)} = \frac{\gamma A}{p \cdot c(A)} = 28.3\%$.

The rate of debt retirement w is set at 0.05, which corresponds to an average debt maturity of 20 years. In addition, since the short-term (3 year) Treasury-note yield in year 2002 varied from 2.23% to 4.14% and averaged 3.1%, we set the risk free rate at $r = 3\%$. We set the costs of issuing equity and debt at $C_{Equity} = 5\%$ and $C_{debt} = 2\%$, which are roughly consistent with approximations reported in the literature.[24]

The empirical literature on financial distress provides some guidance on the proportional distress cost, $C_{Distress}$, parameter, and the interest coverage ratio which triggers distress, s. Opler and Titman (1994) show that during industry downturns, more highly leveraged firms experience a drop in operating income which is more than 10% greater than the drop experienced by less highly levered firms. Andrade and Kaplan (1998) document that financial distress results in

[24] Transaction costs parameter of debt issue is along the lines with corresponding parameter values used in Fischer et al. (1989a), Leland (1998) and Goldstein et al. (2001).

a 10–20% decline in operating and net cash flow margins, and Altman (1984) estimates that in the three years prior to bankruptcy, the average (median) decline of earning per share compared to analysts' expectations is 289% (50%). It is difficult to directly assign empirically observed values to parameter $C_{Distress}$ because in the model the distress costs increase proportionally with the level of cash shortfall. We expect that distress costs for gold-mining firms should be relatively low and thus set the proportional distress costs parameter $C_{Distress}$ at 50% for the base case. Since the financial distress costs are applied to the difference between the firm's required coupon payments and its net earnings (see Equation 9), at this level, the distress parameter leads to relatively low distress-related cash flow losses. However, in our comparative static analysis we will be considering scenarios where financial distress costs have a much stronger effect on cash flows.

In our base case, we assume that the firm is in distress if its net income falls below its coupon payments, $\frac{p \cdot c(A)-b}{d} < 1$, so that the threshold interest coverage ratio s equals one ($s = 1$). In our comparative statics we consider threshold coverage ratios of 3 and 4. The median coverage ratio for a firm with a BBB rating is about 3.4, so one might assume that a drop below 3 could cause the firm's rating to drop below investment grade. This could in turn affect the firm's ability to transact with its stakeholders, which can create what we describe as financial distress costs.

4.3 FINANCING CHOICES FOR THE BASE CASE PARAMETERS

We measure a firm's leverage in two different ways. The first measure is the market *debt-to-value ratio*, $\frac{D}{(E+D)}$, which captures the future expectations of the firm's cash flows and thus measures its long-term credit worthiness. For any given initial debt level, that is, the face value of the debt F (or the coupon rate d), the debt-to-value ratio is different for the different firm types since the market value of both equity and debt depend on the firm's type. The second measure is the *interest coverage ratio*, which is the ratio of the firm's current net income to interest (coupon) payment of the debt, $\frac{p \cdot c(A)-b}{d}$. This ratio measures the current ability of the firm to meet its debt obligations, and at least initially, for a given initial debt level, this ratio is the same for all firm types.

Initial Target Leverage

Table IV reports firm values for different initial debt levels and interest coverage ratios for each of the two firm types described earlier. The underlined numbers in each table represent the variables that correspond to the target capital structures which maximize the value of debt plus equity. The initial target ratio is 21.7% and 39% for the equity- and the value-maximizing firms respectively. Intuitively, the equity-maximizing firm initially chooses to be less levered because it has the option to later issue additional debt, but generally is not inclined to reduce its debt owing to the wealth transfer to the debtholders. For the value-maximizing firms the incentives to increase and decrease leverage is more symmetric and as a result their initial target ratio is substantially higher than the corresponding ratio for the equity-maximizing firm. In fact, since recapitalization costs are assumed to be less than bankruptcy and financial distress costs with our base case parameters, the value-maximizing firm always chooses to issue equity and pay down debt when it is doing poorly and thus never chooses to go bankrupt.

Investment Choice

The capacities of both the value- and equity-maximizing firm are initially set at their initial steady state levels. This implies that the value-maximizing firm will initially invest at a rate that exactly offsets the depreciation rate to keep its capacity at the optimal level. In contrast, given its incentive to underinvest, the equity-maximizing firm invests to offset depreciation only when its debt ratio is very low, and otherwise invests nothing, and allows its capacity to depreciate to a level that is optimal for a levered equity-maximizing firm (see Column 7, Table IV).[25]

The Incentive to Move Towards the Target

In this section we take transaction costs into account and examine the extent to which a firm with a capital structure that deviates from its target debt ratio will take steps to move towards its target. Our analysis is based on a target debt ratio calculated with the base case parameters along with the initial capacity level and product price.

The last columns in Table IV describe whether the firm will take actions to increase or decrease its outstanding debt for each debt ratio we consider. The table shows that when the debt ratio is sufficiently low (below 18% for the value-maximizing firm and below 9% for the equity-maximizing firm) the firm increases its debt instantly. When the debt ratio exceeds this level, but is below its target debt ratio, it is not optimal to immediately increase debt since the transaction costs more than offset the benefit of moving towards the target debt ratio.

[25] Since firms have different objective functions, the boundaries at which firms invest from internal cash and/or from issuing equity are different. For more details see the boundary conditions for investments in the footnote on page 415.

Table IV. This table reports market values and several financial ratios that are generated by our model. The variables are reported for various debt ratios, with the underlined values corresponding to the initially optimal debt ratios. The model's parameter values are as in the base case reported in Table III

	Equity-maximizing firm							Value-maximizing firm				
Coupon level	Interest coverage ratio[1]	Firm value[2]	Leverage ratio[3]	Credit spread (%)	Agency cost[4] (%)	Investment rate per year[5]	Debt decision[6]	Firm value[2]	Leverage ratio[3]	Credit spread (%)	Investment rate per year[5]	Debt decision[6]
0	–	7,238	0.000	0.00	5.0	–	Increase	7,616	0.00	0.00	81.9	Increase
20	8.96	7,238	0.092	0.00	5.0	–	Increase	7,616	0.09	0.00	81.9	Increase
40	4.48	7,265	0.180	0.06	4.6	81.9	No change	7,616	0.18	0.00	81.9	Increase
49	3.66	7,271	0.217	0.11	4.6	81.9	No change	7,622	0.21	0.00	81.9	No change
60	2.99	7,269	0.264	0.13	4.7	81.9	No change	7,631	0.26	0.00	81.9	No change
80	2.24	7250	0.348	0.17	5.3	81.9	No change	7,659	0.35	0.00	81.9	No change
90	1.85	7,232	0.39	0.19	5.8	0	No change	7,676	0.39	0.00	81.9	No change
100	1.79	7,197	0.430	0.23	6.2	0	No change	7,673	0.43	0.00	81.9	No change
120	1.49	7,100	0.507	0.33	7.4	0	No change	7,666	0.52	0.00	81.9	No change
140	1.28	6,895	0.587	0.46	9.8	0	No change	7,648	0.61	0.00	81.9	No change
160	1.12	6,600	0.648	0.74	13.0	0	No change	7,589	0.70	0.00	81.9	No change
180	1.00	6,260	0.687	1.19	16.5	0	No change	7,497	0.80	0.00	81.9	Reduce
200	0.90	5,760	0.750	1.63	22.3	0	No change	7,409	0.90	0.00	81.9	Reduce
220	0.81	5,090	0.829	2.21	30.6	0	No change	7,339	0.99	0.00	81.9	Reduce
240	0.75	3,250	0.97	4.64	39.1	0	No change	5,341	0.99	1.51	81.9	No change
260	0.69	2,922	1.000	–	–	–	Default	2,922	–	–	–	Default

[1] Interest coverage ratio is the ratio of the net income to coupon payment, $\frac{pc(A)-b}{d}$. [2] Firm value is $D+E$. [3] Leverage ratio is $\frac{D}{D+E}$. [4] Agency cost is the difference between the values of the value- and the equity-maximizing firm, as a percentage of the value-maximizing firm. [5] Investment rate per year is numerically calculated from the grid of the numerical algorithm as $[A_{t+1} - A_t(1-\gamma^* dt)]/dt$, where A_{t+1} is the choice of assets at time $t+1$ given that the firm has assets A_t at time t, dt is the time step of the grid. Empty entry "-" implies that the investment rate is not applicable due to instant recapitalization or default. [6] Debt Decision is the firm's strategy with respect to its current debt, i.e., the decision to increase, reissue, retire or default its debt.

