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Bankruptcy, Counterparty Risk, and Contagion*

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Abstract. This paper provides a unifying framework for the modeling of various types of credit risks such as contagion effects. We argue that Markov chains can efficiently be used to tackle these problems. However, our approach is not limited to pricing problems with contagion. On the theoretical side, we derive pricing formulas for three building blocks that are generalizations of contingent claims studied in Lando (1998). These claims can be thought of as atoms forming the basis for all credit risk payments. Furthermore, we demonstrate that, in general, all contingent claims exposed to credit risk satisfy a system of partial differential equations. This is the key result to calculate prices of credit risk claims explicitly and efficiently.

JEL Classification: G13, G33

1. Introduction

Pricing defaultable bonds and credit derivatives has been one of the important topics in finance over the past decade. This development is fueled by a rapid growth in the demand for credit derivatives leading to a \$12.4 trillion market by the end of 2005 (as reported by the Bank for International Settlements (BIS)). Credit default swaps (CDS) and more recently collateralized debt obligations (CDO) have become indispensable tools for the securitization of credit risks.[1] Furthermore, current developments in the automobile industry document the significance of so-called contagion effects: On October 8, 2005 the auto parts maker Delphi Corp. filed for Chapter 11 bankruptcy protection. On December 12, 2005 the rating agency S&P cut GM's corporate credit rating to B, five steps below investment grade. On March 3, 2006 the auto parts maker Dana

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[1] See, e.g., Longstaff (2006).

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Corp. filed for bankruptcy protection defaulting on \$2.5 billion of debt. At the same time, it is argued that the market for credit derivatives is dominated by too few banks, making it vulnerable to a crisis if one of them fails to pay on contracts that insure creditors from companies defaulting. This kind of risk is usually referred to as counterparty risk. The goal of this paper is to offer a framework where all related pricing problems can be addressed in a unified way.

There are a few related papers dealing with the aforementioned issues. Jarrow and Turnbull (1995), Duffie and Singleton (1999a), and Lando (1994, 1998) demonstrate how default risk of a single entity can be handled. Lando (1998) also addresses the important question of default correlation: He puts forth a so-called Cox process framework where default correlation between entities occurs, since all default intensities of the entities depend on the same macroeconomic variables modeled as diffusions. There is an ongoing debate in the literature whether this kind of framework can produce enough default correlation.[2] One of the reasons is that conditioned on the history of macroeconomic variables, defaults occur independently. Hence, there are no feedback effects from defaults of single firms on other firms. To include this kind of contagion effect, there are two suggestions in the literature: The first idea is to make the firm's default intensities dependent on past defaults of other firms in the economy. This idea goes back to Davis and Lo (1999) and Jarrow and Yu (2001). Therefore, conditioned on the history of the macroeconomic variables defaults are no longer independent. The second idea makes use of copulas. Li (2000), Schönbucher and Schubert (2001), as well as Rogge and Schönbucher (2003), among others, assume that conditioned on the macroeconomic variables, defaults are distributed according to some copula function. The first approach resembles the empirically observable fact that upon default of one entity, spreads jump upwards, whereas in the second approach one of the key issues is to find a copula leading to the desired default correlation. In an insightful paper, Yu (2005a) demonstrates that models of the Jarrow-Yu type induce a certain copula and vice versa. Besides, a recursive procedure called "total hazard construction" is presented that overcomes the deficiencies of the primary-secondary approach by Jarrow and Yu (2001) and generalizes the construction of default times by Lando (1998). Nevertheless, in his examples he mostly needs to resort to Monte Carlo methods.

Building on these ideas by Jarrow and Yu, this paper provides a different view of the problems addressed in the aforementioned papers. Instead of starting by constructing default times, we demonstrate that a few relevant problems can be modeled as Markov chains. This includes models for the

[2] See, e.g., Yu (2005b) for a discussion of this problem.

pricing of basket credit default swaps. Starting with the papers by Jarrow et al. (1997) and Lando (1998), so far Markov chains are almost exclusively used to model rating systems.[3] The advantage of our approach is that it gives us access to many useful results on Markov chains simplifying the task of pricing credit risky assets significantly. By relating pricing problems with default risk to a system of partial differential equations, which without default risk collapses into a Black-Scholes-type equation, we demonstrate how these pricing problems are related to ordinary pricing problems without default risk. This is one of our main contributions. At the same time this paper provides a unified framework nesting various credit risk models. One of the main advantages of our approach is that closed-form solutions for several contingent claims exposed to various types of credit risk can easily be obtained. In particular, we also derive pricing formulas for three building blocks that are generalizations of contingent claims studied in Lando (1998).

The remainder of the paper is structured as follows: In Section 2, we briefly describe why Markov chains are useful to model credit risk. In Section 3, our framework is introduced and some pricing relations are established. Section 4 addresses various pricing problems of defaultable bonds. In Section 5, additional applications are discussed. Section 6 concludes. All proofs can be found in the Appendix.

2. Why Markov Chains?

This section is intended to shed some light on the question why Markov chains can be an important tool to model credit risk. We will briefly touch on several possible applications each of which will be analyzed in more detail in one of the following sections.

Starting with the work by Lando (1994, 1998) and Jarrow et al. (1997), Markov chains have been almost exclusively used to model rating classes. The goal of this section is to point out that various kinds of credit risk models can be embedded into a Markov chain framework. The reason is as follows: A Markov chain is a stochastic process jumping back and forth between a

[3] An exception is a recent paper by Di Graziano and Rogers (2005). As in this paper, the authors use a Markov chain to model credit risk. However, they exogenously impose a specific structure on the default probabilities. As a result, their approach does not lead to explicit solutions. In contrast, our approach produces default probabilities endogenously. As we will see later on, the default probabilities of our framework satisfy Kolmogorov forward and backward equations. The resulting probabilities are qualitatively different from the ones in Di Graziano and Rogers (2005). Furthermore, there are three recent papers using Markov chains to study the pricing of credit derivatives, namely Frey and Backhaus (2006), Sidenius et al. (2005), and Schönbucher (2005).

finite number of states. In general, the reader should think of each of these states as a sufficient description of the market for defaultable bonds (and credit derivatives). A jump from one state into another state stands for a credit event (e.g., a firm's default on its obligations) resulting in a new situation for the market of defaultable bonds. The Markov chain corresponding to the classical models for a single defaultable bond by Jarrow and Turnbull (1995) or Duffie and Singleton (1999a) is very simple: Since the issuing firm is initially not in default, but can default upon its debt during the lifetime of the bond, the chain consists of two states (see Figure 2),[4] where state 1 is an absorbing state meaning that once the bond is in default it remains in default forever. This model can be generalized in various directions. One strand of literature adds additional states and interprets them as different rating classes. As mentioned above, this is done by Lando (1994, 1998) as well as Jarrow et al. (1997). But this is not the only possible generalization. In an insightful recent paper by Guo et al. (2005b), it is argued that classical models oversimplify the situation of a financially distressed firm which usually leads to wrongly specified recovery rates. This is because these models end with the default of the respective firm. However, the firm usually continues to operate and there are markets for defaulted debt. For this reason, Guo et al., distinguish between default and bankruptcy. In their framework, default occurs if a state process modeling the firm's financial situation jumps to zero. Given a default has already occurred, the firm enters bankruptcy if its firm value falls below a predefined level. Therefore, their model can be represented by a Markov chain with the three states "no default", "default", and "bankruptcy".

As emphasized in the introduction, on a portfolio level contagion effects play a crucial role. The simplest extension of the classical framework is a model with two defaultable bonds where the default intensity of one firm changes if a second firm defaults on its debt. This can be described by a Markov chain with four states detailed in Figure 3. The chain starts in state 0. Subsequently, either firm A or B can default. If B defaults first (the Markov chain jumps from state 0 into state 2), the default intensity of A jumps from λ^A to $\lambda^{A'}$, whereas if A defaults first (the Markov chain jumps from state 0 into state 1), the default intensity of B jumps from λ^B to $\lambda^{B'}$. If the remaining firm defaults as well, the chain finally jumps into state 3 which is an absorbing state. Apparently, contagion effects in larger baskets of bonds can become even more significant. This has an impact on the pricing of n-to-default baskets. A possible model for this problem is detailed in Figure 7. Finally, as we will see later on, counterparty risks can also be modeled using Markov chains.

[4] We have decided to keep the figures in the corresponding sections.

The following section provides the necessary theoretical results to make our approach work.

3. Framework

The uncertainty in the economy is described by the filtered probability space $(\Omega, Q, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]})$, where $\mathcal{F} = \mathcal{F}_{T^*}$ and Q is an equivalent martingale measure under which discounted asset prices are (local) martingales. We assume the existence of Q, so that assets are priced by arbitrage.[5] Subsequent specifications of the model are all under the equivalent martingale measure Q. The economy is driven by an L-dimensional process $Y = (Y_1, \ldots, Y_L)$ consisting of macroeconomic state variables like interest rates. The ι-th entry of Y follows the diffusion process:

$$dY_\iota(t) = \alpha_\iota(t)dt + \overline{\beta}_\iota(t)d\overline{W}_\iota(t), \tag{1}$$

where $\overline{W}$ is an L-dimensional correlated Brownian motion with $< \overline{W}_i, \overline{W}_j >_t = \rho_{i,j}t$ and α_ι as well as $\overline{\beta}_\iota$ are real-valued functions of t and $Y(t)$. Here and in the following we assume that the coefficients of all stochastic differential equations (SDEs) are predictable processes which are sufficiently integrable such that the SDEs possess unique solutions.[6] Alternatively, the diffusions can be modeled using an L-dimensional standard Brownian motion W with uncorrelated coordinates:

$$dY_\iota(t) = \alpha_\iota(t)dt + \beta_\iota(t)dW(t), \tag{2}$$

where β is an $L \times L$-matrix and β_ι is its ι-th row. In the following, we will use this formulation. In this economy, the short rate is a function of t and $Y(t)$, i.e., $r(t) = r(t, Y(t))$. Of course, the short rate itself can be one of the components of Y. Investors can borrow and lend using a money market account with dynamics

$$dM(t) = M(t)r(t)dt, \quad M(0) = 1, \tag{3}$$

and there exists a market for zero-coupon bonds with risk-neutral dynamics

$$dP(t,T) = P(t,T)[r(t)dt + \sigma(t)dW(t)], \quad P(T,T) = 1. \tag{4}$$

We assume that the price of a zero-coupon bond is a smooth function of time t and macroeconomic state variables $Y(t)$. Therefore, by Ito's lemma, the

[5] See Harrison and Kreps (1979) and Delbaen and Schachermayer (1999) for the essential equivalence of the existence of such a measure and the absence of arbitrage.

[6] See, e.g., Protter (2004, pp. 249ff).

L-dimensional vector σ is a function of t and $Y(t)$. Furthermore, there exists a market for corporate zero-coupon bonds (defaultable bonds). This market is characterized by a finite set of states,[7] $\mathcal{J} = \{0, \ldots, J\}$, where by convention, 0 is the initial state at time 0. For instance, if the market consists of two corporate bonds and each bond can only be in default or in non-default, the state space corresponds to $\mathcal{J} = \{0, 1, 2, 3\}$. Therefore, the model of Jarrow and Yu (2001) is a special case of our framework. Alternatively, if the states are identified with rating classes, this framework nests the models by Jarrow et al. (1997) which is generalized by Lando (1998). Let $Z(t)$ denote the state at time $t \in [0, T^*]$ and let Z be an RCLL process, i.e., a right-continuous process with left limits. Then the associated J-dimensional counting process $N = (N^k)_{k\in\mathcal{J}}$ is an RCLL process, where N^k counts the number of transitions into state k, i.e.

$$N^k(t) = \#\{s | s \in (0, t], Z(s-) \neq k, Z(s) = k\}. \tag{5}$$

We assume that there exist sufficiently regular functions λ^{jk} depending on t and Y such that N^k admits the risk-neutral stochastic intensity process $\{\lambda^{Z(t-)k}\}_{t\in[0,T]}$, i.e.

$$M^k(t) = N^k(t) - \int_0^t \lambda^{Z(s)k}(s)\, ds \tag{6}$$

is a Q-martingale. Let $\mathcal{F}_t^Y = \sigma\{Y(s), s \leq t\}$ and set $\lambda^{j*} = \sum_{k\neq j} \lambda^{jk}$. By construction, conditioned on the history of Y, the process Z is a time-inhomogeneous Markov chain on the state space $\mathcal{J}$. The risk-neutral conditional transition probabilities $q^{jk|Y}(t, u) = Q(Z(u) = k | Z(t) = j, \mathcal{F}_{T^*}^Y)$ satisfy the Kolmogorov backward equations[8]

$$q_t^{jk|Y}(t, u) = \lambda^{j*}(t) q^{jk|Y}(t, u) - \sum_{\nu\neq j} \lambda^{j\nu}(t) q^{\nu k|Y}(t, u) \tag{7}$$

with terminal conditions[9] $q^{jk|Y}(u, u) = \delta^{jk}$ as well as the Kolmogorov forward equations

$$q_t^{jk|Y}(s, t) = \sum_{\nu\neq k} q^{j\nu|Y}(s, t)\lambda^{\nu k}(t) - q^{jk|Y}(s, t)\lambda^{k*}(t) \tag{8}$$

with initial conditions $q^{jk|Y}(s, s) = \delta^{jk}$. Here q_t denotes the partial derivative with respect to t. Let Λ be the generator matrix of the chain defined by

[7] These states should not be mixed up with the macroeconomic state Y.

[8] See Karlin (1969).

[9] The symbol δ^{jk} stands for the Kronecker delta, i.e., $\delta^{jk} = 1$ for $j = k$ and zero else.

$\Lambda^{jk} = \lambda^{jk}$ for $j \neq k$ and $\Lambda^{jj} = -\lambda^{j*}$. The backward and forward Kolmogorov equations can be rewritten in matrix form:

$$\frac{\partial}{\partial t} q^{|Y}(t,u) = -\Lambda(t) q^{|Y}(t,u), \quad \frac{\partial}{\partial t} q^{|Y}(s,t) = q^{|Y}(s,t)\Lambda(t) \tag{9}$$

with $q^{|Y}(u,u) = I$ and $q^{|Y}(s,s) = I$, where $q^{|Y}(t,u)$ denotes the matrix of transition probabilities.

