

Stochastic Dominance Bounds on American Option Prices in Markets with Frictions*

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Abstract. We derive equilibrium restrictions on the range of the transaction prices of American options on the stock market index and index futures. Trading over the lifetime of the options is accounted for, in contrast to earlier single-period results. The bounds on the reservation purchase price of American puts and the reservation write price of American calls are tight. We allow the market to be incomplete and imperfect due to the presence of proportional transaction costs in trading the underlying security and due to bid-ask spreads in option prices. The bounds may be derived for any given probability distribution of the return of the underlying security and admit price jumps and stochastic volatility. We assume that at least some of the traders maximize a time- separable utility function. The bounds are derived by applying the weak notion of stochastic dominance and are independent of a trader's particular utility function and initial portfolio position.

JEL Classification: G13

1. Introduction

Many over-the-counter and exchange-traded call and put options are *American-style*. The CBOE-listed stock and S&P 100 index options are examples of American options. This is the first paper to apply stochastic dominance arguments to derive bounds on the prices of *American* options in the presence of transaction costs. It is also the first paper to derive such bounds on the prices of American *index futures* options, such as the CME-listed S&P 500 index futures options. Since the stock is the only primary risky asset in the traders' portfolios, it has the natural interpretation as the stock market portfolio or index.

Option pricing models often abstract from market *incompleteness*. They also abstract from market *imperfections* such as bid-asked spreads, brokerage

* We thank the referee, the editor and workshop participants for their insightful comments and constructive criticism. We also thank Michal Czerwonko for his able research assistance. We remain responsible for errors and omissions. Constantinides acknowledges financial support from the Center for Research in Security Prices of the University of Chicago and Perrakis from the Social Sciences and Humanities Research Council of Canada.

fees and execution costs, collectively referred to as transaction costs. How important are these abstractions? With dynamic market incompleteness the concept of the no-arbitrage option price is undefined. With market imperfections, the concept of the no-arbitrage option price is ill-defined, even if the market is dynamically complete. For example, in the Black-Scholes (1973) and Merton (1973) setting, if the price of an option differs from its theoretical value, the investor buys the underpriced option or writes the overpriced one. The investor perfectly hedges the position by dynamic trading, thereby realizing as arbitrage profit the difference between the market price and the theoretical value. The dynamic trading policy incurs an infinite volume of trade over the lifetime of the option. This is just fine in the Black-Scholes-Merton model because transaction costs are assumed away. However the total transaction costs of the dynamic trading policy are *infinite*, if there are proportional transaction costs, however small the proportional transaction costs rate may be.

With market incompleteness and imperfections, the transaction prices of options generally differ from the prices that would prevail in a complete and frictionless market. We derive *equilibrium* (as opposed to no-arbitrage) restrictions on the *range* of the transaction prices of *American* options imposed by a class of traders that we refer to as *utility-maximizing* traders. We assume that these traders have heterogeneous endowments and are *risk-averse* with heterogeneous von Neuman-Morgenstern preferences which are otherwise unspecified. Furthermore, we assume, as in most earlier studies, that trading costs in the underlying security are *proportional* to the value of the underlying security that is being traded. These defining characteristics of utility-maximizing traders apply to a broad spectrum of institutional and individual investors.

We find a *range* of prices, such that *any* utility-maximizing trader would be able to exploit a mispricing, net of transaction costs, if the price of the option were to fall outside this range; the frictionless no-arbitrage option price lies within the range. We define *the reservation purchase price* of an option as the maximum price gross of transaction costs below which a given trader in this class increases her expected utility by purchasing the option. Likewise, we define *the reservation write price* of an option as the minimum price net of transaction costs above which a given trader in this class increases her expected utility by writing the option. We identify a lower bound on the reservation purchase prices, and an upper bound on the reservation write prices, of *all* traders in the utility-maximizing class. These bounds are independent of investor characteristics such as wealth or attitude towards risk. The bounds on the reservation purchase price of American puts and the reservation write price of American calls are tight.

For ease of exposition, we assume that the returns between adjacent discrete trading dates are identically and independently distributed. This assumption is

unnecessary and the proofs can be easily extended to accommodate non-*i.i.d.* returns such as the stochastic volatility models of Hull and White (1987), Amin and Ng (1993) and Heston (1993).

The choice of the return distribution is restricted only by the requirement that expected utility exists. In particular, if we think of the return between two adjacent discrete trading dates as generated by a continuous-time process, this process is not limited to a diffusion process but may be a mixed jump-diffusion process such as the one used by Merton (1976), Bates (1991) and Amin (1993).

We argue that all transaction prices of options should be less than or equal to the bound on the reservation write prices and greater than or equal to the bound on the reservation purchase prices. The reasoning leading to this restriction is the following. If a transaction occurs at a price *above* the bound on the reservation write prices, the buyer is acting suboptimally because the buyer could have found any utility-maximizing trader as a willing writer at a lower price. Likewise, if a transaction occurs at a price *below* the bound on the reservation purchase price, the writer is acting suboptimally because the writer could have found any utility-maximizing trader as a willing buyer at a higher price.

We emphasize that we do not make the restrictive assumption that *all* traders belong to the class of utility-maximizing traders. We merely assume that at least *some* traders in the market are utility-maximizing, as defined above. Thus our results are unaffected by the presence of traders with different objectives and preferences and facing a different transaction costs schedule than that of the utility-maximizing traders.

The stochastic dominance bounds derived in this paper complement the utility-based bounds on the prices of American and European options, derived in Constantinides and Zariphopoulou (1999, 2001). They also complement the stochastic dominance bounds on the prices of European options, derived in Constantinides and Perrakis (2002) and tested in Constantinides, Jackwerth and Perrakis (2006). The extension of the methodology in Constantinides and Perrakis (2002) to American options is complicated by the early exercise. In many cases, the transaction costs enter at every possible time of exercise, thus weakening the bounds. In spite of this, we derive bounds in which early exercise or assignment is recognized, but which are virtually insensitive to the presence of transaction costs.

Our results are to be contrasted to the upper and lower bounds on European option prices derived by the super-replication method of Bensaid et. al. (1992) and the approximate replication method of Leland (1985) and to the extension of these bounds to American options by Perrakis and Lefoll (2000, 2004). The size of the bounds derived in these alternative studies depends on the number of trading periods allowed over the life of the option.¹ In particular,

¹ In theory there can be an infinite number of trading periods over the life of an option. In this paper, since the options are American, the derived bounds depend on the number of possible

as the number of trading periods increases, the upper bound tends to the stock price and the lower bound tends to the stock price minus the discounted strike price; thus, the bounds become weak and of little practical use. The stochastic dominance approach to asset pricing adopted here is closely related to earlier results on option bounds in incomplete but frictionless markets, originally derived by Perrakis and Ryan (1984) and extended by Levy (1985) and Ritchken (1985). Perrakis (1986, 1988) and Ritchken and Kuo (1988) extend these bounds to a multiperiod setup as a generalization of the binomial option-pricing model.

When intermediate trading is allowed the tightest of the bounds derived in the earlier studies, the upper bound of Perrakis (1986) and the lower bound of Ritchken and Kuo (1988) are tight and converge to the Black-Scholes model under appropriate limiting conditions. Unfortunately, these bounds do not survive the introduction of transaction costs and are eventually dominated by the weaker bound of Perrakis and Ryan (1984), which is similar to that of Proposition 1 in Constantinides and Perrakis (2002). It is this weaker bound and its counterpart for a put option that form the basis for the American options results presented in this paper, a fact that explains why setting the transaction cost parameter equal to zero in the lognormal case does not yield a unique option price.

The paper is organized as follows. In Section 2, we introduce the model. There are two primary securities, a riskless bond, and a risky stock, interpreted as a stock market index. There is also a cash-settled American call or put option written on the stock that expires some time before the end of the given horizon. The trader maximizes expected utility of wealth at the end of the horizon. Trading of the stock incurs proportional transaction costs. An attractive feature of this economy is that trading is allowed at intermediate points over the horizon.

In Section 3, we derive stochastic dominance bounds on the reservation purchase price (Proposition 1) and the reservation write price (Proposition 2) of American put options. The bounds are illustrated in Tables I and II for the lognormal case and in Tables III and IV for the mixture of a lognormal distribution and a Poisson jump process, the discrete time version of a mixed jump-diffusion process. The bounds on the reservation purchase price are tight. In Section 4, we derive a stochastic dominance bound on the reservation write price of American call options with a known proportional dividend yield (Proposition 3). We also present a bound on the reservation purchase price (Proposition 4) under the assumption that the expiration date of the option coincides with the horizon of at least *some* trader. The bounds are illustrated in Tables III and IV for the lognormal case and in Tables VII and VIII for the

exercise periods over the life of the option. For most index options exercise can effectively take place only once a day. Even if exercise can take place continuously several of our bounds can be shown to converge to a non-trivial value as trading becomes progressively denser.

mixed lognormal-Poisson distribution. The bounds on the reservation write price are very tight. In Section 5, we extend the earlier results to call and put options written on an index futures rather than on an index. Section 6 concludes.

In the remainder of this section, we complete the literature review. Absence of arbitrage implies the existence of a strictly positive *stochastic discount factor*. These ideas are implicit in the option pricing theory of Black and Scholes (1973) and Merton (1973) and are further developed by Ross (1976), Cox and Ross (1976), Constantinides (1978), Harrison and Kreps (1979), Harrison and Pliska (1981), and Delbaen and Schachermayer (1994). Jouini and Kallal (1995) extend the theory to account for transaction costs. Jouini and Kallal (2001) and Bizid and Jouini (2005) address the relationship between no-arbitrage and equilibrium restrictions.

Almost all the work on option pricing under transaction costs has been with European options. Merton (1990), and Boyle and Vorst (1992) consider a self-financing policy that replicates the payoff of a long call option in the presence of proportional transaction costs, when the stock price process is binomial. The cost of the dominating policy tends to the stock price as the density of the binomial steps tends to infinity. Bensaid et. al. (1992) introduce the notion of *super-replication*, which replaces the goal of replicating the payoff of a call option with the goal of dominating it. They show that super-replication coincides with replication for physical delivery options, as well as for all types of options, when the transaction cost rate is low. Hence, in this case as well, the cost of the dominating policy tends to the stock price as the density of the binomial steps tends to infinity. In a fairly general setting, Davis and Clark (1993) conjecture and Soner, Shreve and Cvitanic (1995) prove that the stock price is indeed the minimum-cost dominating policy for the long call option in the presence of proportional transaction costs, however small the (finite) proportional transaction cost rate may be.

Leland (1985) introduces a class of imperfectly replicating policies in the presence of proportional transaction costs. He calculates the total cost, including transaction costs, of an imperfectly replicating policy and the "tracking error", that is, the standard deviation of the difference between the payoff of the option and the payoff of the imperfectly replicating policy. Related work includes Brennan and Schwartz (1979), Figlewski (1989), Hoggard, Whalley and Wilmott (1993), Avellaneda and Paras (1994), Grannan and Swindle (1996), and Toft (1996).

Hodges and Neuberger (1989), and Davis, Panas, and Zariphopoulou (1993) apply the expected utility approach to option pricing under transaction costs and explicitly computed an investor's reservation purchase write prices of a call option. They solve numerically for the optimal multi-period investment

policy in the bond, stock and option, under the assumption that the utility function is exponential, with given absolute risk aversion coefficient.

2. The Model

We consider a market with several types of primary financial assets. We focus, however, on only two of them, a riskless bond and a stock, and we assume that there is a class of traders in the market that invests *only* in these two assets. We refer to these investors as *utility-maximizing* traders. We do not make the restrictive assumption that *all* traders belong to the class of utility-maximizing traders. Thus our results are unaffected by the presence of traders with different objectives and preferences and facing a different transaction costs schedule than that of the utility-maximizing traders. Below we refer to the utility-maximizing traders simply as "traders".

Since the stock is the only primary risky asset in the traders' portfolios, it has the natural interpretation as the stock market portfolio or index. In the following sections we introduce derivative financial assets: an American put option on the stock (Section 3), an American call option on the stock (Section 4), and American futures options (Section 5).