We also examine the conditions under which firms choose to reduce their debt when their debt level exceeds the target. The results in Table IV indicate that the value-maximizing firm repurchases its outstanding debt whenever its debt exceeds 80%, which corresponds to an interest coverage ratio of 1.0. In contrast, for the base case parameters, the equity-maximizing firm never reduces its debt, because the transaction costs and the wealth transfer to debtholders exceed the added value associated with a movement towards the firm's target capital structure. This last result depends on a number of parameters. We find that in a number of situations an equity-maximizing firm will pay down its debt to avoid the costs associated with being overlevered. In particular, the equity-managed firm will in fact pay down debt, to avoid the costs of financial distress, when the financial distress trigger point increases sufficiently. For example, comparative statics show that with a financial distress trigger of $s = 4$, equity-maximizing firms make capital structure choices that are almost identical to the choices made by total-value-maximizing firms.[26]

As a result, firms that are more sensitive to financial distress are subject to less agency costs. It is important to stress that the analysis above suggests that firms can deviate quite dramatically from their target debt ratios without providing an economic incentive to move towards their targets. The incentive to move towards their targets appears to be stronger when the firm is underlevered rather than overlevered. This is especially true for the equity-maximizing firm, which will not move towards its target when it is overlevered, except when financial distress costs are sufficiently large.

Agency Costs

Column 6 in Table IV reports the agency costs associated with the debtholder/equityholder conflict, which we define as the difference between the values of the value-maximizing and the equity-maximizing firms, expressed as a percentage of the value of the value-maximizing firm. When the equity- and value-maximizing firms initially choose their optimal debt ratios, the difference in their values (i.e., the agency costs) is 5.27%.[27] As the table shows, the agency costs can be substantially higher, however, when firms are overlevered, since the value-maximizing firm tends to pay down its debt to avoid financial distress

[26] In the framework where debt is sold only once, Bhanot and Mello (2006) show that a covenant that requires the firm to prepay a fraction of the debt can alleviate the agency costs due to debtholder–equityholder conflict.

[27] In models where debt is static, agency costs driven by debtholder/equityholder conflict of interests are lower. See Mello and Parsons (1992), Parrino and Weisbach (1999), Mauer and Ott (2000) and Titman et al. (2004).

and bankruptcy costs, while the equity-maximizing firm tend not to reduce its debt.[28]

This difference in values reflects the loss in value that arises from the fact that the equity-maximizing firm deviates from the optimal financing and investment strategy. Specifically, the equity-maximizing firm tends to underinvest, and distributes a bigger fraction of its income as dividends. As a result, over time equity-maximizing firms operate with capacity levels that are lower than their optimal levels. In addition, equity-maximizing firm fails to recapitalize when it is doing poorly, and is thus more subject to financial distress costs and bankruptcy costs. Finally, it should be noted that since the equity-maximizing firm is initially less levered, it initially realizes less debt related tax gains than value-maximizing firms.

4.4 COMPARATIVE STATICS

This section examines how changing parameter values affect the firm's initial optimal debt ratio, which maximizes the total value of the firm. This initial debt ratio can also be viewed as the firm's target debt ratio, or equivalently, the debt ratios that firms choose when they bear no costs associated with a recapitalization. In this sense, our comparative statics are directly related to cross-sectional studies of capital structure, like Titman and Wessels (1988), which examine how observed capital structures relate to various proxies that are likely to be related to the firms' target capital structures.

In most of the cases, the comparative statics are computed numerically by calculating the change in the initial optimal debt ratio associated with changing one parameter in the model, while setting the other parameters equal to the level in the base case. The comparative statics results, reported in Table V, can be summarized as follows:

- The initial target debt ratio is positively related to the product price p and negatively related to production costs b. Firms with higher prices (or equivalently lower production costs) have higher profit margins and thus lower operating leverage, which makes the firm less risky and thus increases its optimal target debt ratio. This comparative static illustrates that the target debt ratio evolves over time as prices and the firm's

[28] Our implication that agency costs can be significant in a dynamic model contrasts with implications of Childs et al. (2005), who show that a firm with financial flexibility is subject to relatively low agency costs. In their model the firm cannot recapitalize before the debt matures and, as a result, the suboptimal recapitalization problem is not considered. Thus, in their model the only source of the agency conflict is suboptimal investment strategy.

Table V. Comparative statics this table reports interest coverage ratios, debt ratios and other values for equity- and value-maximizing firms at their initially optimal debt ratios. In each case one parameter value changes while other parameters stay at their base case levels

	Value-maximizing firm				Equity-maximizing firm				
Parameter	Parameter value	Interest coverage ratio[1]	Firm value[2]	Leverage ratio[3] (%)	Interest coverage ratio[1]	Firm value[2]	Leverage ratio[3] (%)	Agency costs[4] (%)	Credit spread (%)
Initial	400	1.69	9,217	45.3	2.37	8,730	31.8	5.28	0.21
Price, p	360	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
	320	2.04	6,161	39.0	3.92	5,853	20.6	5.00	0.10
Production	140	2.58	6,807	33.9	4.84	6,485	18.3	4.73	0.12
Costs, b	110	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
	80	1.65	8,571	42.2	2.67	8,098	26.4	5.51	0.13
	13%	2.28	7,499	33.3	4.27	6,974	18.5	7.00	0.10
Depreciation	10%	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
Rate**	7%	1.55	8,431	47.8	2.80	7,978	26.8	5.38	0.13
	40	2.00	7,676	39.0	2.93	7,243	26.0	5.64	0.25
Maturity,	20	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
Years	10	2.00	7,676	39.0	3.89	7,490	19.9	2.43	0.08
Distress	75%	2.04	7,649	38.3	3.84	7,246	20.8	5.27	0.11
Costs	50%	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
	0%	1.93	7,769	39.9	3.14	7,357	24.7	5.30	0.13
Interest	1.0	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
Coverage	2.0	2.52	7,433	31.8	3.81	7,130	21.5	4.07	0.07
Ratio triggering	3.0	3.32	7,199	25.0	4.17	6,989	20.3	2.92	0.03
Distress, s	4.0	4.84	7,029	17.5	4.84	6,950	17.7	1.13	0.01
Default	75%	2.00	7,676	39.0	4.17	7,092	19.5	7.61	0.11
Costs	50%	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
	25%	2.00	7,676	39.0	3.04	7,493	25.2	2.39	0.12
Transaction	10%	2.13	7,544	37.1	3.89	7,239	20.7	4.04	0.08
Cost of	5%	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
Equity	1%	1.65	7,852	46.1	3.32	7,313	23.7	6.86	0.12
Convenience	2%	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
Yield	3%	3.89	3,292	46.6	5.97	3,066	28.5	6.86	0.43
	4%	6.63	1,623	55.5	9.23	1,574	30.1	3.04	1.09
Transaction	5%	1.44	7,340	56.3	2.33	7,088	33.9	3.43	0.20
Cost of debt	2%	2.00	7,676	39.0	3.66	7,271	21.7	5.28	0.11
	1%	1.89	7,860	40.3	3.89	7,348	20.2	6.51	0.10

** For different depreciation rates we use an initial capacity level at which initial investment equals depreciation rate (initially steady state capacity). [1] Interest coverage ratio is the ratio of the net income to coupon payment, $\frac{pc(A)-b}{d}$ [2] Firm value is $D+E$. [3] Leverage ratio is $\frac{D}{D+E}$. [4] Agency cost is the difference between the values of the value- and the equity-maximizing firm, as a percentage of the value-maximizing firm.

profitability change. In this respect, our model can be contrasted with models in Fischer et al. (1989a) and Leland (1998), which assume the firm moves to the same debt ratio when it recapitalizes.