We consider a corporate contingent claim which is characterized by a payment stream given by

$$dA(t) = c^{Z(t)}dt + \sum_{k \neq Z(t-)} a^{Z(t-)k}(t) dN^k(t) + a^{Z(t)} d\varepsilon_T(t), \tag{10}$$

where for fixed states j and k the variables c^j, a^j, and a^{jk} are functions depending on t and $Y(t)$. Moreover, ε_T is the Dirac mass at T. The function c^j models a continuous coupon payment which can be state-dependent.[10] The function a^{jk} is a payment upon transition from state j into state k and a^j is the final payment given the chain is in state j at time T. To clarify the meaning of a^{jk} and a^j, let us consider a simple example: If we wish to analyze a single zero-coupon bond where the issuing firm can be in non-default (state 0) or in default (state 1), then a^{01} models the recovery payment upon default. The final payment is then either 0 or 1, i.e., $a^1 = 0$ and $a^0 = 1$. In the following sections, it will become clear that these functions can have many other interpretations and that our framework thus nests various types of different credit risk models. For instance, A can also model the loss of a portfolio of loans.

The no arbitrage paradigm dictates that the time-t value of the payment stream is given by

$$\begin{aligned} B(t) &= \mathrm{E}\Big[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s) \Big| \mathcal{F}_t\Big] \\ &= \mathrm{E}\Big[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s) \Big| Y(t), Z(t)\Big], \end{aligned} \tag{11}$$

where the second equality sign follows from the fact that $(Y(t), Z(t))$ is a Markov process. We sometimes use the short-hand notation $\mathrm{E}_t[\cdot]$ for $\mathrm{E}[\cdot|\mathcal{F}_t]$. It is important to realize that $B(t)$ equals the value of the contingent claim's future payments where payments upon transition at time t are disregarded. However, similarly to dividend payments, holding the contingent claim also includes all payments upon transition such as recovery payments. We start with the price dynamics of the claim under the risk-neutral measure.

[10] Lump sum coupon payments can be included in a straightforward manner.

Proposition 1. (Risk-neutral Price Dynamics) *There exist functions $\mathcal{B}^j$, $j \in \mathcal{J}$, such that $B(t) = \mathcal{B}^{Z(t)}(t, Y(t))$. If each $\mathcal{B}^j$ is in $C^{1,2}$, the risk-neutral price dynamics are given by*[11]

$$\begin{aligned} dB(t) &= \left(\mathcal{B}_t^{Z(t)}(t) + \alpha(t)\mathcal{B}_y^{Z(t)}(t) + 0.5\,\mathrm{tr}\left[\beta(t)^T \mathcal{B}_{yy}^{Z(t)}(t)\beta(t)\right]\right)dt \\ &\quad + \mathcal{B}_y^{Z(t)}(t)\beta(t)dW(t) + \sum_{k \neq Z(t-)} (\mathcal{B}^k(t) - \mathcal{B}^{Z(t-)}(t))dN^k(t) \end{aligned} \tag{12}$$

where, to shorten notation, we have omitted the dependence of $\mathcal{B}^j$ on Y.

Remark. If $\mathcal{B}^j$ is in $C^{1,2}$, then $\mathcal{B}^j$ and its derivatives are continuous in t and y implying that, for instance, $\mathcal{B}^j(t-, Y(t-)) = \mathcal{B}^j(t, Y(t))$. Besides, Z jumps at only countably many points, i.e., in Ito and Lebesgue integrals we can replace all $Z(t-)$ by $Z(t)$.

We emphasize that the drift of B under a risk-neutral measure is not equal to r since we are not dealing with a traded asset. System (13) of the following theorem is a generalization of the Black-Scholes PDE satisfied by contingent claims depending on the macroeconomic state variables Y only. If the state process Z cannot jump, this system reduces to the Black-Scholes PDE.

Theorem 1. (System of Pricing Equations) *The functions $\mathcal{B}^j$, $j \in \mathcal{J}$, satisfy the system of PDEs*

$$\mathcal{B}_t^j = r\mathcal{B}^j - \alpha\mathcal{B}_y^j - 0.5\,\mathrm{tr}\left[\beta^T \mathcal{B}_{yy}^j \beta\right] - \sum_{k:k \neq j} \lambda^{jk}\left(a^{jk} + \mathcal{B}^k - \mathcal{B}^j\right) - c^j \tag{13}$$

with terminal conditions $\mathcal{B}^j(T, y) = a^j(y)$, $j \in \mathcal{J}$.

Remarks.

(a) The opposite of the previous theorem is also true: The system of PDEs (13) has the stochastic representation given by (11). This can be interpreted as a generalized version of representation theorems usually referred to as Feynman-Kac theorems.

(b) Our framework can also be used to analyze phenomena referred to as "flight to quality". These are downward jumps of the default-free interest rate as a consequence of defaults of large entities. To model these effects, one needs to replace r in (11) and (13) by a state-dependent short rate r^j. Consequently, default-free zero-coupon bond prices $P(\cdot, T)$ become state-dependent as well.

[11] Throughout the paper we use the following convention: B_s denotes the time-s claim price. Only if $s = t$ we write $B(t)$ to distinguish the time-t price from the partial derivative w.r.t. time, B_t. Besides, tr stands for trace and $\beta(t)^T$ denotes the transpose of the matrix $\beta(t)$.

(c) In the special case when Y and r are constants, the system (13) simplifies into

$$\mathcal{B}_t^j = r\mathcal{B}^j - \sum_{k:k\neq j} \lambda^{jk}\left(a^{jk} + \mathcal{B}^k - \mathcal{B}^j\right) - c^j, \tag{14}$$

where $\mathcal{B}^j(T) = a^j$, $j \in \mathcal{J}$.

(d) If $a^{jk} = 0$ for all $j \neq k$ and $c^j = 0$, then B becomes a traded asset and the drift of B simplifies into r. This can be verified by substituting (13) into the risk-neutral dynamics for B.

Notation. In the sequel, with a slight abuse of notation, we will not distinguish between the process B and the function $\mathcal{B}$ to simplify notation. This means, for instance, that we write B_y instead of $\mathcal{B}_y$.

Theorem 1 suggests that claim prices can be calculated by solving the system of interacting PDEs (13). Instead of doing this, one can also directly tackle the expected value (11) and use the fact that transition probabilities satisfy Kolmogorov equations. The corresponding results are summarized in the following proposition.

Proposition 2. (Pricing via Transition Probabilities) *Assume $Z(t) = j$. If the all expectations are finite, the time-t price of*

(*i*) *the continuous coupon payment $dA_1(t) = c^{Z(t)}(t)dt$ equals*

$$\sum_{k\in\mathcal{J}} \int_t^T E_t\left[e^{-\int_t^s r_u\,du} c^k(s) q^{jk|Y}(t,s)\right] ds \tag{15}$$

(*ii*) *the payments upon transition $dA_2(t) = \sum_{k\neq Z(t-)} a^{Z(t-)k}(t)dN^k(t)$ equals*

$$\sum_{\nu\in\mathcal{J}} \sum_{k\neq\nu} \int_t^T E_t\left[e^{-\int_t^s r_u\,du} a^{\nu k}(s) q^{j\nu|Y}(t,s) \lambda^{\nu k}(s)\right] ds \tag{16}$$

(*iii*) *the final payment $dA_3(t) = a^{Z(t)}(t)d\varepsilon_T(t)$ equals*

$$\sum_{k\in\mathcal{J}} E_t\left[e^{-\int_t^T r_u\,du} a^k(T) q^{jk|Y}(t,T)\right]. \tag{17}$$

For the application of these results it is crucial whether explicit solutions for the transition probabilities can be found, i.e., whether one can solve the Kolmogorov forward or backward equations. This, however, depends on the specific problem. Although the approach via the Kolmogorov equations seems to be simpler at first glance, since one only needs to solve a system of first-order ODEs (7) compared to a system of second-order PDEs (13), this is not completely true: Whereas solving (13) provides a complete solution to the pricing problem, a solution to (7) is only a first step to determine a solution. In a second step, all expected values occurring in Proposition 2 need to be calculated as well. Only if Y is deterministic, this is not necessary, but then (13) collapses into a system of first-order ODEs. Therefore, it actually depends on the problem (and probably also on the reader's taste) which of these approaches is the preferable one.

As indicated above, the representations of the time-t prices provided in Proposition 2 simplify significantly if the intensities λ^{jk} and payments c^j, a^j, as well as a^{jk} are constants. For the convenience of the reader, we formulate the corresponding results as a corollary. Since the transition probabilities do not depend on Y, no conditioning on the history of Y is needed and we thus drop the superscript Y.

Corollary 1. (Constant Intensities) *If the intensities and payments are constant, we obtain the following results:*

(i) The time-t price of the continuous coupon payment equals

$$\sum_{k\in\mathcal{J}} c^k \int_t^T P(t,s) q^{jk}(t,s)\, ds. \tag{18}$$

(ii) The time-t price of the payments upon transition equals

$$\sum_{\nu\in\mathcal{J}} \sum_{k\neq\nu} a^{\nu k} \lambda^{\nu k} \int_t^T P(t,s) q^{j\nu}(t,s)\, ds. \tag{19}$$

(iii) The time-t price of the final payment equals

$$P(t,T) \sum_{k\in\mathcal{J}} a^k q^{jk}(t,T). \tag{20}$$

To demonstrate the differences and similarities of both approaches we now discuss an example of a contingent claim providing only final payments. We assume that the state space is of the form detailed in Figure 1. Starting in state 0, the chain can either jump into states 1 or 2 and finally into state 3. We can think of this claim as a corporate zero-coupon bond providing state-dependent final payments. So apart from the state-dependent payments, this example is similar to the one of Figure 3. In a later section, we will consider a more general situation, but for our current purpose, it is beneficial to assume that Y and thus r as well as the intensities λ^{jk} and the final payments a^j

are constants. Due to (14), we know that $B^j(t) = e^{-r(T-t)}H^j(t)$, where the functions H^j satisfy the following system of ODEs

$$\begin{aligned} H_t^0 &= -\lambda^{01}(H^1 - H^0) - \lambda^{02}(H^2 - H^0), \\ H_t^1 &= -\lambda^{13}(H^3 - H^1), \\ H_t^2 &= -\lambda^{23}(H^3 - H^2), \\ H_t^3 &= 0, \end{aligned} \tag{21}$$

with $H^0(T) = a^0$, $H^1(T) = a^1$, $H^2(T) = a^2$, and $H^3(T) = a^3$. We assume that the chain is currently in state 0. Therefore, solving for the time-t claim price means solving for H^0. Obviously, $H^3(t) = a^3$. Using this result we can calculate $H^j(t) = a^3 + e^{-\lambda^{j3}(T-t)}(a^j - a^3)$, $j = 1, 2$. Finally, we obtain

$$\begin{aligned} H^0(t) &= \int_t^T e^{-\lambda^{0*}(s-t)}(\lambda^{01}H^1(s) + \lambda^{02}H^2(s))\,ds + e^{-\lambda^{0*}(T-t)}a^0 \\ &= a^3(1 - e^{-\lambda^{0*}(T-t)}) + \sum_{k=1}^{2} \frac{\lambda^{0k}(a^k - a^3)}{\lambda^{k3} - \lambda^{0*}} \\ &\quad \times (e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{k3}(T-t)}) + a^0 e^{-\lambda^{0*}(T-t)}, \end{aligned} \tag{22}$$

where we assume that $\lambda^{k3} \neq \lambda^{0*}$. Otherwise, the corresponding term becomes $\lambda^{0k}(a^3 - a^1)e^{-\lambda^{0*}(T-t)}(T - t)$. This result can be derived by integrating or by taking the limit $\lambda^{k3} \to \lambda^{0*}$ and using l'Hospital's rule. On the other hand, from Corollary 1 we know that $H^0(t) = \sum_{k=0}^{3} a^k q^{0k}(t, T)$. Solving the corresponding Kolmogorov backward equations (see Appendix A.4), we arrive

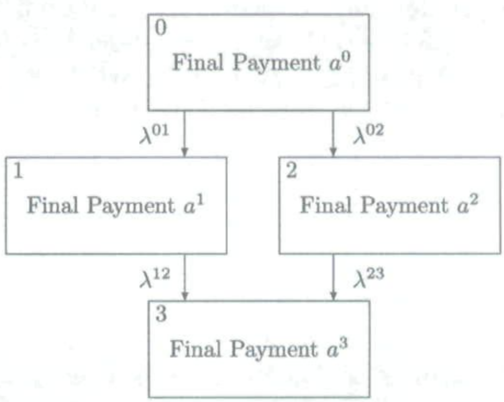

Figure 1. Example with four states.

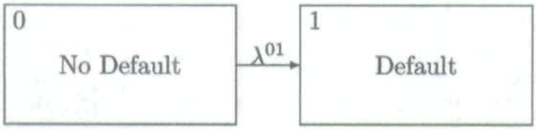

Figure 2. Classical situation.

at:

$$
\begin{aligned}
q^{00}(t,T) &= e^{-\lambda^{0*}(T-t)},\\
q^{0k}(t,T) &= \frac{\lambda^{0k}}{\lambda^{k3}-\lambda^{0*}}\left(e^{-\lambda^{0*}(T-t)} - e^{-\lambda^{k3}(T-t)}\right), \quad k=1,2,\\
q^{03}(t,T) &= 1 - q^{00}(t,T) - q^{01}(t,T) - q^{02}(t,T)
\end{aligned}
\tag{23}
$$

Substituting into $H^0(t) = \sum_{k=0}^{3} a^k q^{0k}(t,T)$ and rearranging gives (22).

4. Pricing of Defaultable Bonds

4.1 RECOVERY ASSUMPTIONS

To make the reader familiar with our framework, we firstly show how the zero-coupon prices for the classical recovery assumptions like recovery of market value (RMV), recovery of treasury (RT), and recovery of par (RP) can be derived in our framework.[12] As mentioned in Section 2, the relevant Markov chain has only two states (see Figure 2): In state 0 the firm is solvent, whereas in state 1 it is in default.