The options have the natural interpretation as index options or index futures options. Each trader makes sequential investment decisions in the primary assets at the discrete trading dates $t = 0, 1, \dots, T'$, where T' is the terminal date and is finite.² A trader may hold long or short positions in these assets. A bond with price one at the initial date has price R , $R > 1$ at the end of the first trading period, where R is a constant. The bond trades do not incur transaction costs.

At date t , the *cum dividend* stock price is $(1 + \gamma_t)S_t$, the cash dividend is $\gamma_t S_t$, and the *ex dividend* stock price is S_t , where the *dividend yield* parameters $\{\gamma_t\}_{t=1, \dots, T'}$ are assumed to satisfy the condition $0 \leq \gamma_t < 1$ and be deterministic and known to the trader at time zero. We assume that $S_0 > 0$ and that the support of the rate of return on the stock, $(1 + \gamma_{t+1})S_{t+1}/S_t$ is the interval $[0, \infty)$. We also assume that the rates of return are *i.i.d.* with constant mean return

$$\bar{R} \equiv E \left[(1 + \gamma_{t+1}) \frac{S_{t+1}}{S_t} \right]. \quad (1)$$

The assumption that the rates of return, $(1 + \gamma_{t+1})S_{t+1}/S_t$ are *i.i.d.* may be relaxed to accommodate non-*i.i.d.* returns. If $\underline{z}_t$ is the vector of state variables at time t and the joint distribution of $((1 + \gamma_{t+1})S_{t+1}/S_t, \underline{z}_{t+1})$, conditional on $\underline{z}_t$, is independent of S_t , then our results hold with minor reformulation. For

² The assumption that the time interval between trading dates is one is innocuous: the unit of time is chosen to be such that the time interval between trading dates is one.

expositional ease, we present proofs only in the case that the rates of return are *i.i.d.*

Stock trades incur proportional transaction costs charged to the bond account. At each date t , the trader pays $(1 + k_1)S_t$ out of the bond account to purchase one *ex dividend* share of stock and is credited $(1 - k_2)S_t$ in the bond account to sell (or, sell short) one share of stock. We assume that $0 \leq k_1 < 1$ and $0 \leq k_2 < 1$.

For future reference, we define the mean return *with the dividend reinvested in the stock, net of transaction costs, long and short*, as³

$$\hat{R}_t \equiv E \left[\left(1 + \frac{\gamma_{t+1}}{1 + k_1} \right) \frac{S_{t+1}}{S_t} \right], \quad \tilde{R}_t \equiv E \left[\left(1 - \frac{\gamma_{t+1}}{1 - k_2} \right) \frac{S_{t+1}}{S_t} \right]. \quad (2)$$

In practice, the distinction between $\bar{R}$ and $\hat{R}_t$ or $\tilde{R}_t$ is negligible, given that both the dividend yield (γ_{t+1}) and the transaction costs rates (k_1, k_2) are small.

We consider a trader who enters the market at date t with dollar holdings x_t in the bond account and y_t/S_t *ex dividend* shares of stock. The endowments are stated net of any dividend payable on the stock at time t .⁴ The trader increases (or, decreases) the dollar holdings in the stock account from y_t to $y'_t = y_t + v_t$ by decreasing (or, increasing) the bond account from x_t to $x'_t = x_t - v_t - \max[k_1 v_t, -k_2 v_t]$. The decision variable v_t is constrained to be measurable with respect to the information up to date t . The bond account dynamics is

$$x_{t+1} = \{x_t - v_t - \max[k_1 v_t, -k_2 v_t]\} R + (y_t + v_t) \frac{\gamma_{t+1} S_{t+1}}{S_t}, \quad t \leq T' - 1 \quad (3)$$

and the stock account dynamics is

$$y_{t+1} = (y_t + v_t) \frac{S_{t+1}}{S_t}, \quad t \leq T' - 1. \quad (4)$$

At the terminal date, the stock account is liquidated, $v_{T'} = -y_{T'}$, and the net worth is $x_{T'} + y_{T'} - \max[-k_1 y_{T'}, k_2 y_{T'}]$. At each date t , the trader chooses investment v_t to maximize the expected utility of net worth, $E[u(x_{T'} + y_{T'} - \max[-k_1 y_{T'}, k_2 y_{T'}]) \mid S_t]$.⁵ We make the plausible assumption

³ Even though the expected return, $\bar{R}$, at time t is time-independent, the expectations $\hat{R}_t$ and $\tilde{R}_t$ in Equation (2) are time subscripted because the dividend yield is time-varying deterministic.

⁴ We elaborate on the precise sequence of events. The trader enters the market at date t with dollar holdings $x_t - \gamma_t y_t$ in the bond account and y_t/S_t *cum dividend* shares of stock. Then the stock pays cash dividend $\gamma_t y_t$ and the dollar holdings in the bond account become x_t . Thus, the trader has dollar holdings x_t in the bond account and y_t/S_t *ex dividend* shares of stock.

⁵ The results extend routinely to the case that consumption occurs at each trading date and utility is defined over consumption at each of the trading dates and over the net worth at the terminal date. See, Constantinides (1979) for details.

that the utility function, $u(\cdot)$, is increasing and concave, and is defined for both positive and negative terminal net worth.⁶

We define the value function recursively as

$$V(x_t, y_t, t) = \max_v E \left[V \left(\{x_t - v - \max[k_1 v, -k_2 v]\} R + (y_t + v) \frac{y_{t+1} S_{t+1}}{S_t}, (y_t + v) \frac{S_{t+1}}{S_t}, t + 1 \right) \right] \quad (5)$$

for $t \leq T' - 1$ and

$$V(x_{T'}, y_{T'}, T') = u(x_{T'} + y_{T'} - \max[-k_1 y_{T'}, k_2 y_{T'}]). \quad (6)$$

We assume that the parameters satisfy appropriate technical conditions such that the value function exists and is once differentiable. We denote by v_t^* the optimal investment decision at date t corresponding to the portfolio (x_t, y_t) . For future reference, we state that the value function $V(x, y, t)$ is increasing and concave in (x, y) , properties inherited from the monotonicity and concavity of the utility function $u(\cdot)$, given that the transaction costs are quasi-linear.⁷

Also for future reference, we define x'_t and y'_t as

$$x'_t = x_t - v_t^* - \max[k_1 v_t^*, -k_2 v_t^*] \quad (7)$$

and

$$y'_t = y_t + v_t^*. \quad (8)$$

Portfolio (x'_t, y'_t) represents the new holdings at t following optimal restructuring of the portfolio (x_t, y_t) . Equations (5), (7) and (8) and the definition of v_t^* imply

$$V(x_t, y_t, t) = V(x'_t, y'_t, t). \quad (9)$$

3. Bounds on the Price of American Puts

We derive a lower bound on the reservation purchase price of an American put option. We enrich the investment opportunity set by introducing an American, cash-settled put option with strike K and expiration date T , $T \leq T'$. The cash payoff of the put exercised at time t is $K - S_t$, $t \leq T$.

⁶ If utility is defined only for non-negative net worth, then the decision variable is constrained to be a member of a convex set, A , that ensures the non-negativity of the net worth. See, Constantinides (1979) for details. This case is studied in Constantinides and Zariphopoulou (1999, 2001). Our bounds apply to this case as well.

⁷ See, Constantinides (1979) for details.

We consider the following sequence of events. A trader enters date t with endowments x_t and y_t in the bond and stock accounts, respectively, and a *long* position in a put option. The endowments x_t and y_t are net of any cash flows that the trader has incurred at date t or at an earlier date in buying the put. We stipulate that, at each date, the trader may either hold on to the put position or exercise it, but is constrained from selling it.⁸ If the trader exercises the put, the trader receives $K - S_t$ in cash from a trader with a short position in the put that is "assigned".

We define the value function $J(x_t, y_t, S_t, t)$ as the expected utility at date t of a trader who has endowments x_t and y_t in the bond and stock accounts, respectively, and a *long* position in a put option. The value function is the expected utility associated with the *constrained optimal policy* that the trader may exercise but not close out his/her position in the put.

Formally, we define the value function recursively as

$$J(x_t, y_t, S_t, t) = \max \left[V(x_t + K - S_t, y_t, t), \right. \\ \left. \max_j E \left[J \left(\left\{ x_t - j - \max[k_1 j, -k_2 j] + (y_t + j) \frac{y_{t+1} S_{t+1}}{S_t} \right\} R, \right. \right. \right. \\ \left. \left. \left. (y_t + j) \frac{S_{t+1}}{S_t}, S_{t+1}, t + 1 \right) \mid S_t \right] \right] \quad (10)$$

for $t \leq T - 1$ and

$$J(x_T, y_T, S_T, T) = V(x_T + [K - S_T]^+, y_T, T). \quad (11)$$

The real number j is the investment in the stock account at time t . For future reference, we note that the optimal investment in the stock account at time t in the maximization problem of Equation (10) may differ from the optimal investment, v_t^* , in the problem of Equation (5).

We define the *reservation purchase price of the American put* as the maximum price below which a given trader increases his/her expected utility by purchasing the put. This reservation purchase price is defined as

$$\max \{ P \mid J(x_t - P, y_t, S_t, t) \geq V(x_t, y_t, t) \}. \quad (12)$$

It is a price that depends on the utility function of the trader, as well as on her portfolio holdings (x_t, y_t) . By definition, a trader who observes a market price lower than her reservation purchase price should establish a long position in the put option. In this section, we provide a *lower bound*, $\underline{P}(S_t, t)$, to the reservation purchase prices of *all* traders, which is *independent* of the form of the utility

⁸ The reservation purchase price of a put is derived under this constrained policy. We demonstrate further on in this section that the reservation price that we derive continues to be valid when the constraint is removed.

function and the trader portfolio. Consequently, any trader who observes at time t a market price $P \leq \underline{P}(S_t, t)$ should establish a long position in the option.

By purchasing the option when $P \leq \underline{P}(S_t, t)$, the trader who is forbidden from selling the put till expiration has expected utility $J(x_t - P, y_t, S_t, t)$ after the purchase, which exceeds her pre-purchase expected utility $V(x_t, y_t, t)$. It is easy to see that $\underline{P}(S_t, t)$ is also a lower bound on the reservation purchase prices of all traders who are *not* constrained from closing out their positions in the option. Let $\bar{J}(x_t, y_t, S_t, t)$ denote the expected utility of the *unconstrained* trader with an open position in the option, which is clearly greater than $J(x_t, y_t, S_t, t)$ for all (x_t, y_t, S_t, t) . Then we also have $\bar{J}(x_t - P, y_t, S_t, t) \geq J(x_t - P, y_t, S_t, t) \geq V(x_t, y_t, t)$ for all $P \leq \underline{P}(S_t, t)$, implying that $\underline{P}(S_t, t)$ is also a reservation purchase price for the unconstrained trader.⁹ A similar argument applies to all other results presented in this paper.

The derivation of bounds relies on the key property that *the marginal utility is non-increasing in the stock price*. This property is obtained under the *monotonicity of wealth condition*: the wealth at the end of every period, including the possible payoff of the derivative, is a non-decreasing function of the stock price. This condition, combined with the assumed concavity of the utility function, implies that the marginal utility is non-increasing in the stock price. The monotonicity of wealth condition is rigorously defined further on in this section, where it is shown that its validity may be guaranteed with a probability arbitrarily close to 1 by *judiciously limiting the size of the position in the derivative relative to the stock and bond positions*.

For $S \geq 0$, we define the auxiliary function $M(S, t)$ recursively as follows:

$$M(S, t) = \frac{1}{R_t} E[\max[K - S_{t+1}, M(S_{t+1}, t + 1)] \mid S_t = S], \text{ for } t \leq T - 1 \quad (13)$$

and

$$M(S, T) = 0. \quad (14)$$

The function $M(S, t)$ has for $t \leq T - 1$ the interpretation as the price of an American put option if there are no transaction costs and the actual distribution of the stock is risk-neutral (that is, if $R = R_t$). For future reference, we state without proof that the function $M(S, t)$ is decreasing and convex in S and that $\lim_{S \rightarrow 0} M(S, t) = K$ for all $t \leq T - 1$. We are now ready to state and prove the main result of this section.