- Firms with higher depreciation rates initially target lower debt ratios. Holding price and costs constant, the higher depreciation rates imply higher operating leverage, which as we just mentioned, leads to lower initial target leverage.[29]
- The trigger point at which the firm becomes financially distressed (parameter s) is quite important for the target debt ratio. When the coverage ratio that triggers distress is higher, the target debt ratio is lower. The financial distress trigger has a stronger effect on the target debt ratio of the value-maximizing firm than the equity-maximizing firm. This is due to the fact that the equity-maximizing firm targets a relatively low debt ratio, for the base case trigger ($s = 1$), and at that lower debt ratio, is only infrequently financially distressed.
- The value-maximizing firm reduces its debt level if prices drop significantly and thus spends little time in financial distress. As a result, distress costs $C_{Distress}$ have very little influence on the initial debt choice of value-maximizing firms. However, the equity-maximizing firm does not reduce its debt (given base case parameters) in the event of financial distress, so the magnitude of distress costs are relevant and act to reduce the firm's initial target debt ratio.
- For the range of parameters we consider, the value-maximizing firm never defaults, so default costs $C_{default}$ do not affect its target choice. In contrast, the equity-maximizing firm does default, and therefore higher default costs lead to lower target debt ratios. The default costs are borne solely by debtholders, and are thus ignored by equityholders ex post who make the financing and investment choices to maximize equity values.
- Firms with lower transaction costs of issuing debt C_{Debt} select lower initial debt ratios. Intuitively, with low transaction costs, the firm chooses to be more conservatively financed because it is less expensive to increase its debt ratio if the product price increases.
- Firms with lower costs of issuing equity C_{Equity} initially have higher target debt ratios, since it is less expensive for them to issue equity to raise funds to pay down their debt if they are doing poorly.

[29] Our results indicate that despite the lower optimal debt-to-value ratios, firms with higher depreciation rates have more volatile debt and equity values.

- To examine the effect of growth, we vary the parameter α, the convenience yield from the price process. The primary effect of α is that it adjusts the risk-neutral growth rate of the price; when α is smaller the growth rate is larger. Firms with faster growing product prices tend to have lower target debt ratios, but lower target coverage ratios. In other words, growth firms target less debt relative to their values but more debt relative to their profits.
- The maturity structure of debt has no influence on either the value or the initial target debt ratio of the value-maximizing firm since the boundary at which the firm recapitalizes is independent of the maturity structure, which in turn implies that the transactions costs associated with recapitalizing is independent of maturity structure. However, for the equity-maximizing firm, the maturity of the debt has two effects that can affect value. First, shorter maturity reduces the under-investment incentives and makes it less costly for the firm to increase its capital structure. These effects make the firm more valuable. However, offsetting this is the fact that shorter maturity debt makes bankruptcy more likely because the per period cash flows to the debt holders are higher because of debt refunding costs. As it turns out, for our base case parameters firm value is higher for shorter maturity debt. In addition, the target debt ratio is lower when the debt has a shorter maturity, reflecting the fact that with shorter maturity debt, the option to increase the debt level in the future is less costly to exercise.

4.5 SIMULATION ANALYSIS AND EMPIRICAL IMPLICATIONS

The previous section reported comparative statics that examined how changes in our parameters influenced the firm's target debt ratio. Although these comparative statics provide some insights about the results of studies of the cross-sectional determinants of capital structure [like Titman and Wessels (1988)] it should be stressed that these studies examine actual debt ratios that change over time rather than their targets. This distinction between target and actual debt ratios is particularly important for empirical studies that examine changes in debt ratios. In practice, debt ratios can potentially change because of changes in target ratios, or because of forces that lead firms to either move towards their targets or away from their targets. As we mentioned in the introduction, one of the main objectives of the recent empirical analysis of capital structure changes is to determine the degree to which firms move towards their target capital structures.

Clearly the concept of a target capital structure is less compelling if firms move very slowly towards their targets, and the authors of some of the recent empirical studies argue that the speed with which firms move towards their targets is quite low. However, up to this point, there has not been much analysis on how quickly observed capital structures should be expected to move towards their targets, which is the subject of this section. We do this by using our model to generate simulated data, which allows us to examine the evolution of actual debt ratios under a variety of conditions. We then estimate regressions from our simulated data and compare these estimates to those observed in the empirical literature which examine actual data. Since our main interest is on the speed at which firms adjust their capital structures towards their targets, our main interest is on the target adjustment models estimated in the empirical literature.

Simulating Data

At each node of the grid (p, A, d), our numerical model generates the values of a number of variables that are of interest. These include the exogenous product price variable as well as the endogenous earnings, investment, capital structure, and firm value variables. To examine how these variables are expected to move over time, we simulate 200 random paths for the product price p, which randomly generates the information embedded in the various nodes. All simulated paths start at the firm's initial optimal capital structure and assume an initial capacity level of $c = 0.8$. Since the price level at which the simulations start may affect the dynamics, the simulations are run with three different starting price levels of $p =$ \$320, \$360 and \$400. Each path is terminated after 100 years. If at any time the simulated price reaches the default boundary, the path is terminated and a new path is started.[30]

Each node t on the simulated path provides an annual data point that we use in our regression analysis. Specifically, at each node we record the firm's annual earnings, its investment choice, its debt level and the market and book value of its assets. In addition, we calculate the firm's market leverage ratio D/V_t (the face value of its debt divided by the market value of its equity plus the face value of debt, $\frac{F_t}{(E_t+F_t)}$), its target debt ratio TL_t (the ratio at which the total market value of debt plus equity reaches its maximum, that is, the debt ratio it would move to if it were to recapitalize), its past stock returns, and its market, to book-ratio MB_t.

We provide separate sets of simulated data for the value-maximizing and equity-maximizing firms. In addition, since we are especially interested in the effect of financial distress on financing decisions, we generate data for firms

[30] As we mentioned, for the base case parameters, only the equity-maximizing firm can default.

with four different levels of the distress trigger s: $s = 1$ (base case), $s = 2$, $s = 3$, and $s = 4$. The other parameters are as set at their base case levels for all simulations.

Summary Statistics for Simulated Data

Table VI, Panel 1, reports summary statistics for the simulated data, segmented by each of the four assumptions about financial distress costs. The table documents that firms with more sensitive distress triggers, that is, higher s have (i) lower and less volatile target debt ratios, (ii) smaller deviations from the corresponding target debt ratios, and (iii) lower and less volatile debt ratios. In addition, it documents that debt ratios vary more for equity-maximizing firms than value-maximizing firms, except for the case where the probability of realizing distress costs is low (i.e., when $s = 1$). Note, that the average leverage deficit (calculated as target minus leverage, $TL_t - D/V_t$ for each year t) for the equity-maximizing firm is negative, implying that on average the firm is overleveraged relative to its target. This is because the equity-maximizing firm tends to recapitalize when it is substantially underlevered, but for most parameters will not recapitalize when it is overlevered. Again, the difference between the ratios of the value- and the equity-maximizing firm ratios declines with parameter s, since equity-maximizing firms act more like value-maximizing firms when distress is more likely.

To provide additional insights, we split the data into two investment regimes: (i) the periods where the firm has excess capacity and chooses not to invest, and (ii) the periods where the firm invests. As Panel 2 reveals, the target ratio tends to be significantly lower in those periods when the firm does not invest. In these periods, the product price tends to be lower, which implies that the firms have greater operating leverages. However, in most cases the realized debt ratio is actually greater in these periods, suggesting that firms with excess capacity tend to be overlevered relative to their targets. The reason is that these firms tend to have experienced declines in product prices (and equity values) in the preceding periods, which increase their actual debt ratios (because of the falling stock price) while decreasing their target debt ratio (because of increased operating leverage). Note also, that the difference between target and actual debt ratios is lower for value-maximizing firms as well as for firms that are more likely to be financially distressed.