The relevant PDE to consider is thus the following:

$$
B_t^0 = rB^0 - \alpha B_y^0 - 0.5\,\mathrm{tr}\left[\beta^T B_{yy}^0 \beta\right] - \lambda^{01}\left(a^{01} + B^1 - B^0\right) \tag{24}
$$

with $a^0(T) = 1$. Depending on the recovery assumption, B^1 and $a^1(T)$ can now be specified. Under RMV we have $B^1 = 0$ and $a^{01} = (1-l)B^0$, where the loss rate $l \in [0,1)$ may depend on t and $Y(t)$. Note that we can allow for lump sum payments to depend on B^0, since B^0 is a function of t and Y. Therefore, (24) becomes

$$
B_t^0 = (r + l\lambda^{01})B^0 - \alpha B_y^0 - 0.5\,\mathrm{tr}\left[\beta^T B_{yy}^0 \beta\right] \tag{25}
$$

which has the stochastic representation $B^0(t) = \mathrm{E}_t[e^{-\int_t^T (r_u + l_u \lambda_u^{01})\,du}]$ derived by Duffie and Singleton (1999a). Under RP we set $B^1 = 0$ and the recovery rate is just a fraction of the notional, i.e. a^{01} is a fraction between 0 and 1 which may depend on t and $Y(t)$. In this case, the solution to (24)

[12] See, e.g., Jarrow and Turnbull (1995), Duffie and Singleton (1999a), and Lando (1998).

has the stochastic representation $B^0(t) = \int_t^T \mathrm{E}_t[e^{-\int_t^T (r_u+\lambda_u^{01})\,du}\lambda_u^{01} a_u^{01}]\,du + \mathrm{E}_t[e^{-\int_t^T (r_u+\lambda_u^{01})\,du}]$. Finally, let us consider the RT assumption which has two variants. Either it is assumed that upon default the bondholder receives a fraction of a default-free zero coupon bond implying $B^1 = 0$ and $a^{01}(t) = k(t)P(t,T)$, $k \in [0,1)$, (recovery at default) or it is assumed that in case of default the bondholder receives k_T at maturity T (recovery at maturity). Then $a^{01} = 0$ and B^1 satisfies the PDE of a default-free zero-coupon bond

$$B_t^1 = rB^1 - \alpha B_y^1 - 0.5\,\mathrm{tr}\big[\beta^T B_{yy}^1 \beta\big] \tag{26}$$

with $a_T^1 = k_T$. Since in our model $k(t)$ may depend on t and $Y(t)$, these two constructions do not necessarily lead to the same bond price. The recovery at default assumption leads to the following bond price: $B^0(t) = \int_t^T \mathrm{E}_t[k_s e^{-\int_t^s (r_u+\lambda_u^{01})\,du}\lambda_s^{01} P(s,T)]\,ds + \mathrm{E}_t[e^{-\int_t^T (r_u+\lambda_u^{01})\,du}]$. In case of the recovery at maturity assumption, $k_s P(s,T)$ needs to be replaced by $B_s^1 = \mathrm{E}_s[k_T e^{-\int_s^T r_u\,du}]$. Obviously, if k is constant, a^{01} and $B^1(t)$ have the same values. Finally, note that all these stochastic representations are usually referred to as Feynman-Kac representations.

4.2 CONTAGION AND COMPETITION EFFECTS

On a portfolio level there is one big issue which is at the center of current research in the area of credit risk and these are so-called contagion effects. Firms' defaults are correlated due to industry or economy wide factors like business relations or liquidity breakdowns. As an effect, a firm's default probability depends on default events of other firms. In general, two effects can occur: Either a firm's default probability goes up if another firm defaults on its debt or it goes down indicating positive or negative default correlation. The former phenomena is usually called contagion, whereas the latter is referred to as a competition effect.[13] In a recent paper, Jorion and Zhang (2005) analyze Credit Default Swap data and find strong evidence of dominant contagion effects for Chapter 11 bankruptcies as well as competition effects for Chapter 7 bankruptcies. Our Markov chain framework is well suited to capture these kinds of dependencies and we consider a problem with two bonds as detailed in Figure 3. Such a model was already discussed in Section 3. The chain starts in state 0 where both firms are solvent and the default intensities are $\lambda^A = \lambda^{01}$ and $\lambda^B = \lambda^{02}$, respectively. If firm B defaults upon its debt first, the default intensity of firm A changes to $\lambda^{A'} = \lambda^{23}$, whereas if firm A defaults first, the

[13] In the following we usually call both effects just contagion effects, since a competition effect can be thought of as negative contagion.

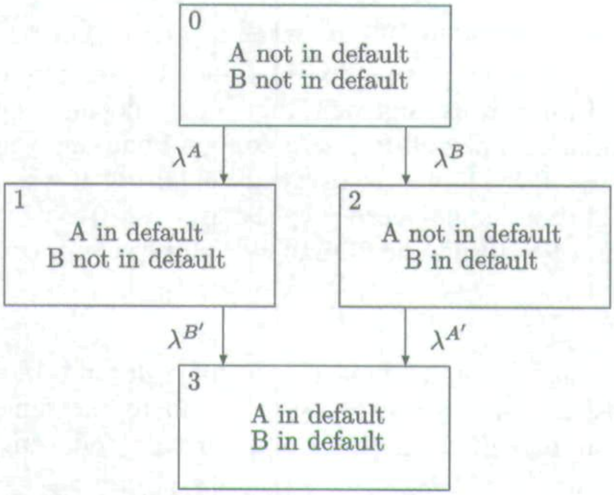

Figure 3. Contagion and competition effects.

default intensity of firm B changes to $\lambda^{B'} = \lambda^{13}$. Therefore, contagion effects occur if $\lambda^{A'} > \lambda^A$ or $\lambda^{B'} > \lambda^B$, while competition effects occur if $\lambda^{A'} < \lambda^A$ or $\lambda^{B'} < \lambda^B$. Starting with Jarrow and Yu (2001), in the literature one usually finds the following model for the intensity of firm A: $\lambda^A_{JY}(t) = a_1 + a_2 \mathbf{1}_{\{\tau^B \leq t\}}$, where τ^B is the default time of firm B. Hence, this model can be embedded into our framework by setting $\lambda^A = a_1$ and $\lambda^{A'} = a_1 + a_2$.

We are now going to price a zero-coupon bond issued by firm A. If firm B has already defaulted upon its debt (state 2), we are back to the situation analyzed in Subsection 4.1, i.e., the bond satisfies PDE (24):

$$B^2_t = rB^2 - \alpha B^2_y - 0.5\,\mathrm{tr}\big[\beta^T B^2_{yy}\beta\big] - \lambda^{A'}\left(a^{A'} + B^3 - B^2\right) \tag{27}$$

with $B^2(T) = 1$ and $a^{A'} = a^{23}$. Without loss of generality, we set $B^3 = 0$ so that recovery is solely captured by $a^{A'}$ implying a recovery at default regime. Then the solution to the above PDE reads

$$B^2(t) = \int_t^T \mathrm{E}_t\big[\lambda^{A'}_s a^{A'}_s e^{-\int_t^s r_u + \lambda^{A'}_u\, du}\big]\, ds + \mathrm{E}_t\big[e^{-\int_t^T r_u + \lambda^{A'}_u\, du}\big]. \tag{28}$$

Depending on the recovery assumption, this expression can be further simplified. Specifically under RMV we have $a^{A'}_s = (1 - l^{A'}_s)B^2_s$ with $l^{A'}$ being a function of t and $Y(t)$ implying

$$B^2(t) = \mathrm{E}_t\big[e^{-\int_t^T r_u + l^{A'}_u \lambda^{A'}_u\, du}\big]. \tag{29}$$

According to (13), in state 0 the PDE for the bond price of firm A reads:

$$B_t^0 = rB^0 - \alpha B_y^0 - 0.5\,\mathrm{tr}[\beta^T B_{yy}^0 \beta] - \lambda^A \left(a^A + B^1 - B^0\right) - \lambda^B \left(B^2 - B^0\right) \tag{30}$$

with $B^0(T) = 1$. Again set $B^1 = 0$ so that $a^A = a^{01}$ exclusively captures the recovery payment given firm A defaults first. Therefore, we obtain the following result.

Proposition 3. (Bond Price under Contagion) *In a setting with contagion or competition effects the bond price of firm A has the stochastic representation*

$$B^0(t) = \int_t^T E_t\big[(\lambda_s^A a_s^A + \lambda_s^B B_s^2)e^{-\int_t^s r_u + \lambda_u^A + \lambda_u^B \, du}\big]\, ds + E_t\big[e^{-\int_t^T r_u + \lambda_u^A + \lambda_u^B \, du}\big]. \tag{31}$$

Remarks.

(a) We wish to emphasize that (31) provides an explicit representation for B^0 since B^2 is given by (28).

(b) The special case with constant interest rates and RT was already considered in Section 3. Let δ denote the recovery rate. By Corollary 1, the solution follows by defining the payoffs as $a^0 = a^2 = 1$ and $a^1 = a^3 = \delta$ and multiplying them with the corresponding probability (23).

Jarrow and Yu (2001) analyze the following simplified version called primary-secondary framework: A default of firm A does not affect the default intensity of firm B, whereas a default of firm B still does effect the intensity of firm A. In our framework, this is equivalent to setting $\lambda^B = \lambda^{B'}$. Bielecki and Rutkowski (2002) and Yu (2005a) have already remarked that the bond price of firm A does not depend on $\lambda^{B'}$ and thus the assumption $\lambda^B = \lambda^{B'}$ can be dropped without changing the bond price. They analyze the default time τ^A and demonstrate that its marginal density does not contain $\lambda^{B'}$. This result is reflected in our pricing formula (31) since neither B^2 nor B^0 depend on $\lambda^{B'}$. Our model offers an alternative explanation for this: In the first place, the intensity $\lambda^{B'}$ enters the PDE for B^1. However, since upon transition from state 0 to state 1 the recovery payment is made and the bond disappears, without loss of generality we were able to set $B^1 = 0$. This implies that $\lambda^{B'}$ has no effect on B^0. At this point, we wish to emphasize that this result relies upon the simplifying assumption that default and bankruptcy happen at the same time. However, Guo et al. (2005b) argue that this assumption oversimplifies the situation of a financial distressed firm and leads to wrongly specified recovery rates. Using our Markov chain framework, one can show that $\lambda^{B'}$ does play a role if a firm continues to operate after default.

Furthermore, an inspection of (31) shows that under more specific assumptions on the dynamics of the macroeconomic variables and the intensity processes as well as the recovery payment, $a^{A'}$, the bond price can be calculated explicitly (at least up to a numerical integration in the first term). For instance, in an affine framework the results of Duffie et al. (2000) can be used to calculate all occurring expectations. However, since the so-called no-jump condition of Duffie et al. (1996) and Duffie and Singleton (1999a) is violated, a handy representation like (29) cannot be expected to hold under the risk-neutral measure.[14] Recall that in an affine framework, the dynamics for the macroeconomic variables are specified as follows:

$$\alpha(t) = K_0 + K_1 Y(t), \qquad (\beta(t)\beta(t)^T)_{ij} = (H_0)_{ij} + (H_1)_{ij} Y(t) \tag{32}$$

with $K = (K_0, K_1) \in \mathbb{R}^L \times \mathbb{R}^{L\times L}$ and $H = (H_0, H_1) \in \mathbb{R}^{L\times L} \times \mathbb{R}^{L\times L\times L}$. We additionally assume a RMV regime, i.e.

$$a^A = (1 - l^A)B^0, \qquad a^{A'} = (1 - l^{A'})B^2. \tag{33}$$

To achieve an affine framework, suppose that $R^A = r + l^A\lambda^A + \lambda^B$ and $R^{A'} = r + l^{A'}\lambda^{A'}$ are affine in Y, i.e., there exist (ρ_0^A, ρ_1^A), $(\rho_0^{A'}, \rho_1^{A'}) \in \mathbb{R} \times \mathbb{R}^L$ such that

$$R^A = \rho_0^A + \rho_1^A Y, \qquad R^{A'} = \rho_0^{A'} + \rho_1^{A'} Y. \tag{34}$$

This last assumption is satisfied if l^A and $l^{A'}$ are constants and the short rate and the intensities are affine functions of Y.

Proposition 4. (Bond Price under Contagion in an Affine Framework) *Assume (32), (33), and (34) to hold. Additionally assume that $\lambda^B = \rho_0^B + \rho_1^B Y$ with $(\rho_0^B, \rho_1^B) \in \mathbb{R} \times \mathbb{R}^L$. Then the bond price is given by*

$$\begin{aligned} B^0(t) = \int_t^T &\left\{\rho_0^B + \rho_1^B\left[\overline{A}^0(t,s) + \overline{B}^0(t,s)Y(t)\right]\right\} e^{\tilde{A}^2(s,T)+\hat{A}^0(t,s)+\hat{B}^0(t,s)Y(t)} \\ &+ e^{\tilde{A}^0(t,T)+\tilde{B}^0(t,T)Y(t)}, \end{aligned} \tag{35}$$

[14] Collin-Dufresne et al. (2004) show that even if the no-jump condition is violated, similar relations hold under an artificial probability measure putting zero probability on paths where default occurs prior to the maturity. This measure is thus only absolutely continuous with respect to the risk-neutral probability measure.

where (i) $(\tilde{A}^0, \tilde{B}^0)$ *satisfies*[15] *(76) with* $(\rho_0, \rho_1) = (\rho_0^A, \rho_1^A)$ *and* $\tilde{A}^0(T,T) = 0$, $\tilde{B}^0(T,T) = 0$, *(ii)* $(\tilde{A}^2, \tilde{B}^2)$ *satisfies (76) with* $(\rho_0, \rho_1) = (\rho_0^{A'}, \rho_1^{A'})$ *and* $\tilde{A}^2(T,T) = 0$, $\tilde{B}^2(T,T) = 0$, *(iii)* $(\hat{A}^0, \hat{B}^0)$ *satisfies (76) with* $(\rho_0, \rho_1) = (\rho_0^A, \rho_1^A)$ *and* $\hat{A}^0(s,s) = 0$, $\hat{B}^0(s,s) = \tilde{B}^2(s,T)$, *and (iv)* $(\overline{A}^0, \overline{B}^0)$ *satisfies (77) with* $\overline{A}^0(s,s) = 0$, $\overline{B}^0(s,s) = 1$.

Remark. Jarrow and Yu (2001) consider the special case where Y equals the short rate following Vasicek dynamics.

Finally, it is worthwhile to study which correlations between the default times of two entities are implied by the model detailed in Figure 3. Said differently, we wonder how for given intensities λ^A and λ^B the intensities $\lambda^{A'}$ and $\lambda^{B'}$ need to be adjusted such that a certain correlation is achieved. Let $\mathbf{1}_{\{\tau_A \le t\}}$ and $\mathbf{1}_{\{\tau_B \le t\}}$ denote the default indicators of firms A and B where τ_A and τ_B denote the corresponding default times. Default correlation between firm A and B over the time interval $[t, T]$ is defined as[16]

$$\rho_{A,B}(t,T) = \frac{\mathrm{Cov}_t(\mathbf{1}_{\{\tau_A \le T\}}, \mathbf{1}_{\{\tau_B \le T\}})}{\sqrt{\mathrm{Var}_t(\mathbf{1}_{\{\tau_A \le T\}})}\sqrt{\mathrm{Var}_t(\mathbf{1}_{\{\tau_B \le T\}})}}. \tag{36}$$

The following proposition provides a closed-form expression for $\rho_{A,B}$ in terms of transition probabilities:

Proposition 5. (Default Correlation) *In the model with contagion, the default correlation between firms A and B is given by:*

$$\rho_{A,B} = \frac{q^{03} - (q^{01} + q^{03})(q^{02} + q^{03})}{\sqrt{(q^{01} + q^{03})(1 - q^{01} - q^{03})}\sqrt{(q^{02} + q^{03})(1 - q^{02} - q^{03})}}, \tag{37}$$

where, for notational convenience, we have dropped the arguments (t, T).