Proposition 1. *If the monotonicity of wealth condition holds, $\underline{P}(S_t, t)$ is a lower bound on the reservation purchase price of an American put option at time t ,*

⁹ The above argument implies that if the no-sale constraint were to be removed the reservation purchase price would increase. Unfortunately removing the constraint would also require modeling the stochastic evolution of the option price till expiration. The required assumptions as to option market equilibrium would severely limit the generality of our results.

where

$$\underline{P}(S_t, t) \equiv \max \left[K - S_t, \frac{1 - k_2}{1 + k_1} M(S_t, t) \right], \quad t \leq T. \quad (15)$$

A formal proof is presented in Appendix A. The proof uses the following auxiliary equation, whose validity is shown as part of the proof:

$$J \left(x_t, y_t - \frac{1}{1 + k_1} \max[K - S_t, M(S_t, t)], S_t, t \right) \geq V(x_t, y_t, t), \quad t \leq T. \quad (16)$$

We expect transaction costs to have a limited effect on this put lower bound because the roundtrip transaction cost multiplies the option price. By contrast, transaction costs have a major impact on the put option upper bound, as we shall see later on, because the roundtrip transaction costs in that case multiply the strike price.

In Table I, we present values of the lower bound stated in Proposition 1 as a function of the strike-to-price ratio, K/S , for transaction costs rates $k_1 = k_2 = 0.1, 0.5$ and 1%. The stock price is assumed lognormal, generated by a geometric Brownian motion. The parameter values are: expiration 30 days, stock price 100, annual risk-free rate 3%, annual arithmetic expected stock return 8%, annual volatility 20%, and annual dividend yield 1%. The price of the same *American* put in the absence of transaction costs is also presented in Table I and is referred to as the Black-Scholes price because it shares with Black-Scholes the assumptions of lognormality and zero transaction costs rate.

Table I. Lower bounds on the reservation purchase price of an American put implied by Proposition 1 and upper bounds on the reservation write price implied by Proposition 2, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under lognormal return distribution. Parameter values: expiration 30 days, annual risk-free rate 3%, annual expected stock return 8%, annual volatility 20%, and annual dividend yield 1%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	1.082	2.623	5.817	1.825	3.391	6.602	2.752	4.352	7.581
Black-Scholes	0.525	2.178	5.495	0.525	2.178	5.495	0.525	2.178	5.495
Lower bound	0.462	2.012	5.292	0.458	1.996	5.250	0.454	1.976	5.198

Table II. Lower bounds on the reservation purchase price of an American put implied by Proposition 1 and upper bounds on the reservation write price implied by Proposition 2, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under lognormal return distribution. Parameter values: expiration 90 days, annual risk-free rate 3%, annual expected stock return 8%, annual volatility 20%, and annual dividend yield 1%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	2.891	4.672	7.427	3.622	5.427	8.2	4.531	6.368	9.164
Black-Scholes	1.67	3.643	6.631	1.67	3.643	6.631	1.67	3.643	6.631
Lower bound	1.39	3.193	6.091	1.379	3.168	6.043	1.366	3.136	5.983

Table III. Lower bounds on the reservation purchase price of an American put implied by Proposition 1 and upper bounds on the reservation write price implied by Proposition 2, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under a mixed lognormal-jump distribution. Parameter values: expiration 30 days, annual risk-free rate 3%, annual expected stock return 8%, annual total volatility 20%, annual dividend yield 1%, annual jump frequency 1/5, lognormal jump amplitude with mean -1% , and volatility 7%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	1.214	2.727	5.895	2.016	3.529	6.697	3.096	4.609	7.778
Merton	0.544	2.173	5.494	0.544	2.173	5.494	0.544	2.173	5.494
Lower bound	0.492	2.038	5.323	0.488	2.022	5.280	0.484	2.002	5.228

The lower bound is tight. For example, with transaction cost rates $k_1 = k_2 = 0.5\%$ (50 basis points), the lower bound of the at-the-money 30-day put is 1.996, compared to the Black-Scholes American price of 2.178. In Table II, we present the same information but for the 90-day put. The lower bound is reasonably tight. For example, with transaction costs rates $k_1 = k_2 = 0.5\%$, the lower bound of the at-the-money 90-day put is 3.168, compared to the Black-Scholes American price of 3.643.

In Tables III and IV, we assume that the stock price process is a mixed lognormal-jump process. We target the same parameters of the total return as in Tables I and II. Specifically, we lower the volatility of the lognormal component of the return such that the total return volatility remains 20%.

Table IV. Lower bounds on the reservation purchase price of an American put implied by Proposition 1 and upper bounds on the reservation write price implied by Proposition 2, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under a mixed lognormal-jump distribution. Parameter values: expiration 90 days, annual risk-free rate 3%, annual expected stock return 8%, annual total volatility 20%, annual dividend yield 1%, annual jump frequency 1/5, lognormal jump amplitude with mean -1% and volatility 7%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	3.141	4.895	7.628	3.935	5.689	8.422	5.005	6.758	9.492
Merton	1.696	3.652	6.636	1.696	3.652	6.636	1.696	3.652	6.636
Lower bound	1.464	3.279	6.179	1.452	3.252	6.130	1.438	3.220	6.069

The jump amplitude of the Poisson process is lognormally distributed.¹⁰ The parameters chosen for the jump component are within the range of values in the various scenarios presented by Liu et al. (2005) as representative of the S&P 500 jumps: annual arrival frequency 1/5, expected log amplitude -1% , and log amplitude volatility 7%. Unlike the lognormal process, this process does not yield a unique option price in the absence of transaction costs even at the limit of continuous trading, and various assumptions have been used in the literature in order to "complete" the market and generate an option price. We use as a value of reference in the tables the American option price generated by the Merton (1976) assumption, that the jump component is fully diversifiable and, thus, not priced.¹¹

In Tables III and IV, we observe that the put lower bound is close to the Merton price and is, therefore, tight for both the 30- and the 90-day put. Thus, the lower put bound remains tight in the presence of stock price jumps.

Although $\underline{P}(S_t, t)$ is a reservation purchase price for all risk averse traders, there are trader-specific limits on the size of the long position in the put option. The relative size of the long option position to the trader's portfolio holdings depends on the satisfaction of the monotonicity condition with probability sufficiently close to 1. To see this we note that the monotonicity condition is used in the proof of Proposition 1 in Appendix A to establish the fact that the expression

¹⁰ This mixed process has *i.i.d.* returns and can be easily computed by using the tree method developed by Amin (1993).

¹¹ This assumption, although widely used, is rather controversial, since many rare events generally reflect economy-wide shocks; see Amin (1993). Unfortunately, an option price for the mixed process can only be derived if one assumes a particular value of the risk aversion parameter of a "representative" investor.

$$V_x \left(x'_{t-1} R, y'_{t-1} \frac{S_t}{S_{t-1}} + \frac{h(S_{t-1}, S_t, t)}{1 + k_1}, t \right)$$

is a decreasing function of S_t in its second argument for any $t \leq T - 1$. Replacing $h(S_{t-1}, S_t, t)$ and using (13) as well as the fact that $V(x, y, t)$ is concave, we observe that the monotonicity condition is strictly equivalent to the requirement that the following function

$$Q(S_t) \equiv \left[y'_t - \left(1 + \frac{\gamma_\tau}{1 + k_1} \right) \frac{M(S_{t-1}, t - 1)}{1 + k_1} \right] \frac{S_t}{S_{t-1}} + \frac{\max[K - S_t, M(S_t, t)]}{1 + k_1} \quad (17)$$

must be monotone increasing. A sufficient condition for this is

$$\left[y'_t - \left(1 + \frac{\gamma_\tau}{1 + k_1} \right) \frac{M(S_{t-1}, t - 1)}{1 + k_1} \right] \frac{1}{S_{t-1}} > \frac{1}{1 + k_1}. \quad (18)$$

It is clear from (5)–(8) that the left-hand-side of (18) is an increasing function of the portfolio holdings (x_{t-1}, y_{t-1}) , which, in turn, are increasing functions of the initial portfolio holdings at the time the long position is established. Hence, the probability that the monotonicity condition will be satisfied at any time prior to option expiration is also an increasing function of the initial portfolio holdings. Relations similar to (17)–(18) also define the monotonicity condition for all the other results presented in this paper.

The probability that (18) is satisfied depends also on the trader's utility function through the quantity v_{t-1}^* . A closed form expression for this probability does not exist, even for the simplest case of a CPRA utility and stock dynamics following a diffusion process. Nonetheless, numerical simulations can easily define in any given situation a *minimal* portfolio size for each newly established option position that would guarantee the virtual satisfaction of the monotonicity condition. In an appendix available from the authors upon request, we present numerical results that show that this minimum portfolio size is quite modest for the long put option position in the benchmark case of CPRA utility and diffusion stock dynamics. The key variable is the ratio (y_0/S_0) , the starting number of shares for one long put option in the trader portfolio. The probability that (18) is satisfied is essentially one under all parameter values and conditions for a value of that ratio equal to 1.3, or for one long option for each 1.3 shares of the stock in the trader portfolio.

In the next proposition, proved in Appendix B, we present an *upper* bound $\bar{P}(S_t, t)$ on the reservation write price of an American put option. The reservation write price of a given trader is defined as

$$\min\{P \mid J(x_t + P, y_t, S_t, t) \geq V(x_t, y_t, t)\} \quad (19)$$

and, as with the reservation purchase price, it depends on the utility function of the trader, as well as on her portfolio holdings (x_t, y_t) . The upper bound $\bar{P}(S_t, t)$ that we derive is at least as large as the maximum reservation write price for all traders, implying that any trader observing a write price $P \geq \bar{P}(S_t, t)$ can improve her utility by writing the option.

We denote by $J(x_t, y_t, S_t, t)$ the value function of the expected utility at date t of a trader who has endowments x_t and y_t in the bond and stock accounts respectively, and a *short* position in a put option. This function is defined as follows:

$$J(x_t, y_t, S_t, t) = \min \left\{ V(x_t - (K - S_t)^+, y_t, t), \right. \\ \left. \max_j E \left[J \left(\left\{ x_t - j - \max(k_1 j, -k_2 j) + (y_t + j) \frac{y_{t+1} S_{t+1}}{S_t} \right\} R, \right. \right. \right. \\ \left. \left. \left. (y_t + j) \frac{S_{t+1}}{S_t}, S_{t+1}, t + 1 \right) \mid S_t \right] \right\} \quad (20)$$

for $t \leq T - 1$, and

$$J(x_T, y_T, S_T, T) = V(x_T - (K - S_T)^+, y_T, T). \quad (21)$$

Since $V(x_t, y_t, t)$ is increasing and concave in x_t and y_t , it can be shown by induction that $J(x_t, y_t, S_t, t)$ is also increasing and concave in S_t . The function $J(x_t, y_t, S_t, t)$ embodies the assumption that the trader may not close her position till option expiration unless she is assigned. As with the put option, removing this constraint would result in an increased expected utility. Hence, we can show by an argument similar to the one used for the put lower bound that $\bar{P}(S_t, t)$ is also a reservation write price for an unconstrained trader.¹²

Proposition 2. *If the monotonicity of wealth condition holds, $\bar{P}(S_t, t)$ is an upper bound on the reservation write price of an American put option at time t , where*

$$\bar{P}(S_t, t) = \max[K - S_t, L(S_t, t)], \quad t \leq T, \quad (22)$$

$$L(S_t, t) = \frac{K}{R} - \frac{1 - k_2}{1 + K_1} \frac{K}{\bar{R}_t} \\ + \frac{1}{\bar{R}_t} E \left[\max \left[\frac{1 - k_2}{1 + k_1} [K - S_{t+1}]^+, L(S_{t+1}, t + 1) - \left(1 - \frac{1 - k_2}{1 + k_1} \right) \frac{K}{R} \right] \right], \quad (23)$$

¹² The properties of the value function are quite similar to those of the equivalent function used in the demonstration of the call upper bound in Proposition 3 in the next section, where there is a more extended discussion. Note also that the definitions of the option reservation purchase and write prices presented in this section for the put option are also valid for the call option and the futures options presented in the following sections.

if $t \leq T - 1$, and

$$L(S_T, T) = 0. \quad (24)$$

In the above expressions, the roundtrip transaction costs multiply the strike price, which is of a much larger order of magnitude than the option price. For this reason, and in contrast to the put lower bound of Proposition 1, we expect transaction costs to have a major effect upon the size of the put upper bound.