Regression Results Applied to Model-simulated Data

In this section we apply regression analysis to the model-simulated data. We concentrate on partial adjustment regression models and analyze factors

Table VI. This table presents summary statistics for the model-simulated data using base case parameters along with four different levels of the distress trigger s: s = 1 (base case), s = 2, s = 3 and s = 4, where TL_t is the target leverage at time t, $(D/V)_t$ is the leverage ratio, $TL_t - (D/V)_t$ is the leverage deficit.
Summary statistics are reported for three subsamples: Panel 1 reports statistics for all time periods. Panel 2 reports statistics for the periods where the firm's investments are positive, and Panel 3 reports statistics for the periods when investments are zero

Panel 1: All time periods

Parameter s	s = 1				s = 2				s = 3				s = 4			
Firm type	Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing	
	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev	Mean	St. Dev
$(D/V)_t$	0.32	0.19	0.36	0.20	0.26	0.18	0.27	0.15	0.17	0.14	0.17	0.10	0.12	0.08	0.13	0.10
TL_t	0.22	0.14	0.31	0.22	0.19	0.15	0.22	0.16	0.16	0.15	0.16	0.11	0.12	0.09	0.14	0.13
$TL_t - (D/V)_t$	−0.10	0.24	−0.04	0.20	−0.07	0.24	−0.05	0.13	−0.01	0.15	−0.01	0.09	0.00	0.07	0.00	0.12

Panel 2: Periods when the firm's investments are positive

Parameter s	s = 1				s = 2				s = 3				s = 4			
Firm type	Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing	
	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev
$(D/V)_t$	0.3	0.16	0.37	0.20	0.23	0.14	0.24	0.15	0.18	0.1	0.17	0.09	0.11	0.08	0.13	0.07
TL_t	0.23	0.13	0.34	0.21	0.19	0.11	0.24	0.16	0.17	0.11	0.17	0.10	0.11	0.09	0.13	0.08
$TL_t - (D/V)_t$	−0.08	0.22	−0.03	0.19	−0.05	0.03	0.00	0.13	0.01	0.11	0.00	0.09	0.00	0.08	0.00	0.07

Panel 3: Periods when the firm's investments are zero

Parameter s	s = 1				s = 2				s = 3				s = 4			
Firm type	Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing		Equity-maximizing		Value-maximizing	
	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev	Mean	St.Dev
$(D/V)_t$	0.49	0.25	0.27	0.23	0.38	0.24	0.31	0.19	0.3	0.2	0.12	0.10	0.16	0.18	0.08	0.07
TL_t	0.18	0.15	0.21	0.20	0.18	0.24	0.14	0.16	0.16	0.26	0.11	0.14	0.15	0.23	0.08	0.14
$TL_t - (D/V)_t$	−0.32	0.3	−0.06	0.23	−0.2	0.31	−0.17	0.16	−0.14	0.23	−0.01	0.09	−0.01	0.20	0.00	0.07

discussed in Fama and French (2002), Welch (2004), Flannery and Rangan (2006), and Kayhan and Titman (2007).[31] These simulated regressions allow us to examine the extent to which the patterns of the model generated data resembles the actual data and provide additional insights about how the parameters of our model affects the speed with which debt ratios move towards their targets.

One–year Partial Adjustment Model

We start with a simple regression of the change in the debt ratio on the difference between the firm's actual and target debt ratio:

$$\left(\frac{D}{V}\right)_{t+1} - \left(\frac{D}{V}\right)_t = \lambda \cdot \left(TL_t - \left(\frac{D}{V}\right)_t\right) + \varepsilon_t,$$

where TL_t and $\left(\frac{D}{V}\right)_t$ are the target and the realized (market) debt ratio at time t. In this regression, the coefficient λ measures the speed of adjustment. A coefficient of $\lambda = 1$ implies a 100% adjustment within 1 year, while a coefficient of 0 implies no adjustment. The target debt ratios are measured in two ways in this regression. We first use the actual targets of the firms, which we calculate as the debt ratio the firm would move to if it could adjust its capital structure without incurring transaction costs. In actual empirical work, the target, of course, is not observable and must be estimated.[32]

Our second measure of the target is the average debt ratio in our sample, which resembles the target debt ratios used in some empirical studies. Using the average debt ratio instead of the true target debt ratio in this regression allows us to estimate the type of bias that is generated by such an approximation.

We first run the regressions on data generated by different firm types, which have different distress levels and different objective functions (i.e., value- and equity-maximizing firms). This pooled sample more closely resembles the actual samples used in the empirical studies, which include a broad cross-section of firms. As reported in Panel 1, Table VII (column 2), using the actual targets, the estimated speed of adjustment across all firms is 7.1%, which is within the range of the empirically estimated values reported in

[31] There are a number of papers that estimate speed of adjustment. See for example Hovakimian et al. (2001), Baker and Wurgler (2002), Hovakimian et al. (2004) and Liu (2005).

[32] Typically, the target used in regressions is estimated from a first stage of the two-stage regressions where control variables used on the first stage include size, R&D expenses, Industry dummy, Property, Plant And Equipment (PP&E), etc. Since most of these variables are not represented in the model, we do not replicate the first stage regressions for the simulated data.

Table VII. These tables report the speed of adjustment for the two partial adjustment models both estimated (1) for combined model generated data of firms with different objective functions and four different distress triggers s, and (2) separately for the equity- and the value maximizing firm for four levels of distress trigger s: s = 1 (base case), s = 2, s = 3 and s = 4. Panel 1 reports regression coefficients for the adjustment rate to a firm's target leverage ratio. Panel 2 reports regression coefficients for the adjustment rate to a firm's long-term average leverage ratio. Panel 3 reports regression coefficients for the target adjustment rate for different maturity parameters.

Panel 1: The speed of adjustment to a firm's target leverage ratio.

$(D/V)_{t+1} - (D/V)_t = \lambda^*(TL_t - (D/V)_t) + e_t,$

where TL_t is the target leverage, and $(D/V)_t$ is the leverage ratio at time t.

Parameter s	All values of s combined	s = 1		s = 2		s = 3		s = 4	
Firm type	Both types combined	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
$TL_t - (D/V)_t$	0.071	0.046	0.068	0.057	0.091	0.11	0.19	0.14	0.20
R-squared	0.05	0.03	0.11	0.017	0.11	0.04	0.17	0.06	0.15

Panel 2: The speed of adjustment to a firm's average leverage ratio

$(D/V)_{t+1} - (D/V)_t = \lambda^*(\overline{(D/V)} - (D/V)_t) + e_t,$

where $\overline{(D/V)}$ is the average leverage across all periods in the sample, $(D/V)_t$ is the leverage ratio at time t.

Parameter s	All values of s combined	s = 1		s = 2		s = 3		s = 4	
Firm type	Both types combined	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
$\overline{(D/V)} - (D/V)_t$	0.068	0.066	0.081	0.084	0.093	0.1	0.10	0.10	0.11
R-squared	0.029	0.033	0.05	0.028	0.06	0.071	0.055	0.053	0.062

Table VII. (Continued)

Panel 3: The speed of adjustment to a firm's target leverage ratio for different maturity parameters. The remaining parameter values are as in the base case. (s = 1)
$(D/V)_{t+1} - (D/V)_t = \lambda * (TL_t - (D/V)_t) + e_t$,
where TL_t is the target leverage, and $(D/V)_t$ is the leverage ratio at time t.

Maturity = 1/w	Maturity = 5 y		Maturity = 10 y		Maturity = 20 y		Maturity = 30 y	
Firm type	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
$TL_t - (D/V)_t$	0.062	0.068	0.061	0.068	0.046	0.068	0.044	0.068
R-squared	0.07	0.11	0.17	0.11	0.03	0.11	0.032	0.11

Panel 4: The speed of adjustment to a firm's target leverage ratio for different maturity parameters for the firm with high distress trigger (s = 4). The remaining parameter values are as in the base case.
$(D/V)_{t+1} - (D/V)_t = \lambda * (TL_t - (D/V)_t) + e_t$,
where TL_t is the target leverage, and $(D/V)_t$ is the leverage ratio at time t.