Remark. In Appendix A.4, it is discussed how to calculate the transition probabilities q^{0k} explicitly.

If no contagion effects are present, i.e., $\lambda^A = \lambda^{A'}$ and $\lambda^B = \lambda^{B'}$, the correlation is zero. On the other hand, if for instance $\lambda^{A'} = \lambda^A + b$ and $\lambda^{B'} = \lambda^B + b$ with a constant $b \ge 0$, then $\lim_{b\to\infty} \rho_{A,B} = 1$. For this reason, one can achieve any correlation between 0 and 1 by choosing an appropriate b. Using some algorithm to find the null of an equation (e.g., the Newton method), one

[15] The respective ODEs are provided in the proof of the proposition in the Appendix.

[16] There are reasons why the ordinary correlation coefficient is an inferior measure compared to Spearman's Rho or Kendall's Tau. A discussion of this point is not the focus of our paper. See, e.g., Li (2000).

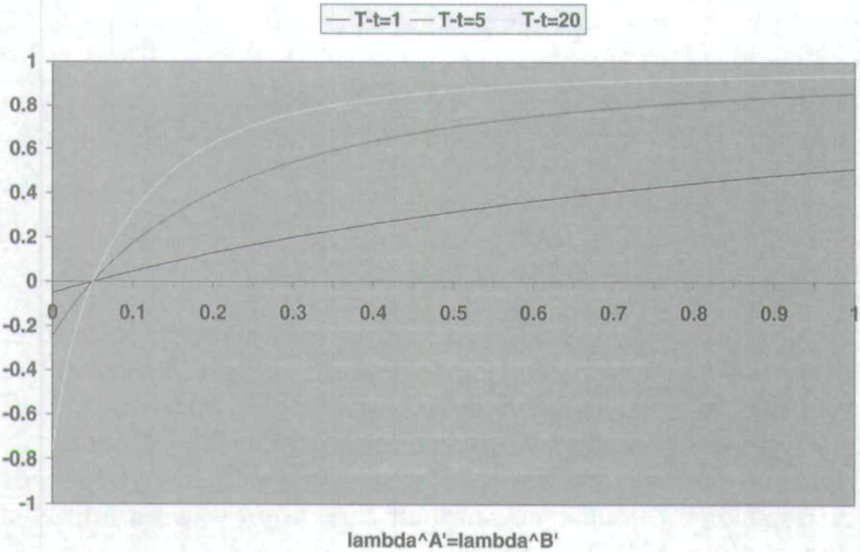

Figure 4. Default correlation via contagion effects.

can solve for b given a certain correlation. Note that choosing $\lambda^{A'} = \lambda^{B'} = 0$ usually does not lead to a correlation of -1. Let us finally consider a numerical example. Assume that $\lambda^A = \lambda^B = 0.05$. For three different time horizons $T - t$, Figure 4 shows how default correlation varies with different values of $\lambda^{A'} = \lambda^{B'}$. It can be seen that the correlation goes to one if b goes to infinity and that the correlation is greater than minus one for zero intensities. Thus there is a lower bound for which correlations can be achieved.

So far we have analyzed situations where only two entities, firm A and B, are involved. However, as shown by Jorion and Zhang (2005), defaults can have industry-wide effects on bond prices. Therefore, we now consider a model where in addition to firm A there are K other firms and upon default of one of these firms the default intensity of firm A may change, i.e., each default causes a contagion effect on firm A's bond prices.[17] To achieve a tractable model, we assume that the additional firms are homogeneous.

We thus get the model depicted in Figure 5. In the absorbing state $K + 1$, firm A is in default, whereas in every state $j \leq K$ firm A is solvent and j of the other firms have already defaulted. If the chain is currently in some state $j \in \{0, 1 \ldots, K - 1\}$, then either firm A or one of the additional firms can default, i.e., the chain can either jump into the absorbing state $K + 1$ or in the next state $j + 1$. The intensities for jumping into the absorbing or the next

[17] We thank the referee for suggesting this application.

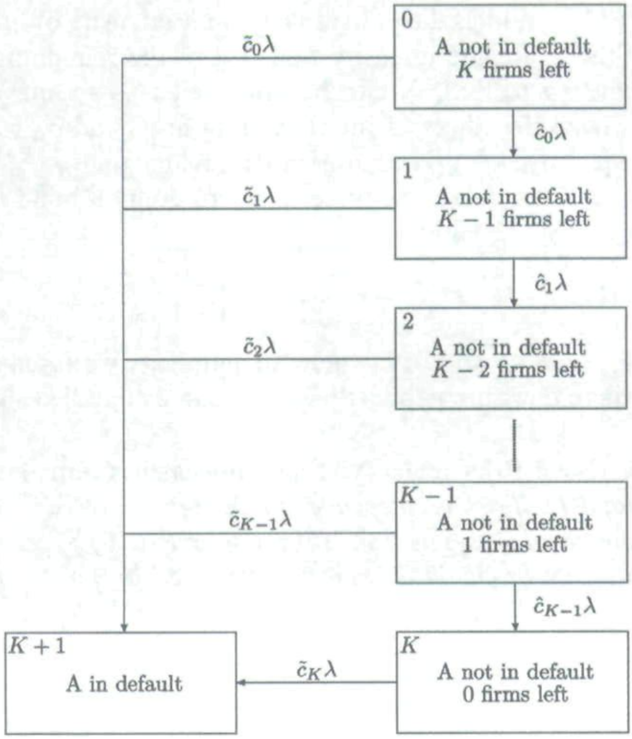

Figure 5. Model for multiple contagion.

state are assumed to be of the form

$$\lambda^{jK+1} = \tilde{c}_j\lambda, \qquad \lambda^{jj+1} = \hat{c}_j\lambda, \qquad j = 0, 1, \ldots, K, \tag{38}$$

where $\tilde{c}_j$ and $\hat{c}_j$ are state-dependent constants and λ is a stochastic process.[18] Note that, by definition, $\tilde{c}_K = \hat{c}_K$. Additionally, the intensity for leaving state j is given by

$$\lambda^{j*} = \overline{c}_j\lambda = \begin{cases} (\hat{c}_j + \tilde{c}_j)\lambda, & j \le K - 1, \\ \tilde{c}_K\lambda, & j = K. \end{cases} \tag{39}$$

Finally, we set $\overline{c}_{K+1} = \hat{c}_{K+1} = \tilde{c}_{K+1} = 0$. The process λ can be interpreted as a stochastic base intensity that is scaled by $\hat{c}_j$ or $\tilde{c}_j$. For instance, if the additional firms are homogeneous, then it makes sense to choose $\hat{c}_j = (K - j)k_1 + k_2 j$ with constants k_1 and k_2. We will discuss this particular parametrization in the following section in more detail.

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[18] In general, λ can be any stochastic process satisfying $\int_t^T (\hat{c}_j + \tilde{c}_j)\lambda_s\, ds < \infty$ a.s. for all j. This is for instance satisfied by processes with continuous paths.

In general, λ^{jK+1} models the individual default intensity of firm A, whereas λ^{jj+1} models the aggregate intensity that one of the remaining firms $K-j$ defaults. According to (38), $\hat{c}_j$ can be interpreted as a sum $\sum_\nu \hat{c}_{j\nu}$, where this sum goes over all indices of the remaining firms and $\hat{c}_{j\nu}$ is the scalar of firm ν. Consequently, λ^{jj+1} is usually much greater than λ^{jK+1} if j is small compared to K. To calculate the price of a zero-coupon bond issued by firm A, let $(k,l) = \hat{c}_k/(\overline{c}_k - \overline{c}_l)$ and

$$[k,l]_{t,T} = (k,l)\left\{e^{-\int_t^T \overline{c}_l \lambda_u \, du} - e^{-\int_t^T \overline{c}_k \lambda_u \, du}\right\} \tag{40}$$

for $k, l \in \{0, 1, \ldots, K\}$ with $k < l$, where for simplicity we assume $\overline{c}_k \neq \overline{c}_l$. The special case where there are pairs with $\overline{c}_k = \overline{c}_l$ can be handled similarly.

Proposition 6. (Bond Price under Multiple Stochastic Contagion) *Under the assumption that default risk is independent of interest rate risk,*[19] *the time-t price of a zero-coupon bond issued by firm A is given by* $P(t,T)\sum_{k=0}^{K} E_t[q^{0k|\lambda}(t,T)]$, *where the transition probabilities can be calculated by the following forward algorithm: Firstly,* $q^{jj|\lambda}(t,T) = e^{-\int_t^T \overline{c}_j \lambda_u \, du}$ *and* $q^{jj+1|\lambda}(t,T) = [j, j+1]_{t,T}$. *Iteratively, for every* $m \geq j+2$ *the transition probability* $q^{jm+1|\lambda}(t,T)$ *can be calculated from* $q^{jm|\lambda}(t,T)$ *by replacing every* (k,l) *by* $(k+1, l+1)$ *and every* $[k,l]_{t,T}$ *by*

$$(k+1, l+1)\cdot\left\{[k, l+1]_{t,T} - [k, k+1]_{t,T}\right\}. \tag{41}$$

Remarks.

(a) Note that the model (38) leads to closed-form solutions for the conditional transition probabilities $q^{0k|\lambda}(t,T)$. These solutions can be calculated by applying the forward algorithm of the proposition. For instance,

$$\begin{aligned} q^{01|\lambda} &= [0,1] \\ q^{02|\lambda} &= (1,2)\{[0,2]-[0,1]\} \\ q^{03|\lambda} &= (2,3)(1,3)\{[0,3]-[0,1]\} - (2,3)(1,2)\{[0,2]-[0,1]\}, \end{aligned} \tag{42}$$

where we have suppressed all dependencies on (t,T).

(b) Depending on the dynamics of λ, it may be possible to calculate $\mathrm{E}_t[q^{0k|\lambda}(t,T)]$ explicitly. Since (k,l) is a constant, it is crucial whether one can compute the expected value of $[k,l]_{t,T}$. If λ is *affine*, then this

[19] This is a common assumption in the credit risk literature. Otherwise, the result holds if expectations are taken under the T-forward measure.

is possible:

$$\mathrm{E}_t\big[[k,l]_{t,T}\big] = \frac{c_k}{c_k - c_l}\left\{e^{A_l(t,T)+B_l(t,T)\lambda_t} - e^{A_k(t,T)+B_k(t,T)\lambda_t}\right\} \tag{43}$$

with appropriate deterministic functions A_l, B_l, A_k, and B_k. Then, for instance,

$$\begin{aligned}
&\mathrm{E}_t[q^{02|\lambda}(t,T)]\\
&\quad = \frac{c_1}{c_1 - c_2}\frac{c_0}{c_0 - c_2}\left\{e^{A_2(t,T)+B_2(t,T)\lambda_t} - e^{A_0(t,T)+B_0(t,T)\lambda_t}\right\}\\
&\qquad - \frac{c_1}{c_1 - c_2}\frac{c_0}{c_0 - c_1}\left\{e^{A_1(t,T)+B_1(t,T)\lambda_t} - e^{A_0(t,T)+B_0(t,T)\lambda_t}\right\}.
\end{aligned} \tag{44}$$

Therefore, in an affine model we obtain closed-form solutions for the zero-coupon bond price of firm A.

5. Further Applications

5.1 CDS SPREADS UNDER COUNTERPARTY RISK

The most popular credit derivatives are so-called credit default swaps (CDS) allowing to insure against a default of a certain entity (e.g., firm or country). The protection seller B promises to make a payment of one dollar to the protection buyer A if a certain reference entity C defaults during the lifetime of the CDS.[20] In exchange for that, the protection buyer continuously pays a fee to the protection seller.

We are now going to analyze such contracts if counterparty risk is present, i.e., we allow that A or B fail to fulfill their contractual obligations. Therefore, the protection payment is made only if the protection seller has not defaulted up to the default time of the reference entity. Besides, the protection buyer continues to make his fee payments only if he has not defaulted yet. Since most CDS contracts traded in practice are settled at the default of the reference entity, we also make this assumption.[21] The relevant model is detailed in

[20] Usually a CDS pays the loss given default of a certain corporate bond. CDS contracts paying a fixed amount of dollars are sometimes referred to as digital CDS. This assumption is made for simplicity.

[21] Jarrow and Yu (2001) and Yu (2005a) assume that the contracts are settled at maturity. Jarrow and Yu (2001) consider a special case where A is default free. Yu (2005a) uses Monte Carlo simulation to calculate CDS spreads.

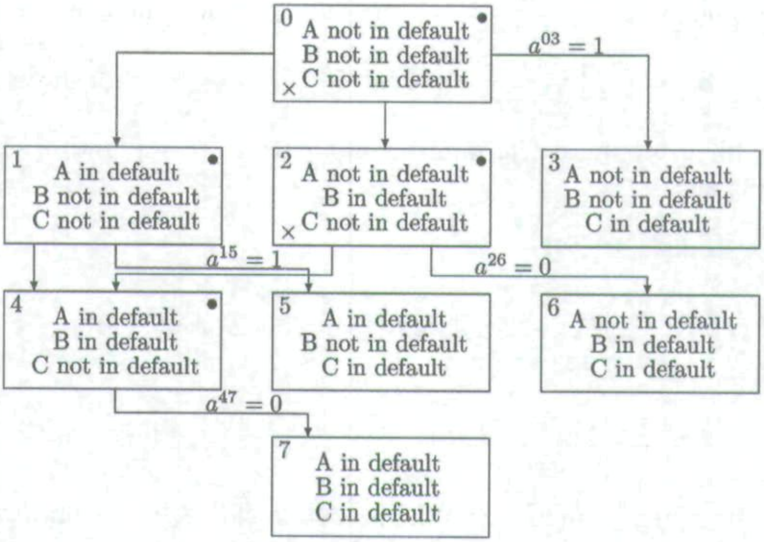

Figure 6. CDS payments when the contract is settled at default.

Figure 6. The states where the CDS contract is not settled yet are marked by •, whereas the states marked by × are the ones where the buyer still makes his fee payments. For simplicity, we additionally impose the following assumption that is also made by Jarrow and Yu (2001) and Yu (2005a): If A defaults before C defaults, B needs to fulfill his obligations as long as B is not in default and, independently of whether B is in default or not, A continues to make his fee payments until its own default, a default of C, or maturity whichever occurs first.