In Table I, we illustrate the upper bound stated in Proposition 2 when the stock price is assumed lognormal. The upper bound is not tight. For example, with transaction costs $k_1 = k_2 = 0.5\%$ (50 basis points), the upper bound of the at-the-money 30-day put is 3.391 in Table I, compared to the Black-Scholes American price of 2.178. In Table II, we present the same information but for the 90-day put. Again, the upper bound is not tight. For example, with transaction costs $k_1 = k_2 = 0.5\%$, the upper bound of the at-the-money 90-day put is 5.427, compared to the Black-Scholes American price of 3.643. In Tables III and IV, we present the upper bound of Proposition 2 for the case of the mixed lognormal-jump distribution, for the 30-day and the 90-day put, respectively. Again, the upper bound is not tight.

4. Bounds on the Price of American Calls

We derive an upper bound on the reservation write price of an American call option. We consider again a market as defined in Section 2, in which we introduce an American, cash-settled call option on the stock with strike price K and expiration date T , $T \leq T'$. If a holder of a call exercises her call at time t , $t \leq T$, when the *cum dividend* stock price is $(1 + \gamma_t)S_t$, then she receives $(1 + \gamma_t)S_t - K$ in cash from a trader with a short position in the call that is "assigned".

We consider the following sequence of events at date t . A trader enters date t with endowments x_t and y_t in the bond and *ex dividend* stock accounts, respectively, and a *short* position in a call option. The endowments are stated net of any cash flows that the trader has incurred at date t or at an earlier date in writing the call, and net of the dividend payable on the stock at time t . First, the trader is informed whether she has been "assigned" or not. If the trader has been "assigned", then the trader pays $(1 + \gamma_t)S_t - K$ in cash and has her position in the call closed out. If the trader has been "assigned", the value of the cash account becomes $x_t - \{(1 + \gamma_t)S_t - K\}$.¹³ Second, the

¹³ We elaborate on the precise sequence of events. The trader enters the market at date t with dollar holdings $x_t - \gamma_t y_t$ in the bond account and y_t/S_t *cum dividend* shares of stock. The trader is informed whether she has been "assigned" or not. If the trader has not been "assigned", the stock pays cash dividend $\gamma_t y_t$ and the dollar holdings in the bond account become x_t . Thus, the trader has dollar holdings x_t in the bond account and y_t/S_t *ex dividend* shares of stock. If the

trader increases (or, decreases) the dollar holdings in the stock account by j , by decreasing (or, increasing) the dollar holdings of the bond account by $j + \max[k_1 j, -k_2 j]$.

We define the value function $J(x_t, y_t, S_t, t)$ as the expected utility at date t of a trader who enters date t with endowments x_t and y_t in the bond and *ex dividend* stock accounts, respectively, and a *short* position in a call option. At certain times, it may well be optimal for the trader to close out the short position rather than leave it open. However, we stipulate, as in the previous section, that at each date the trader is constrained from closing out the short position in the call. The value function is the expected utility associated with this *constrained optimal policy*. The upper bound on the reservation write price of a call is derived under this constrained policy. As noted earlier, this same price would also be a reservation price when the constraint is removed, albeit perhaps not the tightest one. The trader's expected utility depends on his/her expectations regarding the probability that the trader is "assigned". Since we wish to avoid making any assumptions regarding the exercise policy of those having long positions in the call, we define the value function as the trader's expected utility under the worst-case scenario from the perspective of the trader.¹⁴

$$J(x_t, y_t, S_t, t) = \min \left[V(x_t - \{(1 + \gamma_t)S_t - K\}, y_t, t), \right. \\ \left. \max_j E \left[J \left(\{x_t - j - \max[k_1 j, -k_2 j]\}R + (y_t + j) \frac{\gamma_{t+1} S_{t+1}}{S_t}, \right. \right. \right. \\ \left. \left. \left. (y_t + j) \frac{S_{t+1}}{S_t}, S_{t+1}, t + 1 \right) \mid S_t \right] \right] \quad (25)$$

for $t \leq T - 1$ and

$$J(x_T, y_T, S_T, T) = V(x_T - [(1 + \gamma_T)S_T - K]^+, y_T, T). \quad (26)$$

For future reference, we note that the optimal investment, j , in the maximization problem of Equation (25) may differ from the optimal investment, v_t^* , in the problem of Equation (5).

We define the *reservation write price of the American call* as the minimum price above which any trader increases his/her expected utility by writing the

trader has been "assigned", the trader pays $(1 + \gamma_t)S_t - K$ and not $S_t - K$ because the call is exercised before the stock goes *ex dividend*. Then the stock pays cash dividend $\gamma_t y_t$. Thus, the trader has dollar holdings $x_t - \gamma_t y_t + \gamma_t y_t - \{(1 + \gamma_t)S_t - K\} = x_t - \{(1 + \gamma_t)S_t - K\}$ in the bond account and y_t/S_t *ex dividend* shares of stock.

¹⁴ Clearly, any other assumption about the exercise policies of the option holders would result in a *larger* value function for the option writer and, hence, in a lower reservation write price for the call. Hence, the reservation write price derived under the worst-case scenario is also a reservation price under alternative exercise assumptions.

call. In this section, we provide an *upper bound*, $\bar{C}(S_t, t)$, to the reservation write price.

As in Section 3, the derivation of bounds relies on the key property that *the marginal utility is non-increasing in the stock price*. This property is obtained under the *monotonicity of wealth condition*: the wealth at the end of every period, including the possible payoff of the derivative, is a non-decreasing function of the stock price. As with the put option in Section 3, there are relations similar to (17) and (18) that show that the monotonicity condition is guaranteed by *judiciously limiting the size of the position in the derivative relative to the stock and bond positions*. This relative size is trader-specific.

For $S \geq 0$, we define the auxiliary function $N(S, t)$ recursively as follows:

$$N(S, t) = \frac{1}{\hat{R}_t} E[\max\{(1 + \gamma_{t+1})S_{t+1} - K, N(S_{t+1}, t + 1)\} \mid S_t = S] \quad (27)$$

for $t \leq T - 1$, and

$$N(S, T) = 0. \quad (28)$$

Recall that $\hat{R}_t$ is defined in Equation (2) and that, in practice, the distinction between R_t and $\hat{R}_t$ is negligible, given that both the dividend yield and the transaction cost rate are of the order of a few percent. The function $N(S, t)$ has the interpretation as the price of an American call option, if there are no transaction costs and the expected rate of return on the stock equals the risk free rate, $R_t = R$. For future reference, we state without proof that the function $N(S, t)$ is increasing and convex in S , $\lim_{S \rightarrow 0} N(S, t) = 0$,

$$\lim_{S \rightarrow 0} \frac{\partial N(S, t)}{\partial S} = 0, \quad \text{and} \quad 0 \leq \frac{\partial N(S, t)}{\partial S} \leq 1.$$

We are now ready to state and prove the main result of this section.

Proposition 3. *If the monotonicity of wealth condition holds, $\bar{C}(S_t, t)$ is an upper bound on the reservation write price of an American call at time t , where¹⁵*

$$\bar{C}(S_t, t) = \frac{1 + k_1}{1 - k_2} \max[N(S_t, t), (1 + \gamma_t)S_t - K], \quad t \leq T. \quad (29)$$

¹⁵ Naturally, at the maturity date T , $[(1 + \gamma_T)S_T - K]^+$ is a tighter upper bound on the reservation write price of an American call. The upper bound of Equation (29) does not become tighter in the special case $T = T'$.

Table V. Lower bounds on the reservation purchase price of an American call implied by Proposition 4, and upper bounds on the reservation write price implied by Proposition 3, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under lognormal return distribution. Parameter values: expiration 30 days, annual risk-free rate 3%, annual expected stock return 8%, annual volatility 20%, and annual dividend yield 1%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	6.088	2.627	0.788	6.137	2.648	0.794	6.199	2.674	0.802
Black-Scholes	5.755	2.408	0.695	5.755	2.408	0.695	5.755	2.408	0.695
Lower bound	5.607	2.129	0.27	5.607	2.129	0.27	5.607	2.128	0.27

Table VI. Lower bounds on the reservation purchase price of an American call implied by Proposition 4, and upper bounds on the reservation write price implied by Proposition 3, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under lognormal return distribution. Parameter values: expiration 90 days, annual risk-free rate 3%, annual expected stock return 8%, annual volatility 20%, and annual dividend yield 1%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	8.188	4.975	2.715	8.254	5.015	2.737	8.337	5.065	2.764
Black-Scholes	7.348	4.324	2.276	7.348	4.324	2.276	7.348	4.324	2.276
Lower bound	6.784	3.505	1.177	6.783	3.504	1.177	6.782	3.503	1.175

The formal proof is given in Appendix C. The proof uses the following auxiliary equation, whose validity is also shown in Appendix C:

$$J\left(x_t, y_t + \frac{\max[(1 + \gamma_t)S_t - K, N(S_t, t)]}{1 - k_2}, S_t, t\right) \geq V(x_t, y_t, t), \quad t \leq T. \quad (30)$$

As with the put lower bound of Proposition 1, we note that in Proposition 3 the roundtrip transaction costs multiply the option price, implying that they are expected to have a limited effect on the size of the call option upper bound.

In Table V, we illustrate the upper bound stated in Proposition 3 as a function of the strike-to-price ratio, K/S , for transaction cost rates $k_1 = k_2 = 0.1, 0.5$ and 1%. The stock price is assumed lognormal, generated by a geometric Brownian motion. The parameter values are: expiration 30 days, stock price 100, annual risk-free rate 3%, annual arithmetic expected stock return 8%,

Table VII. Lower bounds on the reservation purchase price of an American call implied by Proposition 4, and upper bounds on the reservation write price implied by Proposition 3, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under a mixed lognormal-jump distribution. Parameter values: expiration 30 days, annual risk-free rate 3%, annual expected stock return 8%, annual volatility 20%, annual dividend yield 1%, annual jump frequency 1/5, lognormal jump amplitude with mean -1% and volatility 7%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	6.043	2.580	0.770	6.091	2.600	0.776	6.153	2.627	0.784
Merton	5.780	2.407	0.697	5.780	2.407	0.697	5.780	2.407	0.697
Lower bound	5.617	2.106	0.253	5.616	2.106	0.253	5.616	2.106	0.252

Table VIII. Lower bounds on the reservation purchase price of an American call implied by Proposition 4, and upper bounds on the reservation write price implied by Proposition 3, as functions of the transaction costs rate and the strike-to-price ratio, K/S , under a mixed lognormal-jump distribution. Parameter values: expiration 90 days, annual risk-free rate 3%, annual expected stock return 8%, annual volatility 20%, annual dividend yield 1%, annual jump frequency 1/5, lognormal jump amplitude with mean -1% and volatility 7%.

	Transaction cost rate								
	0.1%			0.5%			1%		
	K/S			K/S			K/S		
	0.95	1.00	1.05	0.95	1.00	1.05	0.95	1.00	1.05
Upper bound	8.044	4.856	2.639	8.109	4.895	2.660	8.190	4.944	2.686
Merton	7.390	4.345	2.294	7.390	4.345	2.294	7.390	4.345	2.294
Lower bound	6.789	3.483	1.147	6.788	3.482	1.146	6.786	3.481	1.145

annual volatility 20%, and annual dividend yield 1%. The price of the same *American* call in the absence of transaction costs is also presented in Table I and is referred to as the Black-Scholes price because it shares with Black-Scholes the assumptions of lognormality and zero transaction costs. The upper bound is very tight for in-the-money and at-the-money calls, but less tight for out-of-the-money calls. In Table VI, we present the same information but for the 90-day American call. The conclusion is the same.