Maturity = 1/w	Maturity = 5 y		Maturity = 10 y		Maturity = 20 y		Maturity = 30 y	
Firm type	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
$TL_t - (D/V)_t$	0.193	0.2	0.190	0.20	0.14	0.20	0.14	0.20
R-squared	0.02	0.15	0.139	0.15	0.06	0.15	0.15	0.15

Kayhan and Titman (2007) and Fama and French (2002) but is somewhat slower than the estimates found in Flannery and Rangan (2006).[33]

In Panel 1, Table VII, we also report the results of separate regressions run for firms with different distress trigger levels and different objectives. These results indicate that the speed of adjustment increases as the distress trigger increases. When $s = 1$, the estimated adjustment speed for the equity-maximizing (value-maximizing) firm is $\lambda = 4.6\%$ (6.8%); when the distress trigger increases to $s = 4$, the speed of adjustment increases to $\lambda = 14\%$ (20%). These results also indicate that the speed of adjustment is slower for equity-maximizing firms than for value-maximizing firms, which is because the equity-maximizing firm is slow (if at all) to reduce the size of its debt when the firm is doing poorly. However, the difference in the adjustment speed (on a percentage basis) between firms with different objectives decreases as the distress trigger increases. Intuitively, what is happening is that the equity-maximizing firms with more sensitive distress triggers act more like the value-maximizing firms, and thus recapitalize more often.

In Panel 2, Table VII, these same regressions were estimated using the second target measure—which is the average debt ratio across firms in the sample—in place of the actual target. For the regression on the entire sample, the speed of adjustment is slightly lower (6.8%), which is consistent with the idea that the errors in variables bias the coefficient towards zero. However, in some subsamples, the speed of adjustment tends to be higher when the estimated target is used. For example, when parameter s is low, debt ratios tend to adjust more quickly towards their average debt ratios than towards their actual targets.

The reason why debt ratios sometimes adjust more quickly towards average debt ratios than target debt ratios is that changes in the debt ratio, which are caused by changes in equity values, tend to be negatively correlated with changes in the target. To understand this, note that product price increases cause equity values to increase, and thus decrease the firm's market debt ratio. However, product price increases also decrease the firm's operating leverage, which increases its target debt ratio. Hence, as a result, there is an inherent negative correlation between changes in the target debt ratio and changes in the actual debt ratio. In other words, for some realizations, the distance between

[33] The speed of adjustment coefficients in the Fama and French (2002) and Kayhan and Titman (2007) are likely to be somewhat downward biased because the target debt ratio is measured with error. Flannery and Rangan (2006) estimates a much faster speed of adjustment coefficient, using a model that has firm fixed effects. However, their speed of adjustment coefficient may be biased upwards, because their estimation technique effectively uses future data to estimate the firm's target debt ratio (i.e., firms that are more levered in the future will have higher current target debt ratios).

the target and actual debt ratios widen, which dampens the estimated speed at which firms move towards their true targets. However, since the firm's average debt ratio is constant, this effect does not arise when we use the average debt ratio as a proxy for the target. As a result, for some parameters, debt ratios are estimated to move faster towards average debt ratios than target debt ratios. For example, when the financial distress trigger is low, the speed of adjustment towards the average debt ratio is faster than the speed of adjustment towards the target. However, firms that recapitalize more frequently, that is, firms with a greater financial distress trigger s, move to their true target more often, which explains why these firms tend to exhibit faster reversion to their targets than to their average leverages.

In Panel 3 and 4, Table VII, we present results of separate regressions run for firms with different maturities for two cases of the distress trigger ($s = 1$ and $s = 4$). For the value-maximizing firm, changes in maturity have no effect on adjustment speed because its debt is risk-free. The equity-maximizing firm moves to its target faster because its cost of adjusting its debt level is lower when its debt maturity is shorter. Specifically, when its debt maturity is shorter, the value transfer to the debtholders is lower when the firm repurchases its debt, which we require in our specification if the firm is either increasing or decreasing its debt ratio. This increase in the adjustment speed as maturity declines is more pronounced when the firm has a high distress trigger.

The Effect of Changes in Earnings

In addition to examining the tendency of firms to move towards their target ratios, we use our simulations to estimate the effect of economic shocks that can move firms away from their target debt ratios. We do this by adding variables that proxy for these shocks to our simulated regression model. We start with variables that are considered in Fama and French (2002) and estimate the following regression:

$$\left(\frac{D}{V}\right)_{t+1} - \left(\frac{D}{V}\right)_t = c + \lambda \cdot \left(TL_t - \left(\frac{D}{V}\right)_t\right) + b \cdot \frac{EBIT_{t+1} - EBIT_t}{A_{t+1}} + d \cdot \frac{A_{t+1} - A_t}{A_{t+1}} + \varepsilon_t,$$

where $EBIT_t$ is earnings before interest, taxes, and depreciation and A_t is the book value of assets.[34] In Fama and French (2002), the target debt ratio TL_{t+1}

[34] Fama and French (2002) examine earnings before interest but after taxes as well as lagged changes in earnings and assets.

is the fitted values from the regression on various factors including R&D expenses, depreciation expenses, etc., whereas in our simulated regression, we use the actual target.

As reported in Panel 1, Table VIII, the estimate of the adjustment rate λ for the model-generated data varies between 7.8% and 21% for parameters $s = 1$ and $s = 4$, which are similar to the coefficients estimated in the simple regressions reported in the previous table. The coefficients of the other variables, which measure the changes in earnings and changes in assets, have the same signs and comparable economic significance to the regression coefficients reported in Fama and French (2002). For example, the coefficient for $\frac{EBIT_{t+1}-EBIT_t}{A_{t+1}}$ is negative for all cases, which reflects the fact that firms do not always adjust their capital structures when an increase (decrease) in earnings causes firm values to increase (decrease) and market debt ratios to decrease (increase). The sensitivity of capital structure changes to earnings changes is less for value-maximizing firms as well as firms with more sensitive financial distress triggers, reflecting the fact that these firms recapitalize more often.[35]

Results for Periods When Firms Actively Recapitalize

The evidence of a negative relation between changes in earnings and the debt ratio appears to be inconsistent with our comparative statics, which indicates that the optimal initial debt ratio is positively related to the product price. Indeed, when firms in our model recapitalize after an increase (or decrease) in price, they do in fact move to a higher (lower) debt ratio. However, when firms do not recapitalize, an increase (decrease) in price leads to an increase (decrease) in firm value, which in turn leads to a decrease (increase) in the debt ratio.

To look at the relation between capital structure changes and earnings more closely, we select only periods when the firm changes its outstanding debt, by either repurchasing or issuing debt. The regression results for this subsample are reported in Panel 2, Table VIII. A comparison of coefficients from this regression with the previous regression on data from all periods (see Panel 1, Table VIII), reveals that changes in earnings has either a much weaker effect

[35] In unreported regressions we also incorporated other variables that have been used to proxy for the forces that move firms away from their target debt ratios including the financial deficit FD, [see Shyam-Sunder and Myers (1999), Frank and Goyal (2003), and Kayhan and Titman (2007)] the market-to-book ratio MB, and the profitability measured as $EBIT$ divided by assets, $\frac{EBIT}{A}$. These results are available upon request.