In general, there is a kind of asymmetry in the payments made by A and B. Whereas A is continuously making payments, B makes no payments except for a potential protection payment at default of the reference entity C. A default of counterparty A immediately becomes visible, whereas the ability of B to make a protection payment is not revealed until C defaults. Therefore, the assumption that A makes fee payments although B has already defaulted is not an unreasonable one, whereas the assumption that B needs to make a protection payment although A has already stopped making fee payments seems to be debatable. To circumvent this problem, in our numerical examples we also provide the results if the default probability of A is zero. To calculate CDS spreads, we need to determine the values of the fee leg and the protection leg. Since a CDS contract is settled upon default of the reference entity, the protection payment is modeled as a payment upon transition a^{jk}, whereas the fee leg is modeled as a state-dependent coupon payment c^j. Let $y(t)$ denote the fair CDS spread and assume that all intensities λ^{jk} are constant. The default intensities of A, B, and C given that no entity has defaulted yet are denoted by $\lambda^A = \lambda^{01}$, $\lambda^B = \lambda^{02}$, $\lambda^C = \lambda^{03}$. Besides, for instance, $\lambda^{C|B} = \lambda^{26}$ denotes the default intensity of C given that B is in default, but A is not in default. By Corollary 1, the value of the *fee leg* is given by

$$y(t)B^0(t) = y(t) \sum_{k\in\mathcal{J}_F} \int_t^T P(t,s)\,q^{0k}(t,s)\,ds, \tag{45}$$

where $\mathcal{J}_F = \{0, 2\}$ consists of the states in which A is not in default. Furthermore, the value of the *protection leg* is given by

$$\overline{B}^0(t) = \lambda^C \int_t^T P(t,s)\,q^{00}(t,s)\,ds + \lambda^{C|A} \int_t^T P(t,s)\,q^{01}(t,s)\,ds. \tag{46}$$

The value of the fair CDS spread is then given by $y(t) = \overline{B}^0(t)/B^0(t)$.[22] Under the assumption of constant interest rates, this expression becomes explicit. The following proposition provides the corresponding closed-form solution. The relevant $\tilde{g}$-functions are defined as follows (see also Appendix A.4):

$$\begin{aligned} \tilde{g}^j(t,T) &= \frac{1-e^{-(r+\lambda^{j*})(T-t)}}{r+\lambda^{j*}}, \quad j = 0,1,2, \\ \tilde{g}^{0k}(t,T) &= \frac{\tilde{g}^0(t,T)-\tilde{g}^k(t,T)}{\lambda^{k*}-\lambda^{0*}}, \quad k = 1,2. \end{aligned} \tag{47}$$

Set $w^0 = \tilde{g}^0/(\tilde{g}^0 + \lambda^B\tilde{g}^{02})$ and $w^1 = \tilde{g}^{01}/(\tilde{g}^0 + \lambda^B\tilde{g}^{02})$.

Proposition 7. (CDS Spreads under Counterparty Risk) *Assume interest rates to be constant. Then we obtain $y(t) = w^0(t,T)\lambda^C + w^1(t,T)\lambda^{C|A}\lambda^A$. The effect of counterparty risk vanishes if time approaches the maturity of the CDS contract, i.e., $\lim_{t\to T} y(t) = \lambda^C$.*

Remarks.

(a) If $\lambda^A = 0$, the effect of counterparty risk is captured by w^0. Note that $w^0 \in (0, 1]$.

(b) By assumption, recovery rates in case of a default of B are zero. For this reason, $a^{26} = 0$ and $a^{47} = 0$. This assumption can be relaxed by choosing positive values for these payments upon transition. This increases the value of the protection leg. For instance, for $a^{26} \in (0, 1)$

[22] If the contract is settled at maturity, the value of the fee leg is given by $y(t)B^0(t) = y(t)\sum_{k\in\mathcal{J}_F^M}\int_t^T P(t,s)q^{0k}(t,s)\,ds$ with $\mathcal{J}_F^M = \{0,2,3,6\}$ and the value of the protection leg equals $\overline{B}^0(t) = (q^{03}(t,T) + q^{05}(t,T))P(t,T)$.

one needs to add the following term to the value of the fee leg:

$$a^{26}\lambda^{C|B}\int_t^T P(t,s)\,q^{02}(t,s)\,ds = a^{26}\lambda^{C|B}\lambda^{B}\tilde{g}^{02}(t,T), \tag{48}$$

where the payment upon transition a^{26} equals the recovery rate.

Let us consider the numerical example of a CDS contract with 5 years to maturity and a constant default-free interest rate of 5% per year. We compare our above three settings using three different scenarios: (i) no counterparty risk and no contagion, (ii) counterparty risk, but no contagion, and (iii) counterparty risk and contagion. In scenario (i), all intensities except for $\lambda^C = 0.03$ are assumed to be zero. In scenario (ii), we set $\lambda^A = 0.01$, $\lambda^B = 0.01$, and $\lambda^C = 0.03$ and assume that there exist no contagion effects, i.e., for instance that $\lambda^C = \lambda^{C|B}$. In scenario (iii), the default intensities of A and B increase by 0.005 after the first default and by additional 0.005 after the second default of any entity. The intensities of C increase by 0.01 and by additional 0.01 after the second default. If $\lambda^A = 0.01$, then the spreads are 300 basis points (bp) in scenario (ii) and 302 bp in scenario (iii). On the other hand, if $\lambda^A = 0$, then the spreads are 300 bp in scenario (i) and 293 bp in scenarios (ii) and (iii). In this example, the impact of contagion is small and around 2.3% if A is default-free and it becomes negligible if the default intensity of A is equal to 0.01. There is an important reason for this result: Since the CDS contract is at latest settled if the reference entity C has defaulted, this setting excludes contagion effects stemming from $\lambda^{B|C}$. In practice, however, a default of C can trigger a default of B. This can be modeled by splitting up state 3 into two states: In state 3_1 the reference entity C defaults and B remains solvent, whereas in state 3_2 the reference entity C defaults triggering a default of B as well.[23] If no recovery payment is made in state 3_2, one can basically disregard this state and use our above model with an adjusted default intensity for C. For instance, if B defaults with probability 0.03 if C defaults, the adjusted intensity equals $0.97\lambda_C$. This clearly widens the spread and has first-order effects. Let us consider our numerical examples from above and assume that in scenario (ii) and (iii) the protection seller B defaults with a probability of 0.015 if C defaults. If $\lambda^A = 0.01$, then the spreads are 296 bp in scenario (ii) and 298 bp in scenario (iii). On the other hand, if $\lambda^A = 0$, then the spreads are 300 bp in scenario (i) and 289 bp in scenarios (ii) and (iii). This shows that the impact is now around 3.7% if A is default-free.

[23] Formally, we allow for joint default events in the sense of Duffie and Singleton (1999b).

5.2 *N*-TO-DEFAULT BASKETS AND CONTAGION

In this section, we apply our approach to the valuation of n-to-default baskets (syn. basket credit default swaps), which are OTC-traded credit derivatives. The underlying of an n-to default basket is a pool of defaultable bonds and a payment is triggered if the n-th entity in the pool defaults. It is worthwhile to note that studying the valuation of basket CDS is interesting in itself, but the results can also be used to price the different tranches of a collateralized debt obligation (CDO). It is apparent that default correlations play a crucial role for the pricing of a basket CDS. Specifically, contagion effects influence its value significantly. For this reason, we will focus in our analysis on the pricing implications of these effects.

If our results should also be useful for the pricing of CDOs, one needs to be able to deal with baskets of a pretty large size (e.g., 100 firms). On the other hand, modeling every single entity separately and allowing for contagion at the same time increases the dimension tremendously. Therefore, it is useful to divide the pool into a number of subsets and to assume that within the subsets the firms are homogeneous. In the simplest version, one can assume that given a specific number of already defaulted firms the remaining firms are homogeneous, i.e., they default with the same intensity. Contagion can be included in this model by making the default intensities dependent on the number of defaults so far. This already allows for a couple of degrees of freedom and it needs to be tested empirically if it is really necessary to distinguish between different subsets of firms in the pool to explain prices of basket CDS. For instance, for a 20-to-default basket there are 20 different intensities which can be used to fit market prices. We wish to emphasize that a user of such a model has the option to distinguish between 20 different intensities, but in practical applications, it seems to be reasonable to reduce the number of parameters. For instance, one can assume that the default intensity jumps upwards after every fifth default or only after the first and second defaults. In our numerical examples we will try to shed some light on this issue. It is worthwhile noting that even for the simplest model with a constant default intensity for each entity the aggregate intensity is *not* constant because the individual default intensity needs to be multiplied by the number of solvent firms. Only in the very special case where the contagion effect exactly compensates this shrinking of the pool is the overall intensity constant. For a pool consisting of a large number of entities, the shrinking effect on an n-to-default basket is negligible if n is small. But for higher n or in general for smaller pools this effect may be pricing relevant.

Firstly, we consider basket credit default swaps where the firms are assumed to be homogeneous. An n-to-default basket can then be represented as a

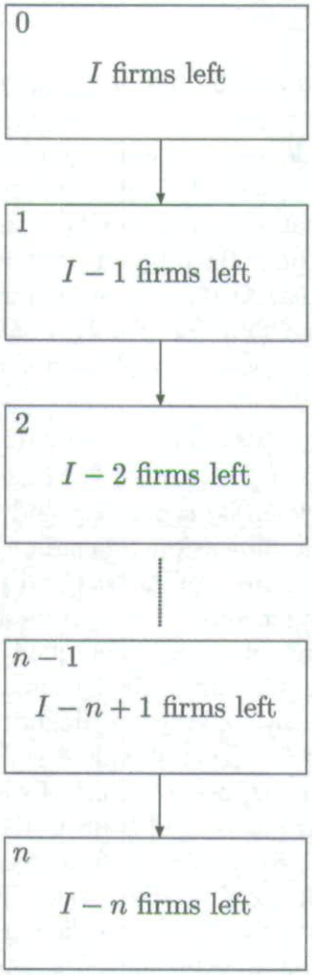

Figure 7. n-to-default basket.

Markov chain with state space $\mathcal{J} = \{0, 1, \ldots, n\}$. In this model, the labels of the states have the intuitive interpretations as the numbers of defaults so far (see Figure 7). The corresponding contract leads to a payment of one dollar, say, if during its lifetime state n is reached. In practice, one expects that this payment is made upon default of the n-th firm, but pricing can be further simplified under the assumption that the payment is made at the maturity of the contract. If the premium is paid up front, pricing of a basket CDS is equivalent to calculating the transition probabilities $q^{0n}(t, T)$ (up to discounting). The strength of our approach is that all results about calculating transition probabilities of Markov chains can be used to compute prices of n-to-default baskets. Specifically results on pure death processes can be applied.From this perspective, one can alternatively interpret the problem as follows: For a basket of I firms our problem is equivalent to computing the probability of n deaths in a population of I individuals within a certain time span. If we abstract from contagion, we are facing a linear pure death process, i.e., $\lambda^{jj+1} = (I - j)\lambda$, where λ denotes the risk-neutral default intensity of a single firm and $I - j$ equals the number of non-defaulted firms. Let us consider a problem where a contagion effect is present. It is assumed that the aggregate default intensity is linear in the number of defaults, i.e.

$$\lambda^{jj+1} = (I - j)\lambda + cj = I\lambda - j(\lambda - c) \tag{49}$$

with c modeling the contagion effect. For $\lambda > c$, the shrinking effect dominates the contagion effect, whereas the opposite is true for $\lambda < c$. Both effects cancel out if $\lambda = c$. The advantage of specification (49) is that it allows for an explicit solution of the price of basket CDS. To simplify notation, we set $\lambda^j = \lambda^{jj+1}$.

Proposition 8. (Basket CDS with Linear Contagion) *Assume λ and c to be constants. For specification (49), the transition probabilities are given by $(j + k < n)$*

$$q^{jj+k}(t, T) = \frac{\prod_{\nu=j}^{j+k-1} \lambda^{\nu}}{k!} \left(\frac{1 - e^{-(c-\lambda)(T-t)}}{c - \lambda} \right)^{k} e^{-\lambda^{j}(T-t)}, \tag{50}$$

where $\prod_{\nu=j}^{j-1} \cdots = 1$. For $\lambda = c$, this collapses into $q^{jj+k}(t, T) = \frac{(I\lambda(T-t))^k}{k!} e^{-I\lambda(T-t)}$. The time-t price of an n-to-default basket paying one dollar at maturity is given by $P(t, T)(1 - \sum_{k=0}^{n-1} q^{0k}(t, T))$.

Remarks.

(a) The following is important: Our Markov chain ends with state n which is assumed to be an absorbing state. For this reason, state n encompasses all situations where n or *more* firms have defaulted. Consequently, $q^{0n} = 1 - \sum_{k=0}^{n-1} q^{0k}$.

(b) If $c = 0$, the transition probability is equal to the transition probability of a pure death process:[24]

$$q^{0n}(t, T) = \sum_{k=0}^{I-n} \binom{I}{k} e^{-k\lambda(T-t)} (1 - e^{-\lambda(T-t)})^{I-k}. \tag{51}$$

[24] See, e.g., Karlin (1969).

The right-hand side is the sum over all events where n or more individuals have died. Note that if $U \sim B(I, e^{-\lambda(T-t)})$ is binomially distributed, the probability for $U \leq I - n$ equals $q^{0n}(t, T)$.

(c) For $\lambda = c$, the result can be rephrased as follows: If $\tilde{U}$ is exponentially distributed with parameter $I\lambda(T-t)$, the probability for $\tilde{U} = I - k$ equals $q^{0k}(t, T)$. Hence, the n-th default time is gamma-distributed with parameters n and $I\lambda$.

Note that (49) is different from a contagion effect which is linear in the individual default intensities. In this case, we obtain an aggregate intensity of

$$\tilde{\lambda}^{jj+1} = (I-j)(\lambda + \tilde{c}j) = I\lambda - j(\lambda - \tilde{c}(I-j)). \tag{52}$$

Since $\tilde{c}$ is multiplied by $I - j$, a term of order j^2 becomes relevant. To obtain the individual default intensities, one needs to divide by the number of remaining firms, $I - j$. In the first case corresponding to (49) one obtains $\lambda + cj/(I-j)$ and in the second case corresponding to (52) one gets $\lambda + \tilde{c}j$. For $c = \tilde{c}$, the contagion effect per entity induced by (49) is smaller at the beginning, $\lambda^{jj+1} \leq \tilde{\lambda}^{jj+1}$, but catches up at the end such that $\lambda^{I-1I} = \tilde{\lambda}^{I-1I}$. However, if c is chosen to be large enough, both intensities behave similarly even at the beginning and it depends on the magnitude of n which transition intensities are actually relevant. Figure 8 details the individual default intensities for the parameter choices $I = 25$, $\lambda = 0.035$, and $T - t = 5$. Specification (52) with $\tilde{c} = 0.002$ results in the straight line in the middle, whereas the two curves follow from (49) with $c = 0.03$ and $c = 0.002$.