In Tables VII and VIII, we assume that the stock price process is a mixed lognormal-jump process. We use the same parameters as in Tables III and IV, namely total expected stock return of 8%, annual dividend yield 1%, and total annual return volatility 20%, and for the jump component annual arrival frequency 1/5, expected log amplitude -1% , and log amplitude volatility 7%.

The values for the upper bound of Proposition 3 are uniformly somewhat tighter, although similar in magnitude to those of the pure lognormal case shown in Tables V and VI.

Finally, we provide a *lower* bound on the reservation *purchase* price of a European or American call, in the special case that there exists at least one trader with investment horizon coinciding with the expiration date of the call option, $T = T'$. The proof is given in Appendix D.

Proposition 4. *Assume that the monotonicity of wealth condition holds and $T = T'$. Then $\underline{C}(S_t, t)$ is a lower bound on the reservation purchase price of an American or European call option at time t , where*

$$\underline{C}(S_t, t) = \left[\frac{S_t}{\prod_{s=t+1}^T \left(1 + \frac{\gamma_s}{1 - k_2}\right)} - \frac{K}{R^{T-1}} + \frac{E[(K - S_T)^+ | S_t]}{\prod_{s=t}^{T-1} \hat{R}_s} \right]^+, \quad t \leq T. \quad (31)$$

The lower bound stated in Proposition 4 is illustrated in Tables V and VI, as a function of the strike-to-price ratio, K/S , for transaction cost rates $k_1 = k_2 = 0.1, 0.5$, and 1%. Clearly, the lower bound is loose and is of little use. In Tables VII and VIII we assume that the stock price process is a mixed lognormal-jump process. The values for the lower bound given by Proposition 4 are also similar and equally loose as those for the pure lognormal process of Tables V and VI.

5. Bounds on the Price of American Index Futures Options

Bounds on American-style calls and puts on index *futures*, such as the highly liquid CME-listed S&P 500 index futures options are of independent interest. Practically all of the stochastic dominance bounds on index options extend to index futures options with appropriate modification. In this section, we present without proof the extended versions of the two most powerful results derived above: the lower bound on the reservation purchase price of an American put (Proposition 1) and the upper bound on the reservation write price of an American call (Proposition 3).

In the market of Section 2, we introduce a cash-settled *futures* contract with maturity T , $T \leq T'$. We assume that the futures price F_t is linked to the stock price as

$$F_t = \alpha_t S_t + \varepsilon_t, \quad t \leq T \quad (32)$$

where α_t is a time-dependent parameter and the random variables $\{\varepsilon_t\}$ are independent of the stock price series $\{S_t\}$. In a frictionless market, a standard

static no-arbitrage argument yields the cost-of-carry relation

$$\alpha_t = R^{T-t} \prod_{s=t+1}^T (1 + \gamma_s)^{-1}, \quad \varepsilon_t \equiv 0. \quad (33)$$

In a market *with* transaction costs, absence of arbitrage implies that the restriction (33) holds at $t = T$ but not necessarily at earlier dates. A plausible description of the market at dates prior to the delivery date is to say that the cost-of-carry relation is *unbiased* but there is some basis risk.¹⁶ This translates into the statement that the time-dependent parameters α_t satisfy the relation (33) and the random variables $\{\varepsilon_t\}$ have zero mean and variance reflecting the basis risk. In what follows, we do not limit ourselves to this or any other specific assumption about the parameters α_t and the distribution of the random variables $\{\varepsilon_t\}$. We present option pricing bounds as functions of α_t and the parameter $\bar{\varepsilon}$, defined as the upper bound to the random variables $\{\varepsilon_t\}$. We assume that the parameter $\bar{\varepsilon}$ is observable from historical data.

We begin by introducing an American, cash-settled futures put option with strike K and expiration date T , same as the delivery date of the futures. We consider the following sequence of events. A trader enters date t with endowments x_t and y_t in the bond and stock accounts, respectively, and a *long* position in a futures put. The endowments x_t and y_t are net of any cash flows that the trader has incurred at date t or at an earlier date in buying the put. We stipulate that, at each date, the trader may either hold on to the put position or exercise it, but is constrained from selling it. If the trader exercises the put at time t , she receives cash payoff $K - F_t$.

The following proposition provides a lower bound on the *reservation purchase price of the American put*, in terms of the auxiliary function $M(S, t)$ defined recursively for $S \geq 0$, as follows:

$$M(S, t) = \frac{1}{R_t} E[\max[K - (\alpha_{t+1} S_{t+1} + \bar{\varepsilon}), M(S_{t+1}, t + 1)] \mid S_t = S], \quad t \leq T - 1, \quad (34)$$

and

$$M(S, T) = 0. \quad (35)$$

The function $M(S, t)$ has the interpretation as the price of an American futures put if there are no transaction costs, the expectation is taken with respect to the risk-neutral distribution (that is, setting $R_t = R$), and the cost of carry has an error of $\bar{\varepsilon}$. The proof is omitted as it is similar to that of Proposition 1.

¹⁶ See, for instance, Modest and Sundaresan (1983).

Proposition 5. *If the monotonicity of wealth condition holds, $\underline{P}(F_t, S_t, t)$ is a lower bound on the reservation purchase price of an American futures put option at time t , where*

$$\underline{P}(F_t, S_t, t) \equiv \max \left[K - F_t, \frac{1 - k_2}{1 + k_1} M(S_t, t) \right], \quad t \leq T. \quad (36)$$

Next we introduce an American, cash-settled futures *call* option, instead of a *put* option. The call has strike K and expiration date T , same as the delivery date of the futures. We consider the following sequence of events. A trader enters date t with endowments x_t and y_t in the bond and *ex dividend* stock accounts, respectively, and a *short* position in a futures call option. The endowments are stated net of any cash flows that the trader has incurred at date t or at an earlier date in writing the call, and net of the dividend payable on the stock at time t . First, the trader is informed whether she has been "assigned" or not. If the trader has been "assigned", then the trader pays $F_t - K$ in cash and has her position in the call closed out. If the trader has been "assigned", the value of the cash account becomes $x_t - (F_t - K)$.

The following proposition provides an upper bound on the *reservation write price of the American call*, in terms of the auxiliary function $N(S, t)$ defined recursively for $S \geq 0$, as follows:

$$N(S, t) = \frac{1}{R_t} E[\max\{\alpha_{t+1} S_{t+1} + \bar{\varepsilon} - K, N(S_{t+1}, t+1)\} \mid S_t = S], \quad t \leq T-1 \quad (37)$$

and

$$N(S, T) = 0. \quad (38)$$

Proposition 6. *If the monotonicity of wealth condition holds, $\bar{C}(F_t, S_t, t)$ is an upper bound on the reservation write price of an American futures call option at time t , where*

$$\bar{C}(F_t, S_t, t) = \frac{1 + k_1}{1 - k_2} \max[N(S_t, t), F_t - K], \quad t \leq T. \quad (39)$$

As with the corresponding bounds for index options, the bounds given in (36) and (39) are also reservation prices for *unconstrained* traders, who are allowed to close their futures options positions prior to expiration. The proofs of Propositions 5 and 6 also rely on monotonicity restrictions similar

to (18) and (19). These can be satisfied by imposing trader-specific limits on the position in the option relative to the stock and bond holdings.

6. Concluding Remarks

We derive upper and lower bounds on the reservation purchase price and the reservation write price of American put and call options when trading costs in the underlying security are proportional to the value of the underlying security that is being traded. The lower bound on the reservation purchase price of American puts is tight. In addition, the upper bound on the reservation write price of American calls is tight for in-the-money and at-the-money calls, but less so for out-of-the-money calls.

Violations of the bounds trigger investment strategies that increase the expected utility of *any* risk averse investor. The only restrictions are that the investor holds a two-asset portfolio containing the underlying asset and the risk free asset, and that the monotonicity of wealth condition holds. The latter can be achieved by limiting the derivative position in the adopted strategies to a small proportion of the underlying asset holdings in the initial portfolio. Since we do not make the restrictive assumption that *all* traders belong to the class of utility-maximizing traders, our results are unaffected *and indeed may be strengthened* by the presence of traders with different objectives and preferences and facing a different transaction costs schedule than that of the utility-maximizing traders.

For which markets and traders are the results of this paper most applicable? It is useful to consider separately the following cases: individual investors in organized exchanges; institutional investors in organized exchanges; dealers in over-the-counter markets for plain-vanilla derivatives; and dealers in over-the-counter markets for client-customized derivatives. Two important assumptions made in this paper are that the traders have exposure in the payoff of the derivatives that is small compared to their net worth and that traders maximize their utility. These assumptions are most applicable to individual investors in exchange-traded derivatives.

Consider next institutional investors in exchange-traded derivatives and in over-the-counter, plain-vanilla derivatives. Institutional investors typically take large positions in the derivatives and we cannot plausibly argue that these investors have sufficient reserves to afford to leave the book exposed to substantial risk through endogenous trading. However, because these derivatives are standardized, institutional investors may oftentimes hedge a long exposure in such a derivative by taking a short exposure in a similar derivative in a transaction with a third party. Thus, investors may have to hedge only the *residual* exposure of the book to risk and this may be done with low transaction costs. Transaction costs play only a minor role in these trades because it is easy to hedge the book with offsetting trades.

For example, the bid-asked spread on plain-vanilla swaps is just a few basis points.

Unlike the case with over-the-counter, plain vanilla derivatives, a long position in a client-customized derivative cannot typically be hedged with an offsetting trade in a similar derivative with a third party. The reason is that these derivatives are not standardized and it is difficult to match their contingent cash flows in pairs. In practice, dealers impose upon themselves tight exposure limits to the various sources of risk, such as delta and vega risk. They hedge the derivative dynamically on a stand-alone basis and incur substantial transaction costs that are passed on to the client in the form of price quotations that are substantially different than the theoretical value. The approach discussed in this paper may provide dealers with a different perspective, reduce the transaction costs, and result in price quotations that are closer to the theoretical value.

Appendix A: Proof of Proposition 1

(i) For $S \geq 0$ and $t \leq T$ we define the auxiliary function $h(S_{t-1}, S_t, t)$ as

$$h(S_{t-1}, S_t, t) = \max[K - S_t, M(S_t, t)] - \hat{R}_{t-1}^{-1} \frac{S_t}{S_{t-1}} \left(1 + \frac{\gamma_t}{1 + k_1} \right) \times E[\max[K - S_t, M(S_t, t)] | S_{t-1}]. \quad (\text{A.1})$$

By the definition of $\hat{R}_{t-1}$, we have

$$\frac{1}{\hat{R}_{t-1}} E \left[\frac{S_t}{S_{t-1}} \left(1 + \frac{\gamma_t}{1 + k_1} \right) | S_{t-1} \right] = 1 \quad (\text{A.2})$$

and, therefore, $E[h(S_{t-1}, S_t, t) | S_{t-1}] = 0$.

(ii) We prove that there exists a unique function $\hat{S}_t \equiv \hat{S}_t(S_{t-1}, t)$ such that $h(S_{t-1}, S_t, t) > (<)0$, as $S_t < (>)\hat{S}_t$. First note that $E[h(S_{t-1}, S_t, t) | S_{t-1}] = 0$ implies that there exists a value of S_t , say $\hat{S}_t \equiv \hat{S}_t(S_{t-1}, t)$, such that $h(S_{t-1}, \hat{S}_t(S_{t-1}, t), t) = 0$. Second, recall that $M(S_t, t)$ is a decreasing and convex function of S_t . Third, note that $h(S_{t-1}, S_t, t)$ is a decreasing and convex function of S_t because it is the sum of two decreasing and convex functions of S_t . Therefore, the value of S_t such that $E[h(S_{t-1}, S_t, t) | S_{t-1}] = 0$ is unique and $h(S_{t-1}, S_t, t) > (>)0$, as $S_t < (>)\hat{S}_t$.