Table VIII. This table reports regressions results for the model-generated data. The regressions are similar to regressions in Fama and French (2002).
$(D/V)_{t+1} - (D/V)_t = C + \lambda^*(TL_t - (D/V)_t) + b^*(EBIT_{t+1} - EBIT_t)/A_{t+1} + d^*(A_{t+1} - A_t)/A_{t+1} + e_t$,
where TL_t is the target leverage, $(D/V)_t$ is the leverage ratio, At is book value of assets and EBIT are earnings before interest and taxes at time t. Regressions are run on the model generated annual data and are reported for the equity- and the value maximizing firm for 4 levels of distress trigger s: s = 1 (base case), s = 2, s = 3 and s = 4. All variables are statistically significant except ones reported with upper-scripts^{NN}
Panel 1: Results reported for all periods

Parameter s	s = 1		s = 2		s = 3		s = 4	
Firm type	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
C	0.016	0.005	0.011	0.003	0.009	0.000	0.001	− 0.001
$TL_t - (D/V)_t$	0.078	0.091	0.080	0.12	0.14	0.213	0.150	0.199
$(EBIT_{t+1} - EBIT_t)/A_{t+1}$	− 1.027	− 0.843	− 0.977	− 0.598	− 0.634	− 0.395	− 0.375	− 0.270
$(A_{t+1} - A_t)/A_{t+1}$	− 0.357	− 0.124	− 0.319	− 0.053	− 0.276	− 0.038	− 0.064	− 0.039
R-squared	0.46	0.298	0.36	0.272	0.19	0.282	0.114	0.251

Panel 2: Results reported for the subsample of periods when firms undertake active recapitalization

Parameter s	s = 1		s = 2		s = 3		s = 4	
Firm type	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
C	0.240	− 0.040	0.068	0.011^{NN}	− 0.014	0.016	− 0.034	0.003^{NN}
$TL_t - (D/V)_t$	0.228	0.677	0.409	0.728	0.895	0.738	0.914	0.704
$(EBIT_{t+1} - EBIT_t)/A_{t+1}$	-0.148^{NN}	0.251	-0.119^{NN}	0.557	0.415	0.173	-0.021^{NN}	0.276
$(A_{t+1} - A_t)/A_{t+1}$	-0.022^{NN}	− 1.069	-0.046^{NN}	− 0.242	-0.047^{NN}	-0.047^{NN}	− 0.275	− 0.077
R-squared	0.53	0.864	0.18	0.871	0.92	0.905	0.591	0.837

on the debt ratio, or the effect reverses. The evidence of a reversal in signs is especially apparent for the value-maximizing firm.[36]

Stock Return Effect

Following Welch (2004) we examine the influence of equity returns on debt ratio changes. In order to examine how stock returns influence the debt ratio in our model, we construct a variable r_t, which is the 1-year stock return,[37] and estimate the following regression with our simulated data:

$$\left(\frac{D}{V}\right)_{t+1} - \left(\frac{D}{V}\right)_t = c + \lambda \cdot \left(TL_t - \left(\frac{D}{V}\right)_t\right) + b \cdot r_t + \varepsilon_t,$$

These regressions, reported in Table IX, illustrates the strong relation between equity returns and changes in leverage in the data generated by our model. The impact of stock returns is somewhat smaller in absolute value for firms with higher financial distress triggers and is smaller for value-maximizing firms, which again reflects the fact that value-maximizing firms and firms with a potentially greater likelihood of distress recapitalize more frequently.[38]

Empirical Implications

The results of our simulated regressions suggest that future empirical research on the determinants of capital structure changes should estimate regressions on different subsamples representing different categories of firms. As we show, the speed at which a firm's debt ratio will revert to its target, as well as the extent to which it will move away from its target, depends on the firm's susceptibility to finance distress, as well as whether it acts to maximize equity or total value.

[36] This observation is consistent with Hovakimian et al. (2004) who argue that firms that issue a significant amount of both debt and equity in a given year are likely to be near their target debt ratio, and find that for these firms there is no significant relation between profitability and leverage. In addition, Hovakimian et al. (2001) find that in a sample of firms that raise significant amounts of new capital, those that generated high profits in the past tend to issue debt and those that generated low profits tend to issue equity. This evidence suggests that changes in leverage generated by either high or low profits tend to be at least partially reversed when firms raise significant amounts of capital.

[37] Welch (2004) actually constructs a variable that measures how much the firm's debt ratio would change, as a result of the prior years stock return, assuming that the firm does not recapitalize. Empirically, a simple stock return variable, which is highly correlated with the Welch variable, seems to explain changes in capital structure about as well as the Welch variable.

[38] We also analyze the longer-term adjustment speed by running a standard 5-year adjustment regression with and without control variables. Unreported regression demonstrates that the 5-year adjustment rate is faster than 1-year adjustment rate and that the economic impact of the 5-year stock return on leverage changes is lower than the impact of 1-year return.

Table IX. This table reports regression results for the partial adjustment model for the model-generated data. The partial adjustment model includes stock returns as an independent variable.

$$(D/V)_{t+1} - (D/V)_t = C + \lambda * (TL_t - (D/V)_t) + b * r_t + e_t,$$

where TL_t is the target leverage, $(D/V)_t$ is the leverage ratio, and r_t is a stock return at time t. Regressions are run on the model generated annual data and are reported for the equity- and the value-maximizing firm for 4 levels of distress trigger s: s = 1 (base case), s = 2, s = 3 and s = 4. All variables are statistically significant except ones reported with upper-scripts^{NN}

Parameter s	s = 1		s = 2		s = 3		s = 4	
Firm type	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing	Equity-maximizing	Value-maximizing
C	0.007	0.001	0.004	0.000^{NN}	0.003	− 0.001	0.000^{NN}	− 0.001
$TL_t - (D/V)_t$	0.048	0.053	0.058	0.112	0.104	0.202	0.145	0.190
r_t	− 0.177	− 0.023	− 0.173	− 0.036	− 0.147	− 0.026	− 0.084	− 0.018
R-squared	0.80	0.199	0.86	0.213	0.60	0.214	0.301	0.182

Although the relation between management objectives, financial distress costs, and capital structure changes has not been directly analyzed in the literature, there are studies that examine related issues. For example, Welch (2004) and Flannery and Rangan (2006) suggest that adjustment speeds are faster for smaller firms and Fama and French (2002) find that earnings has more of an influence on the capital structures of large firms. These observations are inconsistent with models based solely on adjustment costs, since adjustment costs are likely to be proportionally greater for smaller firms. However, smaller firms are also likely to be more sensitive to financial distress costs, and may also be less subject to debtholder/equityholder conflicts, since they are likely to obtain more of their debt financing from banks, rather than by issuing bonds.[39] Hence, these findings are consistent with our simulated regressions that indicate that adjustment speeds are faster for firms that are more sensitive to financial distress and less exposed to debtholder/equityholder conflicts.

5. Conclusion and Extensions of the Model

There has been a recent effort to quantify models of optimal capital structure by using the methodology that was originally developed to price derivative securities. The dynamic capital structure model developed in this paper extends this literature by incorporating continuous investment and financing choices as well as bankruptcy costs, financial distress costs, and transaction costs.

As our model illustrates, the evidence presented in the empirical capital structure literature that suggests that firms move relatively slowly towards their target debt ratios is in fact consistent with theory. The intuition for why firms with conflicts between the interests of debtholders and equityholders choose not to decrease leverage when they are overlevered is well understood. However, as our model illustrates, this conflict of interest is less pronounced for firms that are more subject to financial distress costs, since such firms have a greater incentive to issue equity and pay down debt when they are doing poorly. As a result, our model suggests that firms that are subject to financial distress costs as well as those without conflicts of interest between debtholders and equityholders should adjust more quickly towards their target debt ratios.

[39] Lemmon and Zender (2004) find that smaller firms tend to be financed by banks rather than bondholders. Since banks exert more control over the firm's investment and financing decisions than outside bondholders, smaller firms are likely to make investment and financing choices that are more in line with total value-maximization.

Before concluding, it should be noted that one of the objectives of this research is to develop a model that could potentially provide useful guidance about quantitative issues, for example, how much debt should a firm issue, rather than just the qualitative issues that are addressed in the academic literature. While our model provides some improvement over the related quantitative dynamic models, there are a number of additional improvements that can be considered in future work that would make the model more applicable. For example, if the dimensionality of the problem can be increased, we can consider multiple types of debt, with either different maturities or different seniority. In addition to allowing us to capture the more complicated capital structures that we observe in reality, such a model would better capture the short-term dynamics of the capital structure choice (e.g., in the short-run financing needs that tend to be satisfied with revolving credit agreements with banks). An increase in the dimensionality of the model will also allow us to relax the assumption that product prices follow a random walk. This will allow us to distinguish between financial distress caused by temporary liquidity shocks, and financial distress caused by decreases in the firm's long-term ability to generate cash flows. Moreover, if we relax the assumption that product prices follow a random walk, we can consider situations where investment opportunities improve, holding current profits constant. If we combine this assumption with lags between investment expenditures and increased production, we can capture the fact that firms tend to issue equity when their stock prices increase more than their cash flows. Although some of these issues are quite challenging to tackle without simplifying some other features of the model, we believe they can be addressed in future research.