One can also explicitly calculate the error of disregarding the shrinking effect in the pool. For simplicity, assume that $c = 0$. In this case, disregarding shrinking means that each default occurs with intensity $\lambda^{ij} = I\lambda$ (instead of $\lambda^{ij} = (I-j)\lambda$) implying that the n-th default time is gamma distributed with parameters n and $I\lambda$. The error is thus given by $(q^{0n}(t, T) - F_t^{\gamma}(T))P(t, T)$, where F_t^{γ} denotes the corresponding time-t cumulative distribution function of the gamma distribution. This error can be significant. For instance, if we consider a pool of 100 firms with individual default intensities $\lambda = 0.02$, the pricing error for a 10-to-default basket with time to maturity $T - t = 5$ equals almost 10%, where the yield of the five year zero-coupon bond is assumed to be 4%.

We wish to emphasize that our approach can also be applied using specifications different from (49). For instance, one can assume that the aggregate default intensity jumps after the first, fifth, and tenth default ("step-up contagion"). Let $I = 100$, $\lambda = 0.02$, and $T - t = 5$. The yield of the five year zero-coupon bond is assumed to be 4%. Figure 9 compares prices of

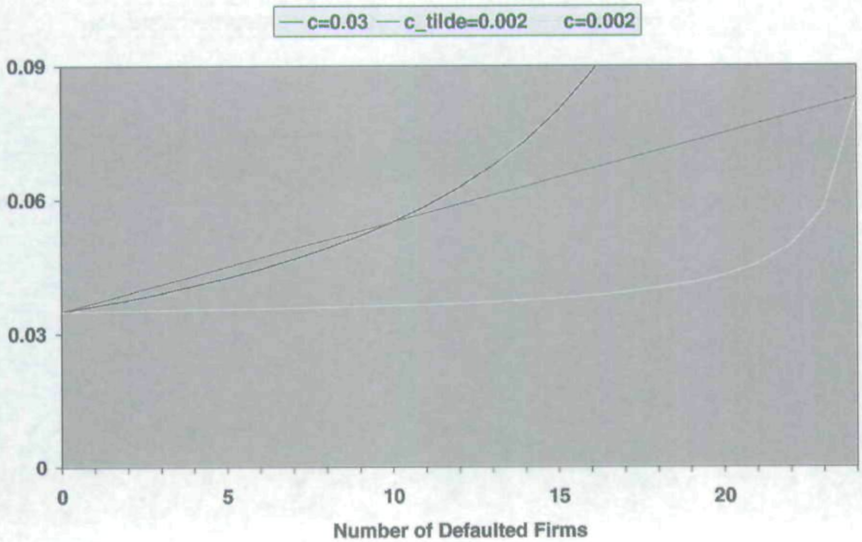

Figure 8. Individual default intensities induced by (49) and (52).

basket CDS for three different assumptions about contagion. The left line characterizes the prices if defaults occur independently. The line which is in the middle for $n \geq 10$ corresponds to (49) with $c = 0.03$. The upper line for $n \geq 10$ results if the aggregate intensity jumps up by 0.0005 after the first default, by 0.001 after the fifth default, and by 0.01 after the tenth default. It can be seen that, due to the last jump, the CDS baskets become more expensive than in the linear case for $n \geq 10$.

We now consider the situation where the pool of loans is subdivided into subpools. To keep the exposition simple, it is assumed that there are two subpools each of which consists of homogeneous firms. Three cases can occur:

(1) There is no contagion within each subpool and no contagion between firms of different subpools.
(2) There is contagion within each subpool, but no contagion between firms of different subpools.
(3) There is contagion within each subpool and contagion between firms of different subpools.

The first two cases are easier to handle than the third one, since defaults in the first pool are independent of defaults in the second pool and vice versa. Under this condition, one can compute transition probabilities separately. The interesting case is the third one where defaults of different pools are dependent. Since in this case no closed-form solutions exist, one needs to solve

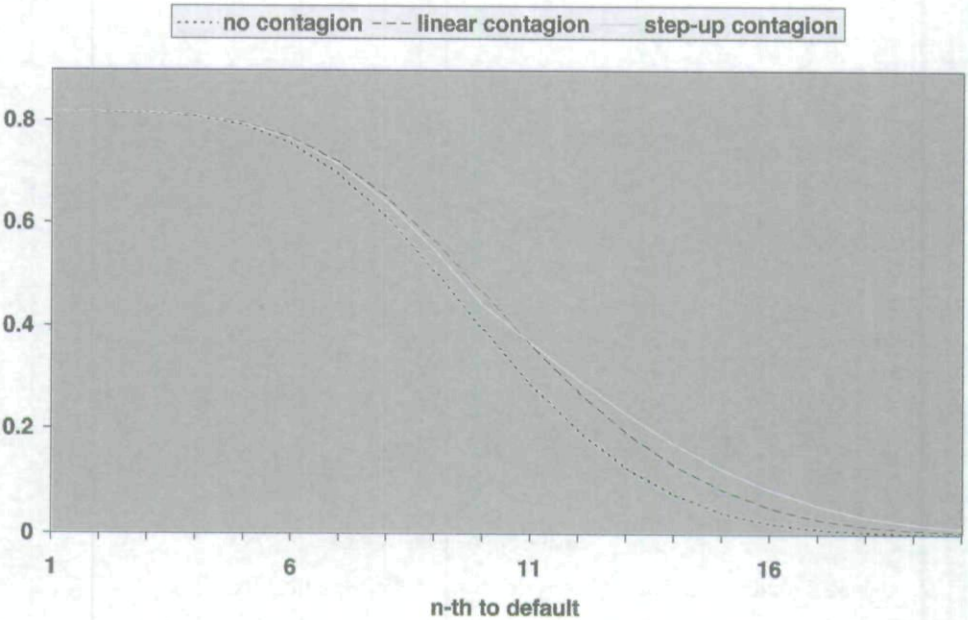

Figure 9. Prices of basket CDS for different assumptions about contagion.

the pricing problem numerically and no computational advantage is achieved by assuming linear contagion effects.[25] However, to make our results easier to compare to the previous results, we consider a generalization of specification (49) where the aggregate default intensities of the first and second subpools are given by

$$\lambda_1^a(j_1, j_2) = (I_1 - j_1)\lambda_1 + c_{11} j_1 + c_{12} j_2$$
$$\text{and} \tag{53}$$
$$\lambda_2^a(j_1, j_2) = (I_2 - j_2)\lambda_2 + c_{21} j_1 + c_{22} j_2.$$

I_m denotes the initial number of firms in subpool m, j_m the number of firms in subpool m that are already in default, λ_m the initial individual default intensity in pool m, and c_{m1} as well as c_{m2} are contagion coefficients. Figure 10 depicts the corresponding Markov chain for a 3-to-default basket. Each state is specified by the pair (j_1, j_2), i.e., by the number of already defaulted firms

[25] In situations where no closed-form solutions are available, we have solved the Kolmogorov backward equation by discretization (see Appendix A.4). This is very efficient and allows for almost real-time applications.

in each subpool. The generator matrix is given by

$$\begin{pmatrix}
-\lambda^{0*} & \lambda^{01} & \lambda^{02} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda^{1*} & 0 & \lambda^{13} & \lambda^{14} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\lambda^{2*} & 0 & \lambda^{24} & \lambda^{25} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda^{3*} & 0 & 0 & \lambda^{36} & \lambda^{37} & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda^{4*} & 0 & 0 & \lambda^{47} & \lambda^{48} & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda^{5*} & 0 & 0 & \lambda^{58} & \lambda^{59} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{54}$$

where λ^{jj} equals minus the sum of the intensities in row j. Since states 6–9 are absorbing states, the corresponding lines have zero entries only. As before, we set $T - t = 5$. The yield of the five year zero-coupon bond is assumed to be 4%. Besides, $I_1 = I_2 = 50$, $\lambda_1 = 0.025$, $\lambda_2 = 0.015$. Figure 11 details the prices of basket CDS for different assumptions about contagion. The case labeled by "no contagion" corresponds to $c_{11} = c_{22} = c_{12} = c_{21} = 0$, whereas the case labeled by "independent subpools" corresponds to $c_{11} = 0.035$, $c_{22} = 0.025$, and $c_{12} = c_{21} = 0$. Since $\lambda_1 - c_{11} = \lambda_2 - c_{22} = 0.01$, the second case is equivalent to a situation with only one pool and $\lambda - c = 0.01$. This follows from the independence between the pools and Proposition 8. Therefore, the no-contagion case of Figure 9 is identical to this second example. This simplifies a comparison of Figures 9 and 11. The last case labeled by "contagion" corresponds to $c_{11} = 0.035$, $c_{12} = 0.02$, $c_{21} = 0.015$, and $c_{22} = 0.025$. The additional contagion effect is obviously pricing relevant. For instance, the

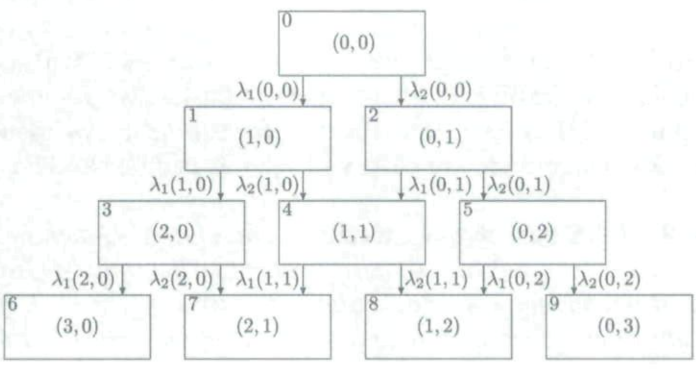

Figure 10. 3-to-default basket for two subpools.

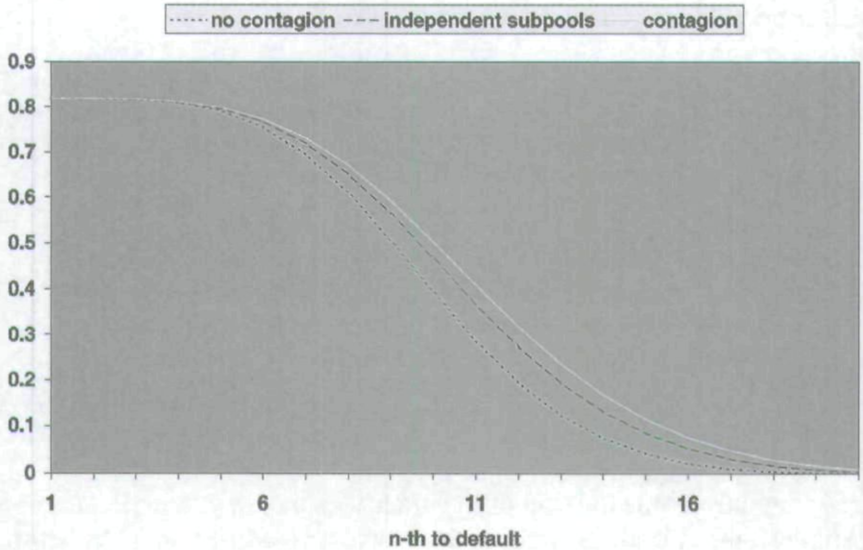

Figure 11. Prices of basket CDS for different assumptions about contagion and two subpools.

price of a 10-to-default basket increases by 7.7%. Finally, Figure 12 shows the prices of a 10-to-default basket for different values of c_{12} and c_{21}. All other variables are fixed as in the contagion case above. It can be seen that contagion effects between the two subpools can have huge pricing impacts.

So far we have assumed that the default intensities of the entities do not depend on macroeconomic variables like interest rates. We relax this assumption and consider intensities[26]

$$\lambda^j = \lambda^{jj+1} = c_j \lambda, \tag{55}$$

where c_j are state-dependent constants and λ is a stochastic process. This is a special case of (38) with $K+1=n$, $\hat{c}_j = c_j$ and $\tilde{c}_j = 0$. For instance, one can choose c_j to be of the form (49), i.e., $c_j = (I-j)k_1 + k_2 j$ with constants k_1 and k_2. To calculate the price of an n-to-default basket, we assume that n is an absorbing state and thus set $c_n = 0$. Further, for simplicity, we assume $c_k \neq c_l$. The special case where there are pairs with $c_k = c_l$ can be handled similarly.

Proposition 9. (CDS Baskets and Stochastic Contagion) *Under the assumption that default risk is independent of interest rate risk, the time-t price of an n-to-default basket paying one dollar at maturity is given by* $P(t,T)E_t[q^{0k|\lambda}(t,T)]$, *where the transition probabilities are given by the following backward algorithm:*

[26] We thank the referee for suggesting this application.

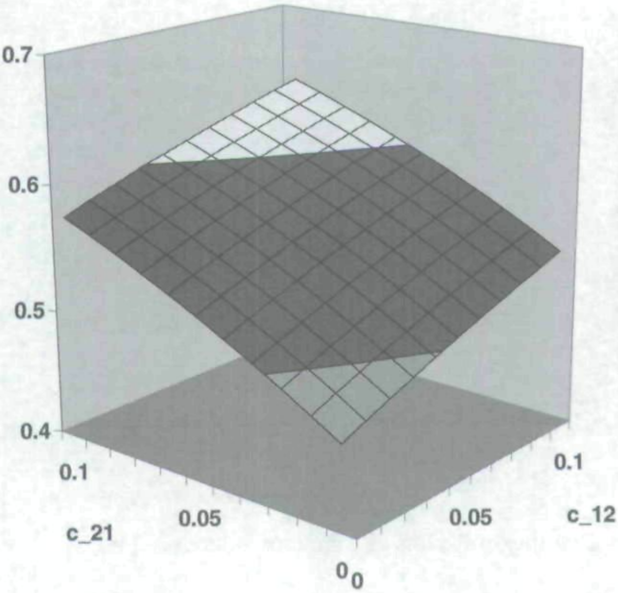

Figure 12. Prices of a 10-to-default basket for two subpools each consisting of 50 firms and different values of c_{12} and c_{21}.