(iii) We prove that $\underline{P}(S_T, T)$ is a lower bound on the reservation purchase price of an American put option at the expiration date T . By Equations (14) and (15), we have $\underline{P}(S_T, T) = [K - S_T]^+$. Therefore,

$$\begin{aligned} J(x_T - \underline{P}(S_T, T), y_T, S_T, T) &= V(x_T - [K - S_T]^+ + [K - S_T]^+, y_T, T) \\ &= V(x_T, y_T, T) \end{aligned} \quad (\text{A.3})$$

and $\underline{P}(S_T, T)$ is a lower bound on the reservation purchase price of an American put option at the expiration date T .

(iv) We prove that Equation (16) holds at $t = T$.

$$\begin{aligned}
 & J\left(x_T, y_T - \frac{1}{1+k_1} \max[K - S_T, M(S_T, T)], S_T, T\right) \\
 &= V\left(x_T + [K - S_T]^+, y_T - \frac{1}{1+k_1} [K - S_T]^+, T\right) \\
 &\geq V\left(x_T, y_T - \frac{1}{1+k_1} [K - S_T]^+ + \frac{1}{1+k_1} [K - S_T]^+, T\right) \\
 &\geq V(x_T, y_T, T).
 \end{aligned} \tag{A.4}$$

Then we proceed by induction. We assume that $\underline{P}(S_t, t)$ is a lower bound on the reservation purchase price for an American put option at date t and that Equation (16) holds at date t .

(v) We prove that $\underline{P}(S_{t-1}, t-1)$ is a lower bound on the reservation purchase price for an American put at date $t-1$. We consider separately the cases $\underline{P}(S_{t-1}, t-1) = K - S_{t-1}$ and $\underline{P}(S_{t-1}, t-1) > K - S_{t-1}$.

First, we assume that $\underline{P}(S_{t-1}, t-1) = K - S_{t-1}$. Then

$$\begin{aligned}
 & J(x_{t-1} - \underline{P}(S_{t-1}, t-1), y_{t-1}, S_{t-1}, t-1) \\
 &\geq V(x_{t-1} - \underline{P}(S_{t-1}, t-1) + K - S_{t-1}, y_{t-1}, t-1) \\
 &\geq V(x_{t-1}, y_{t-1}, t-1)
 \end{aligned} \tag{A.5}$$

and $\underline{P}(S_{t-1}, t-1)$ is a lower bound on the reservation purchase price for an American put at date $t-1$.

Second, we assume that $\underline{P}(S_{t-1}, t-1) > K - S_{t-1}$. Then

$$\begin{aligned}
 & J(x_{t-1} - \underline{P}(S_{t-1}, t-1), y_{t-1}, S_{t-1}, t-1) \\
 &= J\left(x_{t-1} - \frac{1-k_2}{1+k_1} M(S_{t-1}, t-1), y_{t-1}, S_{t-1}, t-1\right)
 \end{aligned} \tag{A.6}$$

(by Equation (15))

$$\geq J\left(x_{t-1}, y_{t-1} - \frac{1}{1+k_1} M(S_{t-1}, t-1), S_{t-1}, t-1\right)$$

(because it is feasible to sell stock of value $\frac{1}{1+k_1} M(S_{t-1}, t-1)$ and increase the cash account by $\frac{1-k_2}{1+k_1} M(S_{t-1}, t-1)$)

$$\begin{aligned} &\geq \max_j E \left[J \left(\left\{ x_{t-1} - j - \max(k_1 j, -k_2 j) \right. \right. \right. \\ &\quad \left. \left. \left. + \left(y_{t-1} + j - \frac{1}{1+k_1} M(S_{t-1}, t-1) \right) \frac{\gamma_{t+1} S_t}{S_{t-1}} \right\} R, \right. \right. \\ &\quad \left. \left. \left(y_{t-1} + j - \frac{1}{1+k_1} M(S_{t-1}, t-1) \right) \frac{S_t}{S_{t-1}}, S_t, t \right) \mid S_{t-1} \right] \end{aligned}$$

(by the definition of J in Equation (10))

$$\begin{aligned} &\geq \max_j E \left[J(\{x_{t-1} - j - \max(k_1 j, -k_2 j)\} R, \right. \\ &\quad \left. \left(y_{t-1} + j - \frac{1}{1+k_1} M(S_{t-1}, t-1) \right) \left(1 + \frac{\gamma_t}{1+k_1} \right) \frac{S_t}{S_{t-1}}, S_t, t \right) \mid S_{t-1} \right] \\ &\geq E \left[J \left((x'_{t-1} R, \left(y'_{t-1} - \frac{M(S_{t-1}, t-1)}{1+k_1} \right) \left(1 + \frac{\gamma_t}{1+k_1} \right) \frac{S_t}{S_{t-1}}, S_t, t \right) \mid S_{t-1} \right] \end{aligned} \quad (\text{A.7})$$

(by the definition of x'_t and y'_t in Equations (7) and (8) respectively and the fact that the optimal investment, j , in the maximization problem of Equation (10) may differ from the optimal investment, v_t^* , in the problem of Equation (5))

$$\begin{aligned} &\geq E \left[V \left(x'_{t-1} R, \left(y'_{t-1} - \frac{M(S_{t-1}, t-1)}{1+k_1} \right) \left(1 + \frac{\gamma_t}{1+k_1} \right) \frac{S_t}{S_{t-1}} \right. \right. \\ &\quad \left. \left. + \frac{\max[K - S_t, M(S_t, t)]}{1+k_1}, t \right) \mid S_{t-1} \right] \end{aligned}$$

(by Equation (16))

$$\geq E \left[V \left(x'_{t-1} R, y'_{t-1} \left(1 + \frac{\gamma_t}{1+k_1} \right) \frac{S_t}{S_{t-1}} + \frac{h(S_{t-1}, S_t, t)}{1+k_1}, t \right) \mid S_{t-1} \right]$$

(by the definition of $h(S_{t-1}, S_t, t)$ and M)

$$\begin{aligned} &\geq E \left[V \left(x'_{t-1} R, y'_{t-1} \frac{S_t}{S_{t-1}}, t \right) + V_y \left(x'_{t-1} R, y'_{t-1} \frac{S_t}{S_{t-1}} + \frac{h(S_{t-1}, S_t, t)}{1+k_1}, t \right) \right. \\ &\quad \left. \times \frac{h(S_{t-1}, S_t, t)}{1+k_1} \mid S_{t-1} \right] \end{aligned}$$

(by the concavity of the function V in y)

$$\geq V(x_{t-1}, y_{t-1}, t-1) + E \left[V_y \left(x'_{t-1}, R, y'_{t-1} \frac{S_t}{S_{t-1}} + \frac{h(S_{t-1}, S_t, t)}{1+k_1}, t \right) \frac{h(S_{t-1}, S_t, t)}{1+k_1}, \mid S_{t-1} \right]$$

(by the definition of V in Equation (5))

$$\geq V(x_{t-1}, y_{t-1}, t-1) + V_y \left(x'_{t-1} R, y'_{t-1} \frac{\hat{S}_t}{S_{t-1}}, t \right) E \left[\frac{h(S_{t-1}, S_t, t)}{1+k_1}, \mid S_{t-1} \right]$$

(by the properties of the function h , the monotonicity condition and the concavity of the function V in y)

$$\geq V(x_{t-1}, y_{t-1}, t-1).$$

(because $E[h(S_{t-1}, S_t, t) \mid S_{t-1}] = 0$).

Therefore, $\underline{P}(S_{t-1}, t-1)$ is a lower bound on the reservation purchase price for an American put at date $t-1$.

(vi) Finally, we prove that Equation (16) holds at $t-1$. We consider separately the cases $K - S_{t-1} \geq M(S_{t-1}, t-1)$ and $K - S_{t-1} < M(S_{t-1}, t-1)$.

First, we assume that $K - S_{t-1} \geq M(S_{t-1}, t-1)$. Then

$$\begin{aligned} & J \left(x_{t-1}, y_{t-1} - \frac{1}{1+k_1} \max[K - S_{t-1}, M(S_{t-1}, t-1)], S_{t-1}, t-1 \right) \\ & \geq V \left(x_{t-1} + K - S_{t-1}, y_{t-1} - \frac{K - S_{t-1}}{1+k_1}, t-1 \right) \\ & \geq V(x_{t-1}, y_{t-1}, t-1). \end{aligned} \tag{A.8}$$

Second, we assume that $K - S_{t-1} < M(S_{t-1}, t-1)$. Then

$$\begin{aligned} & J \left(x_{t-1}, y_{t-1} - \frac{1}{1+k_1} \max[K - S_{t-1}, M(S_{t-1})], S_{t-1}, t-1 \right) \\ & \geq J \left(x_{t-1}, y_{t-1} - \frac{M(S_{t-1}, t-1)}{1+k_1}, S_{t-1}, t-1 \right). \end{aligned} \tag{A.9}$$

Hereafter the proof follows the same steps as the proof in part (v) and is omitted.

Appendix B: Proof of Proposition 2

We defined earlier the function $L(S_t, t)$ with Equations (23) and (24). We state without proof some straightforward properties of this function. $L(S_t, t)$ is

decreasing and convex in S_t ; $L(0, t) = K/R$ for all $t \leq T - 1$; and $L(S_t, t) \rightarrow 0$ as $S_t \rightarrow \infty$.

We define the function $f(S_t, t) \equiv K/R - L(S_t, t)$. Given the above discussion, this function has the following properties: $f(0, t) = 0$; $f(S_t, t)$ is increasing and concave; and $f(S_t, t) \rightarrow K/R$ as $S_t \rightarrow \infty$.

We define the function $H(x_t, y_t, S_t, t)$ as

$$H(x_t, y_t, S_t, t) = \max_j E \left[J \left(\left\{ x_t - j - \max(k_1 j, -k_2 j) + (y_t + j) \frac{\gamma_{t+1} S_{t+1}}{S_t} \right\} R, \right. \right. \\ \left. \left. (y_t + j) \frac{S_{t+1}}{S_t}, S_{t+1}, t + 1 \right) \mid S_t \right] \quad (\text{B.1})$$

and prove the following intermediate result by induction:

$$H \left(x_t + K/R, y_t \frac{f(S_t, t)}{1 - k_2}, S_t, t \right) \geq V(x_t, y_t, t), t \leq T - 1. \quad (\text{B.2})$$

First, we prove that Equation (B.2) holds at $t = T - 1$.

$$H \left(x_{T-1} + K/R, y_{T-1} - \frac{f(S_{T-1}, T - 1)}{1 - k_2}, S_{T-1}, T - 1 \right) \\ \geq E \left[J \left(x'_{T-1} R + K - \frac{\gamma_T S_T}{S_{T-1}} \frac{f(S_{T-1}, T - 1)}{1 - k_2}, \right. \right. \\ \left. \left. y'_{T-1} \frac{S_T}{S_{T-1}} - \frac{S_T}{S_{T-1}} \frac{f(S_{T-1}, T - 1)}{1 - k_2}, S_T, T \right) \mid S_{T-1} \right]$$

(by the definition of x'_{T-1} , y'_{T-1} and the function H)

$$\geq E \left[J \left(x'_{T-1} R + K, y'_{T-1} \frac{S_T}{S_{T-1}} - \frac{\left(1 + \frac{\gamma_T}{1 - k_2}\right) S_T}{S_{T-1}} \frac{f(S_{T-1}, T - 1)}{1 - k_2}, S_T, T \right) \mid S_{T-1} \right] \\ = E \left[V \left(x_T + K - (K - S_T)^+, y_T - \frac{\left(1 + \frac{\gamma_T}{1 - k_2}\right) S_T}{S_{T-1}} \frac{f(S_{T-1}, T - 1)}{1 - k_2}, S_T, T \right) \mid S_{T-1} \right]$$

(by (21), and the fact that $x'_{T-1}R = x_T$ and $y'_{T-1}\frac{S_T}{S_{T-1}} = y_T$)

$$\begin{aligned} &\geq E \left[V \left(x_T, \frac{K - (K - S_T)^+}{1 + k_1} + y_T - \frac{\left(1 + \frac{\gamma_T}{1 - k_2}\right) S_T}{S_{T-1}} \frac{f(S_{T-1}, T-1)}{1 - k_2}, S_T, T \right) | S_{T-1} \right] \\ &\geq V(x_T, y_T, T) + E \left[\frac{\partial V}{\partial y} \left\{ \frac{K - (K - S_T)^+}{1 + k_1} - \frac{\left(1 + \frac{\gamma_T}{1 - k_2}\right) S_T}{S_{T-1}} \frac{f(S_{T-1}, T-1)}{1 - k_2} \right\} | S_{T-1} \right] \end{aligned}$$

(by (4) and (5) and the concavity of the function V).