Appendix A: Valuation

This appendix presents a detailed formulation for the valuation problem of the equity and of the debt for both the value-maximizing firms as well as the value of the all-equity firm. The numerical algorithm that we use for solving stochastic control problems is described in Appendix B.

A.1 VALUATION OF THE ALL-EQUITY FIRM

In this section we consider the valuation of an unlevered firm $E^U(p, A)$. At each state (p, A), the all-equity firm selects its investment strategy $i(p, A)$ that maximizes its value E^U. By Ito's lemma the value of the all-equity firm E^U

satisfies the following optimal stochastic control problem:

$$\max_{i\geq 0}\Big[\frac{1}{2}\sigma_p^2 p^2 E_{pp}^U + (r-\alpha)pE_p^U + (-\gamma A + i)E_A^U - rE^U$$
$$+p\cdot c(A) - b - i - \tau\times \max[0,\ p\cdot c(A) - b - \gamma A] - C_{Equity}$$
$$\times \max[0,\ -p\cdot c(A) + b + i]\Big] = 0, \tag{A1}$$

where the last two terms describe the firm's cash flow. There is an additional boundary condition $E^U > 0$ that corresponds to the case in which the firm uses its option to permanently shut down its operations, if the spot price drops far below its production costs b. Similarly, since we deal with the case of the infinite horizon, the value of the unlevered firm is independent of time $E_t^U = 0$.

A.2 VALUATION OF THE FIRM THAT FOLLOWS THE VALUE-MAXIMIZING STRATEGY

In this section we consider the valuation problem of the firm that maximizes the combined value of its debt and equity. At each state, the firm continuously selects its investment strategy i together with debt restructuring strategy, to maximize its total value $V(p, A, d)$. The firm's value is the solution to the following stochastic control problem:

$$\max_{i\geq 0}\Big[\frac{1}{2}\sigma_p^2 p^2 V_{pp} + (r-\alpha)pV_p + (-\gamma A + i)V_A - rV$$
$$+p\cdot c(A) - b - i - \tau\times \max[0,\ p\cdot c(A) - b - \gamma A - d]$$
$$-C_{Equity}\times \max[0,\ -p\cdot c(A) + b + d + i + wF - w\cdot D^V(p, A, d)]$$
$$-C_{Distress}\times \max[0,\ s\cdot d - p\cdot c(A) + b]\Big] = 0, \tag{A2}$$

where

$$V(p, A, d) = D^V(p, A, d) + E^V(p, A, d),$$
$$E^V(p, A, d) \geq 0,\ D^V(p, A, d) > 0,$$

and where $E^V(p, A, d)$ and $D^V(p, A, d)$ are the equity and the debt value of the value-maximizing firm. Given the choice of investment i, the value of equity E^V and debt D^V satisfy the Equations in (11) and (14), respectively. The second inequality in the problem (A2) ensures that the value of the firm's equity is nonnegative.

Free boundary conditions of this problem are similar to those described in the equity-maximization problem (11).[40] Similarly, in the debt restructuring region, the firm increases or decreases its debt by repurchasing existing debt and issuing a new debt. In this region the boundary conditions are similar to the boundary conditions for the equity-maximizing firm. In the default region, the boundary condition is the same as in the problem (11).

Appendix B: Numerical Algorithm

In this appendix we describe the numerical algorithm that we apply to solve stochastic control problems (11), (A1), and (14). For each case we need to find a solution that satisfies simultaneously the maximization problems and partial differential equations. The algorithm is based on the finite-difference method augmented by a "policy iteration".[41]

The calculations are complicated by the fact these are infinite horizon stochastic optimization problems, where the values of the equity and debt are time independent. Therefore, numerical solutions require reformulating the model into the finite-horizon approximation.[42] We initialize the procedure by approximating (guessing) values for the functions in each node of the terminal time. This reformulation effectively implies that a derivative with respect to time is added to equations of each optimal stochastic control problem. For example, in the valuation problem for the all-equity firm, a new term E_t^U is added to the left-hand side in Equation (A1). The errors that result from the approximation of functions at the terminal time can be reduced by increasing the length of the horizon of the problem and iterating until the derivative E_t^U is indistinguishable from zero for each node on the grid.

For each problem we use a discrete grid and a discrete time step Δt. The state space (p, A, d) is discretized using a four-dimensional grid $N_p \times N_A \times N_d$ with corresponding spacing between nodes in each dimension of $\Delta p, \Delta A$ and Δd and where $\Delta X = \frac{X_{\max} - X_{\min}}{N_x}$ and $X \in \{p, A, d\}$; $X_{\max}$ and $X_{\min}$ are the upper and low boundaries.[43] In each node on the grid (p, A, d) the partial derivatives are computed according to Euler method.

[40] In general, the configuration of regions in the state space (p, A, d) of the equity-maximization problem (11) is different from that of the value maximization problem (A2).

[41] See, for example, Kushner and Dupuis (1992), Barraquand and Martineau (1995), and Langetieg (1986) for the theory and applications of numerical methods to stochastic control problems.

[42] Flam and Wets (1987) and Mercenier and Michel (1994) also discuss the approximation of infinite horizon problems in the deterministic dynamic programming models.

[43] The grid step in each dimension is chosen to achieve the stability of the algorithm.

For example, the first and the second derivatives of the equity value with respect to p are $E_p(p, A, d) = \frac{E(p+\Delta p, A, d) - E(p-\Delta p, A, d)}{2\Delta p}$, $E_{pp}(p, A, d) = \frac{E(p+\Delta p, A, d) - 2E(p, A, d) + E(p-\Delta p, A, d)}{\Delta p \Delta p}$, with appropriate modifications at the grid boundaries.

B.1 CALCULATION OF THE VALUE OF ALL-EQUITY FIRM

In this section we describe the computation of the value of the all-equity firm formulated in (A1). The values of the all-equity firm at each node of the terminal time t are assigned the values of the expected cash flows assuming that the value of fixed (tangible) assets is kept constant at a given level and does not depreciate, that is $E^U_{(t)}(p, A, d) = \mathbb{E}_Q \int_0^\infty [p \cdot c(A) - b - \tau \times \cdot \max(0, p \cdot c(A) - \gamma A)) e^{-rt}] dt$, where $\mathbb{E}_Q$ is the expectation under the risk-neutral measure Q, and the subscript (t) denotes the time of the node.[44] This approximation tends to overvalue the firms with high fixed (tangible) assets (and correspondingly high capacity) and to undervalue the firms with low assets because the approximation ignores investments and depreciation. However, by running backward recursion long enough the values on the grid converge to the steady-state values since the initial mispecifications of the terminal values are "smoothed" away owing to discounting. Thus, working backward in time for each node on the grid according to the explicit finite-difference scheme and taking into account the investment decision, the value of the all-equity firm $E^U_{(t-\Delta t)}$ at each node (p, A, d) at time $t - \Delta t$ is determined as follows:

$$\begin{aligned}
E^U_{(t-\Delta t)}(p, A, d) &= \max_{i \geq 0} \Big[[p \cdot c(A) - i - b]\Delta t - \tau \\
&\quad \times \max[0, p \cdot c(A) - b - \gamma A] - C_{Equity} \\
&\quad \times \max[0, -p \cdot c(A) + b + i] + e^{-r\Delta t} \mathbb{E}_Q [E^U_{(t)}] \Big] \\
&= \max_{i \geq 0} \Big[[p \cdot c(A) - i - b]\Delta t - \tau \times \max[0,\ p \cdot c(A) - b - \gamma A] \\
&\quad - C_{Equity} \times \max[0,\ -p \cdot c(A) + b + i] + E^U_{(t)}(p, A, d) \\
&\quad + \Delta t \mathcal{L}[E^U_{(t)}(p, A - \gamma A \Delta t + i \Delta t, d)] \Big],
\end{aligned} \tag{B1}$$

where $\mathcal{L}[E^U_{(t)}(p, A - \gamma A \Delta t + i \Delta t, d)]$ is the differential operator applied to $E^U_{(t)}$ in the node $(p, A - \gamma A \Delta t + i \Delta t, d)$