Firstly, $q^{n-1n|\lambda}(t,T) = [n-1,n]_{t,T}$. *Recursively,* $q^{k-1n|\lambda}(t,T)$ *can be calculated from* $q^{kn|\lambda}(t,T)$ *by replacing every* $[k,l]_{t,T}$ *by*

$$(k,l)\cdot\left\{[k-1,l]_{t,T}-[k-1,k]_{t,T}\right\}. \tag{56}$$

Remarks.

(a) Note that the backward algorithm also holds in the general case (38).
(b) We have decided to present the backward algorithm as well, since it is faster to implement for our current application than the forward algorithm of Proposition 6.
(c) If λ is affine, we again obtain closed-form solutions for the values of n-to-default baskets (see Remark (b) to Proposition 6).

5.3 A SIMPLE METHOD TO QUANTIFY DEFAULT RISK IN LOAN PORTFOLIOS

Moody's binomial expansion method to quantify the default risk in loan portfolios is widely used in practice.[27] This is an ad hoc approach using an adjusted binomial distribution to model loss distributions. To include

[27] See, e.g., Lando (2004), pp. 227f.

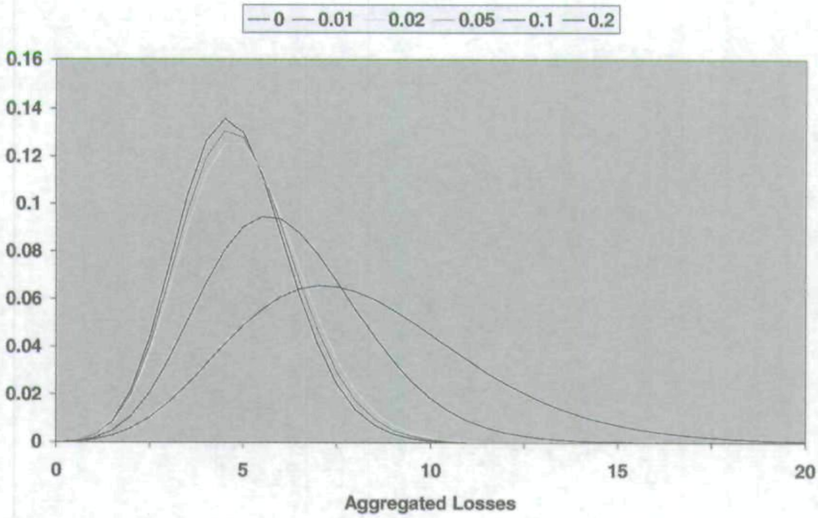

Figure 13. Densities of the loss distributions induced by (49).

contagion effects, firms are grouped into buckets and it is assumed that all entities of one bucket default at the same time. The number of buckets is called the diversity score of the model. Within a bucket there is perfect default dependence, but independence across buckets. The larger the buckets (i.e., the smaller the diversity score), the greater becomes the overall default dependence, but this reduces the number of possible default events as well, i.e., the model and especially the loss distributions become coarser. For a portfolio of 100 entities, grouping into 5 buckets of entities reduces the number of default events from 101 to 21. Furthermore, this approach lacks a sound theoretical justification. Our model induced by specification (49) is in some sense similar to Moody's model because it leads to a binomial distribution in the special case when defaults are uncorrelated. However, in contrast to Moody's approach, contagion is introduced in a model-based way and the grouping procedure of Moody's method is avoided. Nevertheless, all relevant values can be calculated explicitly. We now demonstrate what the loss distributions implied by (49) look like. We assume that the number of obligors in a portfolio equals $I = 100$, the initial intensity is $\lambda = 0.02$, the time to maturity is $T - t = 5$ years, and the loss given default equals 50%. Each loan has a notional of one million dollars. For several choices of the contagion factor c, Figure 13 depicts the loss densities. They are smoothed because actually we are dealing with a discrete distribution. The almost bell-shaped curve on the left-hand side corresponds to the choice $c = 0$, i.e., defaults are uncorrelated. It can nicely be seen how the contagion effect tilts the distributions towards increasing skewness.

6. Conclusion

In this paper, we have developed a unifying framework to handle various types of credit risks such as contagion, counterparty risk, or bankruptcy risk as opposed to default risk. The key idea has been to describe the states of the market for defaultable assets via states of a suitable finite state Markov chain. Credit events can then be interpreted as transitions of the Markov chain from one state into another state. This gives us access to a lot of useful results on Markov chains. One of our key result states that all contingent claims satisfy a particular system of partial differential equations. We have used this result to study several applications including a sophisticated model for a firm being in financial distress and contagion effects. In the literature, it has been argued that the model of Jarrow and Yu (2001) is overparametrized, since contagion effects between every two firms under consideration are modeled by an additional parameter. We have demonstrated that our approach simplifies the task of reducing the number of parameters. As an example, we have presented certain types of linear and stochastic contagion effects simplifying the pricing of credit derivatives tremendously. These parametrizations can also be applied to the problem of quantifying the default risk in a loan portfolio.

Appendix

Proof of Proposition 1. By the definition of a conditional expected value, we obtain

$$B(t) = \mathrm{E}\Big[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s) \Big| Y(t), Z(t)\Big] = \sum_j B^j(t)\mathbf{1}_{\{Z(t)=j\}}, \tag{57}$$

where $B^j(t) := \mathrm{E}[\int_{(t,T]} e^{-\int_t^s r(u)\,du} dA(s) | Y(t), Z(t) = j]$. Due to the Markov property of (Y, Z), there exist functions $\mathcal{B}^j$ such that $B^j(t) = \mathcal{B}^j(t, Y(t))$ implying $B(t) = \mathcal{B}^{Z(t)}(t, Y(t))$. If the functions $\mathcal{B}^j$, $j \in \mathcal{J}$, are in $C^{1,2}$, we can apply Ito's product rule, where we drop the dependencies of $\mathcal{B}^j$ and its derivatives on Y:[28]

$$dB(t) = \sum_j \mathbf{1}_{\{Z(t-)=j\}} d\mathcal{B}^j(t) + \sum_j \mathcal{B}^j(t-) d\mathbf{1}_{\{Z(t)=j\}}$$

[28] Note that we can drop the minus in $\mathbf{1}_{\{Z(t-)=j\}}$ because one can change the integrand of a Lebesgue and an Ito integral on a null set. Besides, we use the fact that B^j is continuous in t and Y.

$$= \sum_j \mathbf{1}_{\{Z(t)=j\}} \Big\{ \big(\mathcal{B}_t^j(t) + \alpha(t)\mathcal{B}_y^j(t) + 0.5\,\mathrm{tr}\big[\beta(t)^T \mathcal{B}_{yy}^j(t)\beta(t)\big]\big)dt$$

$$+\mathcal{B}_y^j(t)\beta(t)dW(t)\Big\} + \sum_j \mathcal{B}^j(t) \sum_{k\neq j} \big(dN^{kj}(t) - dN^{jk}(t)\big), \qquad (58)$$

where $N^{jk}(t) = \#\{s|s \in (0,t], Z(s-) = j, Z(s) = k\}$ counts the number of jumps from state j into state $k \neq j$. We now obtain

$$\sum_j \mathcal{B}^j(t) \sum_{k\neq j} \big(dN^{kj}(t) - dN^{jk}(t)\big) = \sum_j \sum_{k\neq j} (\mathcal{B}^k(t) - \mathcal{B}^j(t))dN^{jk}(t)$$

$$= \sum_j \sum_{k\neq j} (\mathcal{B}^k(t) - \mathcal{B}^{Z(t-)}(t))dN^{jk}(t)$$

$$= \sum_{k\neq Z(t-)} (\mathcal{B}^k(t) - \mathcal{B}^{Z(t-)}(t)) \sum_{j\neq k} dN^{jk}(t)$$

$$= \sum_{k\neq Z(t-)} (\mathcal{B}^k(t) - \mathcal{B}^{Z(t-)}(t))dN^k(t), \qquad (59)$$

since by definition of N^{jk}, we can replace $\mathcal{B}^j$ by $\mathcal{B}^{Z(t-)}$. Hence, we arrive at:

$$dB(t) = \big(\mathcal{B}_t^{Z(t)}(t) + \alpha(t)\mathcal{B}_y^{Z(t)}(t) + 0.5\,\mathrm{tr}\big[\beta(t)^T \mathcal{B}_{yy}^{Z(t)}(t)\beta(t)\big]\big)dt$$

$$+\mathcal{B}_y^{Z(t)}(t)\beta(t)dW(t) + \sum_{k\neq Z(t-)} (\mathcal{B}^k(t) - \mathcal{B}^{Z(t-)}(t))dN^k(t) \qquad (60)$$

which gives the desired result. ■

Proof of Theorem 1. By definition,

$$\mathcal{Y}(t) := \mathrm{E}\Big[\int_0^T e^{-\int_0^s r(u)\,du} dA(s)\Big|\mathcal{F}_t\Big]. \qquad (61)$$

is a Q-martingale which has the representation

$$\mathcal{Y}(t) = \int_0^t e^{-\int_0^s r(u)\,du} dA(s) + e^{-\int_0^t r(u)\,du} B(t) \qquad (62)$$

and is said to be the normalized gain process. To simplify notations, in the following we drop arguments. Due to Proposition 1, we obtain

$$d\,\mathcal{Y} = e^{-\int_0^\cdot r(u)\,du}\Big[dA + dB - Brdt\Big]$$

$$
\begin{aligned}
&= e^{-\int_0^t r(u)\,du}\Big[c^Z dt + \sum_{k\neq Z^-} a^{Z^-k} dN^k + a_T^Z d\varepsilon_T + \mathcal{B}_t^Z dt + \mathcal{B}_y^Z \alpha dt \\
&\quad +0.5\,\mathrm{tr}[\beta^T \mathcal{B}_{yy}^Z \beta]dt + \mathcal{B}_y^Z \beta dW + \sum_{k\neq Z^-} (\mathcal{B}^k - \mathcal{B}^{Z^-}) dN^k - \mathcal{B}^Z r dt\Big] \\
&= e^{-\int_0^t r(u)\,du}\Big[a_T^Z d\varepsilon_T + c^Z dt + \mathcal{B}_t^Z dt + \mathcal{B}_y^Z \alpha dt + 0.5\,\mathrm{tr}[\beta^T \mathcal{B}_{yy}^Z \beta]dt \\
&\quad -\mathcal{B}^Z r dt + \sum_{k\neq Z^-} (a^{Z^-k} + \mathcal{B}^k - \mathcal{B}^{Z^-}) \lambda^{Z^-k} dt \\
&\quad + \sum_{k\neq Z^-} (a^{Z^-k} + \mathcal{B}^k - \mathcal{B}^{Z^-}) dM^k + \mathcal{B}_y^Z \beta dW\Big],
\end{aligned} \tag{63}
$$

where $dM^k = dN^k - \lambda^{Z^-k} dt$. Since $\mathcal{Y}$ is a (local) Q-martingale, the dt-term needs to vanish implying (13). ■

Proof of Proposition 2. We denote the prices by B_i, $i = 1, 2, 3$. The first claim follows from

$$
\begin{aligned}
B_1(t) &= \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du} c^{Z(s)}(s)\,ds\Big] \\
&= \sum_{k\in\mathcal{J}} \int_t^T \mathrm{E}_t\Big[e^{-\int_t^s r_u\,du} c^k(s) \mathbf{1}_{\{Z(s)=k\}}\Big]\,ds \\
&= \sum_{k\in\mathcal{J}} \int_t^T \mathrm{E}_t\big[e^{-\int_t^s r_u\,du} c^k(s) \mathrm{E}_t[\mathbf{1}_{\{Z(s)=k\}} | \mathcal{F}_{T*}^Y]\big]\,ds \\
&= \sum_{k\in\mathcal{J}} \int_t^T \mathrm{E}_t\big[e^{-\int_t^s r_u\,du} c^k(s) q^{jk|Y}(t,s)\big]\,ds.
\end{aligned} \tag{64}
$$

The second claim is valid because

$$
\begin{aligned}
B_2(t) &= \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du} \sum_{k\neq Z(s-)} a^{Z(s-)k}(s)\,dN^k(s)\Big] \\
&= \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du} \sum_{\nu\in\mathcal{J}} \mathbf{1}_{\{Z(s-)=\nu\}} \sum_{k\neq\nu} a^{\nu k}(s)\,dN^k(s)\Big] \\
&= \sum_{\nu\in\mathcal{J}} \sum_{k\neq\nu} \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du} a^{\nu k}(s) \mathbf{1}_{\{Z(s-)=\nu\}}\,dN^{\nu k}(s)\Big]
\end{aligned}
$$

$$= \sum_{\nu\in\mathcal{J}}\sum_{k\neq\nu} \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du} a^{\nu k}(s)\mathrm{E}_t[\mathbf{1}_{\{Z(s-)=\nu\}}|\mathcal{F}^Y_{T*}]\lambda^{\nu k}(s)\,ds\Big]$$

$$= \sum_{\nu\in\mathcal{J}}\sum_{k\neq\nu} \int_t^T \mathrm{E}_t\Big[e^{-\int_t^s r_u\,du} a^{\nu k}(s)\, q^{j\nu|Y}(t,s)\lambda^{\nu k}(s)\Big]\,ds. \tag{65}$$

The third claim follows similarly. ■

CALCULATING TRANSITION PROBABILITIES

Assume that transition intensities λ^{jk} are constant. We wish to calculate q^{0n} for a Markov chain where for every two states l and m a positive probability $q^{lm} > 0$ implies $q^{ml} = 0$ ("there is no way back"). For instance, this condition is satisfied for a pure death process. We define recursively

$$\begin{aligned}
g^j(t,T) &= q^{jj}(t,T) = e^{-\lambda^{j*}(T-t)},\\
g^{jk}(t,T) &= \frac{g^j(t,T)-g^k(t,T)}{\lambda^{k*}-\lambda^{j*}},\\
&\dots\\
g^{i_1\dots i_m jk}(t,T) &= \frac{g^{i_1\dots i_m j}(t,T)-g^{i_1\dots i_m k}(t,T)}{\lambda^{k*}-\lambda^{j*}}, \quad m\geq 1.
\end{aligned} \tag{66}$$

In all our applications we usually have $\lambda^{k*}\neq\lambda^{j*}$, which we assume to hold throughout this appendix. This allows us to avoid a series of tedious subcases that are of minor relevance. Later on, we comment on how these special cases can be handled.