We define the following function:

$$h(S_T, S_{T-1}, T-1) \equiv \frac{K - (K - S_T)^+}{1 + k_1} - \frac{\left(1 + \frac{\gamma_T}{1 - k_2}\right) S_T}{S_{T-1}} \frac{f(S_{T-1}, T-1)}{1 - k_2}.$$

Given S_{T-1} and T , we prove that there exists a unique strictly positive value of S_T , say S_T^* , such that $h(S_T^*, S_{T-1}, T-1) = 0$. Note first that h equals zero at $S_T = 0$ and becomes negative for large values of S_T . Second, the expectation of h with respect to S_T is zero by the definition of the function f . Therefore, h must assume both positive and negative values for $S_T \geq 0$. Since h is a concave function of S_T , it has exactly two zeros. Finally, since the one zero occurs at $S_T = 0$, the second zero occurs at a positive value of S_T .

Hence, we have, by the monotonicity condition,

$$\begin{aligned} &E \left[\frac{\partial V}{\partial y} \left\{ \frac{K - (K - S_T)^+}{1 + k_1} - \frac{\left(1 + \frac{\gamma_T}{1 - k_2}\right) S_T}{S_{T-1}} \frac{f(S_{T-1}, T-1)}{1 - k_2} \right\} | S_{T-1} \right] \\ &\geq \left(\frac{\partial V}{\partial y} \right)_{S_T=S_T^*} E[h(S_T, S_{T-1}, T-1) | S_{T-1}] = 0. \end{aligned}$$

Thus, we have proved that (B.2) holds at $T-1$.

To complete the induction argument, we assume that (B.2) holds at $t+1$, $t \leq T-2$, and prove that it holds at t .

$$H \left(x_t + \frac{K}{R}, y_t - \frac{f(S_t, t)}{1 - k_2}, S_t, t \right) = H \left(x_t + \frac{K}{R}, y_t - \frac{K}{(1 + k_1)\tilde{R}_t} + \frac{1}{(1 - k_2)\tilde{R}_t} \right)$$

$$\times E \left[\max \left\{ \frac{1-k_2}{1+k_1} (K - S_{t+1})^+, L(S_{t+1}, t+1) - \frac{K}{R} \frac{k_1+k_2}{1+k_1} \right\} \mid S_t \right], S_t, t \right)$$

(by (22) and (23) and the definition of the function f)

$$\begin{aligned} &\geq E \left[J \left(x'_t R + K, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{S_t} \frac{1}{1+k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1-k_2}}{(1-k_2)\tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \right. \\ &\quad \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1-k_2}{1+k_1}, L(S_{t+1}, t+1) \right. \right. \\ &\quad \left. \left. \left. - \frac{K}{R} \frac{k_1+k_2}{1+k_1} \right\} \mid S_t \right], S_{t+1}, t+1 \right) \mid S_t \right] \end{aligned} \quad (\text{B.3})$$

(by the definition of x'_t , y'_t and the function H).

By its definition in Equation (20), the function J is equal to the minimum of two right-hand-side terms. Suppose first that the first term is the smaller of the two right-hand-side terms. Then,

$$\begin{aligned} &J \left(x'_t R + K, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{S_t} \frac{1}{1+k_1} \frac{K}{\tilde{R}_t} + \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{(1-k_2)\tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\ &\times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1-k_2}{1+k_1}, L(S_{t+1}, t+1) - \frac{K}{R} \frac{k_1+k_2}{1+k_1} \right\} \mid S_t \right], S_{t+1}, t+1 \right) \\ &= V \left(x'_t R + K - (K - S_{t+1})^+, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{S_t} \frac{1}{1+k_1} \frac{K}{\tilde{R}_t} \right. \\ &\quad \left. + \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{(1-k_2)\tilde{R}_t} \frac{S_{t+1}}{S_t} \right) \end{aligned}$$

$$\begin{aligned}
& \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \Big) \\
& \geq V \left(x'_t R, y'_t \frac{S_{t+1}}{S_t} + \frac{K - (K - S_{t+1})^+}{1 + k_1} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} \right. \\
& \quad \left. + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
& \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \Big).
\end{aligned}$$

(B.4)

Suppose next that the second term is the smaller of the two right-hand-side terms. Then,

$$\begin{aligned}
& J \left(x'_t R + K, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
& \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \Big) \\
& = H \left(x'_t R + K, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
& \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \Big)
\end{aligned}$$

$$\begin{aligned}
&\geq V \left(x'_t R + K - \frac{K}{R}, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{f(S_{t+1}, t + 1)}{1 - k_2} \right. \\
&\quad \left. + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
&\quad \left. \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \right)
\end{aligned}$$

(by the induction assumption that (B.2) holds at $t + 1$)

$$\begin{aligned}
&\geq V \left(x'_t R, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{K - K/R}{1 + k_1} \right. \\
&\quad \left. + \frac{f(S_{t+1}, t + 1)}{1 - k_2} + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
&\quad \left. \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \right).
\end{aligned}$$

(B.5)

We combine the results in Equations (B.4) and (B.5), and obtain the following:

$$\begin{aligned}
&J \left(x'_t R + K, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
&\quad \left. \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right], S_{t+1}, t + 1 \right)
\end{aligned}$$

$$\begin{aligned}
&\geq V \left(x'_t R, y'_t \frac{S_{t+1}}{S_t} + \frac{K}{1+k_1} - \max \left\{ \frac{(K - S_{t+1})^+}{1+k_1}, \frac{K}{(1+k_1)R} - \frac{f(S_{t+1}, t+1)}{1-k_2} \right\} \right. \\
&\quad \left. - \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{S_t} \frac{1}{1+k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1-k_2}}{(1-k_2)\tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \\
&\quad \left. \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1-k_2}{1+k_1}, L(S_{t+1}, t+1) - \frac{K}{R} \frac{k_1+k_2}{1+k_1} \right\} \mid S_t \right], S_{t+1}, t+1 \right).
\end{aligned}
\tag{B.6}$$

We combine the results in Equations (B.3) and (B.6), and obtain the following:

$$\begin{aligned}
&H \left(x_t + K/R, y_t - \frac{f(S_t, t)}{1-k_2}, S_t, t \right) \\
&\geq E \left[J \left(x'_t R + K, y'_t \frac{S_{t+1}}{S_t} - \frac{\left(1 + \frac{\gamma_{t+1}}{1-k_2}\right) S_{t+1}}{S_t} \frac{1}{1+k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1-k_2}}{(1-k_2)\tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \right. \\
&\quad \left. \left. \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1-k_2}{1+k_1}, L(S_{t+1}, t+1) \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{K}{R} \frac{k_1+k_2}{1+k_1} \right\} \mid S_t \right], S_{t+1}, t+1 \right) \mid S_t \right].
\end{aligned}
\tag{B.7}$$

(by Equation (B.3))

$$\begin{aligned}
&\geq E \left[V \left(x'_t R, y'_t \frac{S_{t+1}}{S_t} + \frac{K}{1+k_1} \right. \right. \\
&\quad \left. \left. - \max \left\{ \frac{(K - S_{t+1})^+}{1+k_1}, \frac{K}{(1+k_1)R} - \frac{f(S_{t+1}, t+1)}{1-k_2} \right\} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right)}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \\
& \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) \right. \right. \\
& \quad \left. \left. - \frac{K k_1 + k_2}{R \frac{1}{1 + k_1}} \right\} \mid S_t \right], S_{t+1}, t + 1 \Big) \mid S_t \Big]
\end{aligned}$$

(by Equation (B.6))

$$\begin{aligned}
& \geq V(x_t, y_t, t) + E \left[\frac{\partial V}{\partial y} \left\{ \frac{K}{1 + k_1} \right. \right. \\
& \quad \left. \left. - \max \left\{ \frac{(K - S_{t+1})^+}{1 + k_1}, \frac{K}{(1 + k_1)R} - \frac{f(S_{t+1}, t + 1)}{1 - k_2} \right\} \right. \right. \\
& \quad \left. \left. - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \right. \right. \\
& \quad \left. \left. \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) \right. \right. \right. \right. \\
& \quad \quad \left. \left. \left. - \frac{K k_1 + k_2}{R \frac{1}{1 + k_1}} \right\} \mid S_t \right], S_{t+1}, t + 1 \right\} \mid S_t \right]
\end{aligned}$$

(by Equations (4) and (5) and the concavity of the function V)

$$\geq V(x_t, y_t, t) + E \left[\frac{\partial V}{\partial y} h(S_{t+1}, S_t, t) \right]$$

by the monotonicity condition, where the function $h(S_{t+1}, S_t, t)$ is defined as follows:

$$\begin{aligned}
h(S_{t+1}, S_t, t) \equiv & \frac{K}{1 + k_1} - \max \left\{ \frac{(K - S_{t+1})^+}{1 + k_1}, \frac{K}{(1 + k_1)R} - \frac{f(S_{t+1}, t + 1)}{1 - k_2} \right\} \\
& - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{S_t} \frac{1}{1 + k_1} \frac{K}{\tilde{R}_t} + \frac{1 + \frac{\gamma_{t+1}}{1 - k_2}}{(1 - k_2) \tilde{R}_t} \frac{S_{t+1}}{S_t} \times
\end{aligned}$$

$$\begin{aligned} & \times E \left[\max \left\{ (K - S_{t+1})^+ \frac{1 - k_2}{1 + k_1}, L(S_{t+1}, t + 1) - \frac{K}{R} \frac{k_1 + k_2}{1 + k_1} \right\} \mid S_t \right] \\ & = \frac{\tilde{R}_t f(S_{t+1}, t)}{1 - k_2} - \frac{\left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right) S_{t+1}}{(1 - k_2) S_t} E[f(S_{t+1}, t) \mid S_t]. \end{aligned}$$

Given S_t and t , we prove that there exists a unique strictly positive value of S_{t+1} , say S_{t+1}^* , such that $h(S_{t+1}^*, S_t, t) = 0$. Note first that h equals zero at $S_{t+1} = 0$ and becomes negative for large values of S_{t+1} . Second, the expectation of h with respect to S_{t+1} is zero by the definition of the function f . Therefore, h must assume both positive and negative values for $S_{t+1} \geq 0$. Since h is a concave function of S_{t+1} , it has exactly two zeros. Finally, since the one zero occurs at $S_{t+1} = 0$, the second zero occurs at a positive value of S_{t+1} .

Hence, we have

$$H \left(x_t + K/R, y_t - \frac{f(S_t, t)}{1 - k_2}, S_t, t \right) \geq V(x_t, y_t, t) + E \left[\frac{\partial V}{\partial y} h(S_{t+1}, S_t, t) \mid S_t \right]$$

and, by the monotonicity condition,

$$\begin{aligned} & \geq V(x_t, y_t, t) + \left(\frac{\partial V}{\partial y} \right)_{S_{t+1}=S_{t+1}^*} E[h(S_{t+1}, S_t, t) \mid S_t] \\ & = V(x_t, y_t, t). \end{aligned}$$

Thus, we have proved that (B.2) holds at t .