[44] Notice that values $E^U(p, A, d)$ are the same in dimension d since the all equity firm has no debt.

$$\mathcal{L}[Z] = \frac{1}{2}\sigma_p^2 p^2 Z_{pp} + (r - \alpha)Z_p + (-\gamma A + i)Z_A - rZ,$$

in which all partial derivatives are computed according to Euler method, where $A - \gamma A\Delta t + i\Delta t$ is the value of the firm's fixed (tangible) assets at time t given their value A at time $t - \Delta t$; and the second term reflects the changes in assets values due to depreciation and investments. The maximization over all possible investment choices $i \geq 0$ determines the optimal investment strategy i at time $t - dt$.[45] The second equality in (B1) comes from Euler decomposition of the equation in (A1) in which a new term E_t^U is added, where $E_t^U = \frac{E_{(t)}^U - E_{(t-\Delta t)}^U}{\Delta t}$.

We repeat this backward induction procedure for $t - 2\Delta t$, $t - 3\Delta t, .., t - N\Delta t$ until the value function $E_{(t)}^U$ reaches a steady state in each node on the grid, that is, until $\max\limits_{(p,A,d)} |E_{(t)}^U(p, A, d) - E_{(t-\Delta t)}^U(p, A, d)| < \varepsilon$, where ε is the predetermined accuracy level.[46] We have found this procedure to be robust to the choice of the values at the terminal time.[47]

B.2 CALCULATION OF THE EQUITY AND DEBT VALUES

The computation of the value of equity and debt that solves stochastic control problem of the equity-maximizing firm in (11) and (14) has to be done simultaneously. This is because the value of equity depends on the value of debt, while, at the same time, the value of debt depends upon the decisions of the equityholders. To calculate those values we extend the procedure described in the previous section to incorporate the recapitalization decisions.

As described in the previous section, we first approximate the values of the equity and debt for the "terminal" time t. In each node (p, A, d) we set the terminal values $E_{(t)}(p, A, d)=\max(0,\ E_{(t)}^U(p, A, d) - F)$ and $D_{(t)}(p, A, d)=(1 - C_{default})E_{(t)}^U(p, A, d)$ if $E_{(t)}(p, A, d) = 0$, and $D_{(t)}(p, A, d) = F$, otherwise where $F = \frac{d}{r}$ is the face value of the debt with coupon d. This approximation undervalues both the equity and debt since the tax shield of debt is not incorporated. For the calculation of the equity values,

[45] Given investment rate i, the value of fixed assets $A - \gamma A\Delta t + i\Delta t$ does not fall exactly on a node, therefore we use an interpolation.

[46] Given initial guesses for the values on the "terminal grid" , the procedure for the valuation of the unlevered firm is stable and converges in about 6,000 time steps where each time step $\Delta t = 0.1$ year.

[47] As a test, we checked that for different "reasonable" guesses of the values at the terminal time this procedure converges to the same values, although, the number of iterations may be different.

we separate the decision on the investment rate $i(p, A, d)$ and the decision on whether or not to increase/decrease instantly the debt ratio of the firm.

The value of equity E and debt D of the equity-maximizing firm in each node on the grid (p, A, d) at time $t - \Delta t$ are determined by working backward in time:

$$
\begin{aligned}
E_{(t-\Delta t)}(p, A, d) &= \max_{i\geq 0}\Big[CFE_{(t)}(i)\Delta t + e^{-r\Delta t}\mathbb{E}_Q[E_{(t)}]\Big] \\
&= \max_{i\geq 0}\Big[CFE_{(t)}(i)\Delta t + E_{(t)}(p, A, d) \\
&\quad +\Delta t\widehat{\mathcal{L}}[E_{(t)}(p, A - \gamma A\Delta t + i\Delta t, d)]\Big].
\end{aligned}
\tag{B2}
$$

The value of debt is dependent upon the equityholders decisions

$$
\begin{aligned}
D_{(t-\Delta t)}(p, A, d) &= CFD_{(t)}\Delta t + e^{-r\Delta t}\mathbb{E}_Q[D_{(t)}] \\
&= CFD_{(t)}\Delta t + D_{(t)}(p, A, d) + \Delta t\widehat{\mathcal{L}}[D_{(t)}(p, A - \gamma A\Delta t + i^*\Delta t, d)],
\end{aligned}
\tag{B3}
$$

where

$$
\widehat{\mathcal{L}}[Z] = \frac{1}{2}\sigma_p^2 p^2 Z_{pp} + (r - \alpha)Z_p + (-\gamma A + i^*)Z_A - rZ,
$$

where i^* is the investment strategy that solves (B2) at time $t - \Delta t$. $CFE(i)$ and CFD are cash flows to equityholders and debtholders, respectively.

$$
\begin{aligned}
CFE_{(t)}(i) = {} & p \cdot c(A) - i - d - wF + w \cdot D_{(t)}(p, A, d) - \tau \\
& \times\max[0,\ p \cdot c(A) - d - \gamma A] - C_{Equity} \\
& \times\max[0,\ -p \cdot c(A) + d + i + wF \\
& -w \cdot D_{(t)}(p, A, d)] - C_{Distress} \times max[0,\ s \cdot b - p \cdot c(A) + d] \\
CFD_{(t)} = {} & d + wF - w \cdot D_{(t)}(p, A, d).
\end{aligned}
$$

For each time step we also check whether or not it is optimal for the equityholders to increase/decrease the firm's debt level instantaneously. The firm increases its debt from F to $\widehat{F}$ $(F = \frac{\widehat{d}}{r})$ if the following condition is satisfied

$$
\begin{aligned}
E_{(t)}(p, A, d) < {} & \max_{\widehat{d}>d}[E_{(t)}(p, A, \widehat{d}) + D_{(t)}(p, A, \widehat{d}) \\
& -F - C_{Debt}\widehat{F}],\ E_{(t)}(p, A, \widehat{d}) > 0.
\end{aligned}
\tag{B4}
$$

The firm decreases its debt if

$$E_{(t)}(p, A, d) < \max_{\widehat{d}<d}[E_{(t)}(p, A, \widehat{d}) + D_{(t)}(p, A, \widehat{d})$$
$$-F - C_{Debt}\widehat{F} - C_{Equity} \times [D(p, A, \widehat{d}) - F], E_{(t)}(p, A, d) > 0. \quad \text{(B5)}$$

If inequality (B4) or (B5) is satisfied, then it is optimal to recapitalize, and the value of the equity is set equal to the maximum over all $\widehat{d}$ in the right-hand side of (B4) or (B5). In that case the value of the debt is set to

$$D_{(t)}(p, A, d) = F = \frac{d}{r}.$$

Also at each node we check whether or not the equity value $E_{(t)}(p, A, d)$ is nonnegative. If the equity value becomes negative, the default occurs and we set the value of the debt and equity to

$$D_{(t)}(p, A, d) = (1 - C_{default}) \cdot E^U(p, A) \quad \text{and} \quad E_{(t)}(p, A, d) = 0.$$

Similar to the computation of the all-equity firm, we repeat this iteration until values of the equity and debt reach the steady states in each node, that is until $\max_{(p,A,d)}\{|E_{(t)}(p, A, d) - E_{(t-\Delta t)}(p, A, d)| + |D_{(t)}(p, A, d) - D_{(t-\Delta t)}(p, A, d)|\} < \varepsilon$.

The numerical procedures for the computation of the value-maximizing firm are similar.

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