Denote the set of all paths from state 0 to state k by $P(0,k)$. A path from 0 to k, $p(0,k) = (0, p_1, \dots, p_m, k)$, is a vector of states. For instance, in Figure 1 there are two paths starting in state 0 and ending in state 3, $p^1(0,3) = (0,1,3)$ and $p^2(0,3) = (0,2,3)$ such that $P(0,3) = \{p^1(0,3), p^2(0,3)\}$. Furthermore, for a path $p(0,k)$ we set $\lambda^{p(0,k)} = \lambda^{0p_1}\lambda^{p_1p_2}\dots\lambda^{p_mk}$. Then we get the following result.

Proposition 10. (Transition Probabilities)

$$q^{0k}(t,T) = \sum_{p(0,k)\in P(0,k)} \lambda^{p(0,k)} g^{p(0,k)}(t,T), \tag{67}$$

where we set $\lambda^{p(0,0)} = 1$ *for* $k = 0$.

Proof. Let $p(p_1, k)$ denote the sub-path of $p(0, k)$ starting at p_1 instead of 0. By induction, one can show that

$$\int_t^T e^{-\lambda^{0*}(s-t)} g^{p(p_1,k)}(s, T)\, ds = g^{p(0,k)}(t, T). \tag{68}$$

We now prove the claim of the proposition by induction. For $k = 0$ the claim is obviously valid. For $k > 0$ consider the Kolmogorov backward equation where we drop arguments:

$$\begin{aligned} q_t^{0k} &= \lambda^{0*} q^{0k} - \sum_{\nu \neq 0} \lambda^{0\nu} q^{\nu k} \\ &\overset{\text{Induction}}{=} \lambda^{0*} q^{0k} - \sum_{\nu \neq 0} \lambda^{0\nu} \sum_{p(\nu,k)} \lambda^{p(\nu,k)} g^{p(\nu,k)}. \end{aligned} \tag{69}$$

Integrating leads to

$$\begin{aligned} q^{0k}(t, T) &= \sum_{\nu \neq 0} \sum_{p(\nu,k)} \lambda^{0\nu} \lambda^{p(\nu,k)} \int_t^T e^{-\lambda^{0*}(s-t)} g^{p(\nu,k)}(s, T)\, ds \\ &= \sum_{\nu \neq 0} \sum_{p(\nu,k)} \lambda^{0\nu} \lambda^{p(\nu,k)} g^{0p(\nu,k)}(t, T) \\ &= \sum_{p(0,k)} \lambda^{p(0,k)} g^{p(0,k)}(t, T), \end{aligned} \tag{70}$$

where $0p(\nu, k)$ denotes the path starting in 0 and then following $p(\nu, k)$. ■

Remarks.

(a) One can adapt the algorithm if $\lambda^{k*} = \lambda^{j*}$ for some $j \neq k$. For instance, by applying L'Hospital's rule, we obtain $g^{jk}(t, T) = (T - t)e^{-\lambda^{k*}(T-t)}$.

(b) Alternatively, as outlined in Lando (1998), one can solve the Kolmogorov equations numerically by discretization. Fix a partition $t = t_0 < t_1 < \cdots < t_n = T$ and define recursively

$$\begin{aligned} q^{|Y}(T, T) &= I, \\ q^{|Y}(t_i, T) &= q^{|Y}(t_{i+1}, T) - \Lambda(t_{i+1}) q^{|Y}(t_{i+1}, T)(t_i - t_{i+1}). \end{aligned} \tag{71}$$

This can also be done very efficiently.

It is also of interest to calculate the value of contingent coupon payments as well as payments upon transition. Assume that interest rates are constant and define

$$\tilde{g}^j(t,T) = \frac{1-e^{-(r+\lambda^{j*})(T-t)}}{r+\lambda^{j*}},$$
$$\tilde{g}^{jk}(t,T) = \frac{\tilde{g}^j(t,T)-\tilde{g}^k(t,T)}{\lambda^{k*}-\lambda^{j*}}, \tag{72}$$
$$\ldots$$
$$\tilde{g}^{i_1\ldots i_m jk}(t,T) = \frac{\tilde{g}^{i_1\ldots i_m j}(t,T)-\tilde{g}^{i_1\ldots i_m k}(t,T)}{\lambda^{k*}-\lambda^{j*}}, \quad m \geq 1.$$

According to Corollary 1, we basically need to calculate $\int_t^T e^{-r(s-t)}q^{0k}(t,s)\,ds$ and we get the following result.

Proposition 11. *For constant interest rates, we have*

$$\int_t^T e^{-r(s-t)}q^{0k}(t,s)\,ds = \sum_{p(0,k)\in P(0,k)} \lambda^{p(0,k)}\tilde{g}^{p(0,k)}(t,T), \tag{73}$$

where we set $\lambda^{p(0,0)} = 1$ for $k = 0$.

Proof. By Proposition 10, we obtain

$$\int_t^T e^{-r(s-t)}q^{0k}(t,s)\,ds = \sum_{p(0,k)\in P(0,k)} \lambda^{p(0,k)}\int_t^T e^{-r(s-t)}g^{p(0,k)}(t,s)\,ds. \tag{74}$$

We need to show that $\int_t^T e^{-r(s-t)}g^{p(0,k)}(t,s)\,ds = \tilde{g}^{p(0,k)}(t,T)$, which is done by induction. Integrating leads to $\int_t^T e^{-r(s-t)}g^i(t,s)\,ds = \tilde{g}^i(t,T)$. Using this result and the definitions of $p(0,k)$ as well as $\tilde{g}$, we arrive at

$$\int_t^T e^{-r(s-t)}g^{p(0,p_{m-1})p_m k}(t,s)\,ds$$
$$= \int_t^T e^{-r(s-t)}\frac{g^{p(0,p_{m-1})p_m}(t,s)-g^{p(0,p_{m-1})k}(t,s)}{\lambda^{k*}-\lambda^{p_m*}}\,ds$$
$$\overset{\text{Induction}}{=} \frac{\tilde{g}^{p(0,p_{m-1})p_m}(t,s)-\tilde{g}^{p(0,p_{m-1})k}(t,s)}{\lambda^{k*}-\lambda^{p_m*}} = \tilde{g}^{p(0,k)}(t,T), \tag{75}$$

which gives the desired result. ■

Appendix to the Applications

Proof of Proposition 3. Firstly, recall from Duffie et al. (2000) that the following ODEs play a central role

$$\begin{aligned} \mathcal{B}_t(t,T) &= \rho_1 - K_1\mathcal{B}(t,T) - 0.5\mathcal{B}(t,T)^T H_1 \mathcal{B}(t,T), \\ \mathcal{A}_t(t,T) &= \rho_0 - K_0\mathcal{B}(t,T) - 0.5\mathcal{B}(t,T)^T H_0 \mathcal{B}(t,T), \end{aligned} \tag{76}$$

and

$$\begin{aligned} -\mathcal{B}_t(t,s) &= K_1^T\mathcal{B}(t,s) + \mathcal{B}(t,s)^T H_1 \mathcal{B}(t,s), \\ -\mathcal{A}_t(t,s) &= K_0\mathcal{B}(t,s) + \mathcal{B}(t,s)^T H_0 \mathcal{B}(t,s). \end{aligned} \tag{77}$$

Due to the RMV assumption, PDE (30) can be rewritten as follows:[29]

$$B_t^0 = (r + l^A\lambda^A + \lambda^B)B^0 - \alpha B_y^0 - 0.5\,\mathrm{tr}\big[B_{yy}^0\beta\beta^T\big] - \lambda^B B^2 \tag{78}$$

with $B^2(t) = \mathrm{E}_t[e^{-\int_t^T R_u^{A'}\,du}] = e^{\tilde{A}^2(t,T)+\tilde{B}^2(t,T)Y(t)}$, where due to Duffie et al. (2000) the functions $\tilde{A}^2$ and $\tilde{B}^2$ satisfy the ODEs (76) as stated in the proposition. The solution to (78) reads

$$B^0(t) = \int_t^T \mathrm{E}_t\big[e^{-\int_t^s R_u^A\,du}\lambda_s^B B_s^2\big]\,ds + \mathrm{E}_t\big[e^{-\int_t^T R_u^A\,du}\big]. \tag{79}$$

For the second term, we get $\mathrm{E}_t\big[e^{-\int_t^T R_u^A\,du}\big] = e^{\tilde{A}^0(t,T)+\tilde{B}^0(t,T)Y(t)}$, where $\tilde{A}^0$ and $\tilde{B}^0$ are defined in the proposition. The expectation in the first term can be rewritten as follows:

$$\begin{aligned} \mathrm{E}_t\big[e^{-\int_t^s R_u^A\,du}\lambda_s^B B_s^2\big] &= \rho_0^B e^{\tilde{A}^2(s,T)}\mathrm{E}_t\big[e^{-\int_t^s R_u^A\,du}e^{\tilde{B}^2(s,T)Y(s)}\big] \\ &\quad + \rho_1^B e^{\tilde{A}^2(s,T)}\mathrm{E}_t\big[e^{-\int_t^s R_u^A\,du}Y_s\, e^{\tilde{B}^2(s,T)Y(s)}\big]. \end{aligned} \tag{80}$$

Applying Duffie et al. (2000) again leads to the desired result. ■

Proof of Proposition 4. By shifting the indices, one can show that the forward algorithm of Proposition 6 and the backward algorithm of Proposition 9 are equivalent. It is thus sufficient to prove the backward alorithm, which is done in the proof of Proposition 9. ■

Proof of Proposition 5. The representation of the legs follows from Proposition 2. Under the assumption of constant interest rates, integrals of the form $\int_t^T e^{-r(s-t)}\,q^{0k}(t,s)\,ds$, $k = 0, 1, 2$, need to be calculated which can

[29] Note that $\mathrm{tr}\big[B_{yy}^0\beta\beta^T\big] = \mathrm{tr}\big[\beta^T B_{yy}^0\beta\big]$.

be done by applying Proposition 10:

$$\int_t^T e^{-r(s-t)}\, q^{00}(t,s)\, ds = \tilde{g}^0(t,T)$$

$$\int_t^T e^{-r(s-t)}\, q^{01}(t,s)\, ds = \lambda^A \tilde{g}^{01}(t,T) = \lambda^A \frac{\tilde{g}^0(t,T) - \tilde{g}^1(t,T)}{\lambda^{1*} - \lambda^{0*}}$$

$$\int_t^T e^{-r(s-t)}\, q^{02}(t,s)\, ds = \lambda^B \tilde{g}^{02}(t,T) = \lambda^B \frac{\tilde{g}^0(t,T) - \tilde{g}^2(t,T)}{\lambda^{2*} - \lambda^{0*}} \tag{81}$$

where $\tilde{g}^j(t,T) = \frac{1-e^{-(r+\lambda^{j*})(T-t)}}{r+\lambda^{j*}}$ and $\lambda^{0*} = \lambda^A + \lambda^B + \lambda^C$, $\lambda^{1*} = \lambda^{B|A} + \lambda^{C|A}$, as well as $\lambda^{2*} = \lambda^{A|B} + \lambda^{C|B}$. Since $y(t) = \overline{B}^0(t)/B^0(t)$, using the definitions of w^0 and w^1 leads to the representations for y. Finally, we prove $\lim_{t\to T} y^{(i)}(t) = \lambda^C$ for $i = 2, 3$ by showing that $\lim_{t\to T} w^0(t,T) = 1$ and $\lim_{t\to T} w^1(t,T) = 0$. By l'Hospital's rule, we obtain $\tilde{g}^j(t,T)/\tilde{g}^0(t,T) \to 1$ for $t \to T$ and $j = 1, 2$. This implies $\tilde{g}^{0j}(t,T)/\tilde{g}^0(t,T) = (1 - \tilde{g}^j(t,T)/\tilde{g}^0(t,T))/(\lambda^{j*} - \lambda^{0*}) \to 0$ for $t \to T$ and thus

$$w^0(t,T) = \frac{1}{1 + \lambda^B \frac{\tilde{g}^{02}(t,T)}{\tilde{g}^0(t,T)}} \to 1, \qquad w^1(t,T) = \frac{\frac{\tilde{g}^{01}(t,T)}{\tilde{g}^0(t,T)}}{1 + \lambda^B \frac{\tilde{g}^{02}(t,T)}{\tilde{g}^0(t,T)}} \to 0. \tag{82}$$

■

Proof of Proposition 6. Consider the Kolmogorov backward equation:

$$q_t^{jj+k+1}(t,T) = \lambda^j q^{jj+k+1}(t,T) - \lambda^j q^{j+1j+k+1}(t,T). \tag{83}$$

Since $q^{j+1j+1}(t,T) = e^{-\lambda^{j+1}(T-t)}$, the claim follows for $k = 0$ by integrating the equation. For $k > 0$, by induction, we obtain

$$\begin{aligned} q^{jj+k+1}(t,T) &= \frac{\prod_{\nu=j}^{j+k} \lambda^\nu}{k!}\, e^{-\lambda^j(T-t)} \int_t^T e^{-(c-\lambda)(T-s)} \left(\frac{1-e^{-(c-\lambda)(T-s)}}{c-\lambda}\right)^k ds \\ &= \frac{\prod_{\nu=j}^{j+k} \lambda^\nu}{k!}\, e^{-\lambda^j(T-t)} \frac{1}{k+1} \left(\frac{1-e^{-(c-\lambda)(T-s)}}{c-\lambda}\right)^{k+1}, \end{aligned} \tag{84}$$

which gives the desired result. For $\lambda = c$ an application of L'Hospital's rule leads to the result. ■

Proof of Proposition 7. One can show by induction that every transition probability has a representation

$$q^{kn|\lambda}(t,T)$$

$$= \sum_{\nu} \underbrace{(n-1, m_{1\nu}) \cdot (n-2, m_{2\nu}) \cdot \dots \cdot (k+1, m_{n-k-1\nu})}_{\text{const}} \cdot [k, m_{n-k\nu}]_{t,T}, \tag{85}$$

where $m_{i\nu} \in \{k+1, k+2, \dots, n\}$ and $m_{i\nu} > n-i$, $i = 1, \dots, n-k$. Furthermore, the solution of the Kolmogorov backward equation for $q^{k-1n|\lambda}(t,T)$ reads

$$q^{k-1n|\lambda}(t,T) = \int_t^T e^{-\int_t^s \lambda_u^{k-1}\,du} \lambda_s^{k-1} q^{kn|\lambda}(s,T)\,ds. \tag{86}$$

Hence, the claim is proved if

$$\int_t^T e^{-\int_t^s \lambda_u^{k-1}\,du} \lambda_s^{k-1} [k, m_{n-k\nu}]_{s,T}\,ds$$

$$= (k, m_{n-k\nu}) \big\{ [k-1, m_{n-k\nu}]_{t,T} - [k-1, k]_{t,T} \big\}, \tag{87}$$

which follows by integrating the left-hand side. ■

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