Now we are ready to prove that $\bar{P}(S_t, t)$ is an upper bound on the reservation write price at time t , $t = T$. Obviously, the claim is true at t , $t \leq T - 1$. At any t , $t \leq T - 1$, we have

$$\begin{aligned} & J(x_t + \bar{P}(S_t, t), y_t, S_t, t) \\ & = \min\{V(x_t + \bar{P}(S_t, t) - (K - S_t)^+, y_t, t), H(x_t + \bar{P}(S_t, t), y_t, S_t, t)\} \end{aligned}$$

(by Equation (20))

$$\geq \min\{V(x_t, y_t, t), H(x_t + L(S_t, t), y_t, S_t, t)\}$$

(by the definition of $\bar{P}(S_t, t)$ in Equation (22))

$$\geq \min\{V(x_t, y_t, t), H(x_t + L(S_t, t), y_t, S_t, t)\}$$

$$\geq \min \left\{ V(x_t, y_t, t), H \left(x_t + K/R, y_t - \frac{f(S_t, t)}{1 - k_2}, S_t, t \right) \right\}$$

(by the definition of $L(S_t, t)$)

$$\geq V(x_t, y_t, t)$$

(by Equation (B.2)).

This completes the proof.

Appendix C: Proof of Proposition 3

For $S \geq 0$ and $t \leq T$ we define the auxiliary function $h(S_{t-1}, S_t, t)$ as

$$h(S_{t-1}, S_t, t) = \frac{1+k_1}{1-k_2} \left[\left(1 + \frac{\gamma_t}{1+k_1} \right) \frac{S_t}{S_{t-1}} N(S_{t-1}, t-1) - \max[(1+\gamma_t)S_t - K, N(S_t, t)] \right]. \quad (\text{C.1})$$

We state two properties of the function h that are invoked in the proof of the proposition. First, we take the expectation of the function h with respect to S_t and obtain the following property:

$$E[h(S_{t-1}, S_t, t) \mid S_{t-1}] = 0. \quad (\text{C.2})$$

The second property is that there exists a function $\hat{S}_t \equiv \hat{S}_t(S_{t-1}, t) > 0$ such that $h(S_{t-1}, S_t, t) > (<)0$, as $0 < S_t < (>)\hat{S}_t$. We sketch the proof. For small and positive values of S_t , $h(S_{t-1}, S_t, t)$ is positive because $h(S_{t-1}, 0, t) = 0$ and $\partial h(S_{t-1}, S, t)/\partial S > 0$ at $S = 0$. For large values of S_t , $h(S_{t-1}, S_t, t)$ is negative. Therefore, there exists a strictly positive value of S_t , say $\hat{S}_t(S_{t-1}, t)$, such that $h(S_{t-1}, S_t, t)$ is zero. Since h is concave in S_t , $\hat{S}_t(S_{t-1}, t)$ is unique and $h(S_{t-1}, S_t, t) > (<)0$, as $0 < S_t < (>)\hat{S}_t$.

(i) First, we prove that $\bar{C}(S_T, T)$ is an upper bound on the reservation write price of an American call option at the expiration date T . By Equations (28) and (29), we have

$$\bar{C}(S_T, T) = \frac{1+k_1}{1-k_2} [(1+\gamma_T)S_T - K]^+.$$

Therefore,

$$\begin{aligned} & J(x_T + \bar{C}(S_T, T), y_T, S_T, T) \\ &= V\left(x_T + \frac{1+k_1}{1-k_2} [(1+\gamma_T)S_T - K]^+ - [(1+\gamma_T)S_T - K]^+, y_T, T\right) \quad (\text{C.3}) \\ &\geq V(x_t, y_T, T) \end{aligned}$$

and $\bar{C}(S_T, T)$ is an upper bound on the reservation write price of an American call option at the expiration date T .

(ii) Second, we prove that Equation (30) holds at $t = T$.

$$\begin{aligned}
 & J\left(x_T, y_T + \frac{1}{1-k_2} \max[(1+\gamma_T)S_T - K, N(S_T, T)], S_T, T\right) \\
 &= V\left(x_T - [(1+\gamma_T)S_T - K]^+, y_T + \frac{1}{1-k_2} [(1+\gamma_T)S_T - K]^+, T\right) \quad (C.4) \\
 &\geq V\left(x_T, y_T + \frac{1}{1-k_2} [(1+\gamma_T)S_T - K]^+ - \frac{1}{1-k_2} [(1+\gamma_T)S_T - K]^+, T\right) \\
 &\geq V(x_T, y_T, T).
 \end{aligned}$$

Then we proceed by induction. We assume that $\bar{C}(S_t, t)$ is an upper bound on the reservation write price for an American call option at date t and that Equation (30) holds at date t .

(iii) Third, we prove that $\bar{C}(S_{t-1}, t-1)$ is an upper bound on the reservation write price of a call at date $t-1$. First, we note that

$$\begin{aligned}
 & J(x_{t-1} + \bar{C}(S_{t-1}, t-1), y_{t-1}, S_{t-1}, t-1) \\
 &\geq J\left(x_{t-1}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}, S_{t-1}, t-1\right). \quad (C.5)
 \end{aligned}$$

By Equation (25),

$$J\left(x_{t-1}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}, S_{t-1}, t-1\right)$$

is either equal or smaller than

$$V\left(x_{t-1} - \{(1+\gamma_{t-1})S_{t-1} - K\}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}, t-1\right).$$

We consider separately the two cases. First, we assume that the two expressions are equal. Then

$$\begin{aligned}
 & J(x_{t-1} + \bar{C}(S_{t-1}, t-1), y_{t-1}, S_{t-1}, t-1) \\
 &\geq J\left(x_{t-1}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}, S_{t-1}, t-1\right) \quad (C.6) \\
 &\geq V\left(x_{t-1} - \{(1+\gamma_{t-1})S_{t-1} - K\}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}, t-1\right)
 \end{aligned}$$

$$\geq V\left(x_{t-1}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1} - \frac{\{(1+\gamma_{t-1})S_{t-1} - K\}}{1-k_2}, t-1\right)$$

$$\geq V(x_{t-1}, y_{t-1}, t-1)$$

by Equation (29). Therefore, $\bar{C}(S_{t-1}, t-1)$ is an upper bound on the reservation write price of a call at date $t-1$.

Second, we assume that the first expression is smaller than the second. Then,

$$\begin{aligned} & J(x_{t-1} + \bar{C}(S_{t-1}, t-1), y_{t-1}, S_{t-1}, t-1) \\ & \geq J\left(x_{t-1}, y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}, S_{t-1}, t-1\right) \end{aligned} \quad (C.7)$$

$$\begin{aligned} & \geq \max_j E\left[J\left(\{x_{t-1} - j - \max[k_1 j, -k_2 j]\}R + \left(y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1} + j\right)\right.\right. \\ & \quad \left.\left. \times \frac{\gamma_t S_t}{S_{t-1}}, \left(y_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1} + j\right) \frac{S_t}{S_{t-1}}, S_t, t\right) \mid S_{t-1}\right] \end{aligned}$$

(by Equation (25))

$$\geq E\left[J\left(x'_{t-1}R + \left\{y'_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}\right\} \frac{\gamma_t S_t}{S_{t-1}}, \left\{y'_{t-1} + \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}\right\} \frac{S_t}{S_{t-1}}, S_t, t\right) \mid S_{t-1}\right]$$

(by the definitions in Equations (7) and (8))

$$\geq E\left[J\left(x'_{t-1}R + \frac{y'_{t-1}\gamma_t S_t}{S_{t-1}}, \left\{y'_{t-1} + \left(1 + \frac{\gamma_t}{1+k_1}\right) \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}\right\} \frac{S_t}{S_{t-1}}, S_t, t\right) \mid S_{t-1}\right]$$

$$\begin{aligned} & \geq E\left[V\left(x'_{t-1}R + \frac{y'_{t-1}\gamma_t S_t}{S_{t-1}}, \left\{y'_{t-1} + \left(1 + \frac{\gamma_t}{1+k_1}\right) \frac{\bar{C}(S_{t-1}, t-1)}{1+k_1}\right\} \frac{S_t}{S_{t-1}}\right.\right. \\ & \quad \left.\left. - \frac{1}{1-k_2} \max[(1+\gamma_t)S_t - K, N(S_t, t)], t\right) \mid S_{t-1}\right] \end{aligned}$$

(by the fact that Equation (30) holds at t by the induction hypothesis)

$$\begin{aligned} & \geq E\left[V\left(x'_{t-1}R + \frac{y'_{t-1}\gamma_t S_t}{S_{t-1}}, \left\{y'_{t-1} + \left(1 + \frac{\gamma_t}{1+k_1}\right) \frac{N(S_{t-1}, t-1)}{1+k_2}\right\} \frac{S_t}{S_{t-1}}\right.\right. \\ & \quad \left.\left. - \frac{1}{1-k_2} \max[(1+\gamma_t)S_t - K, N(S_t, t)], t\right) \mid S_{t-1}\right] \end{aligned}$$

(by Equation (29))

$$\geq E \left[V \left(x'_{t-1} R + \frac{y'_{t-1} \gamma_t S_t}{S_{t-1}}, y'_{t-1} \frac{S_t}{S_{t-1}} + \frac{h(S_{t-1}, S_t, t)}{1 + k_1}, t \right) \mid S_{t-1} \right]$$

(by the definition of $h(S_{t-1}, S_t, t)$)

$$\geq V(x_{t-1}, y_{t-1}, t - 1).$$

The steps that lead to the last inequality are identical to the corresponding steps of part (v) of the proof of Proposition 1 (in Appendix A) and are omitted. Thus, we have shown that $\bar{C}(S_{t-1}, t - 1)$ is an upper bound on the reservation write price of an American call at date $t - 1$.

(iv) Finally, we prove, by induction, that Equation (30) holds. Specifically, at this step, we prove that (30) holds at $t - 1$. By Equation (25),

$$J \left(x_{t-1}, y_{t-1} + \frac{1}{1 - k_2} \max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)], S_{t-1}, t - 1 \right)$$

is either equal or smaller than

$$V \left(x_{t-1} - \{(1 + \gamma_{t-1})S_{t-1} - K\}, \right.$$

$$\left. y_{t-1} + \frac{1}{1 - k_2} \max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)], t - 1 \right).$$

We consider separately the two cases. First, we assume that the two expressions are equal. Then

$$J \left(x_{t-1}, y_{t-1} + \frac{1}{1 - k_2} \max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)], S_{t-1}, t - 1 \right) \quad (C.8)$$

$$\begin{aligned} &= V \left(x_{t-1} - \{(1 + \gamma_{t-1})S_{t-1} - K\}, y_{t-1} \right. \\ &\quad \left. + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2}, t - 1 \right) \\ &\geq V \left(x_{t-1}, y_{t-1} + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2} \right) \end{aligned}$$

$$- \frac{(1 + \gamma_{t-1})S_{t-1} - K}{1 - k_2}, t - 1)$$

$$\geq V(x_{t-1}, y_{t-1}, t - 1).$$

Therefore, Equation (30) holds at $t - 1$ in this case.

Second, we assume that the first expression is smaller than the second. Then,

$$J\left(x_{t-1}, y_{t-1} + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2}, S_{t-1}, t - 1\right) \quad (\text{C.9})$$

$$= \max_j E \left[J \left(\{x_{t-1} - j - \max[k_1 j, -k_2 j]\} R \right. \right.$$

$$\left. + \left(y_{t-1} + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2} + j \right) \frac{\gamma_t S_t}{S_{t-1}}, \right.$$

$$\left. \left(y_{t-1} + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2} + j \right) \frac{S_t}{S_{t-1}}, S_t, t \right) \mid S_{t-1} \Big]$$

(by Equation (25))

$$\geq E \left[J \left(x'_{t-1} R + \left(y'_{t-1} + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2} \right) \frac{\gamma_t S_t}{S_{t-1}}, \right. \right.$$

$$\left. \left(y'_{t-1} + \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2} \right) \frac{S_t}{S_{t-1}}, S_t, t \right) \mid S_{t-1} \Big]$$

(by the definitions in Equations (7) and (8))

$$\geq E \left[J \left(x'_{t-1} R + \frac{y'_{t-1} \gamma_t S_t}{S_{t-1}}, \right. \right.$$

$$\left. \left\{ y'_{t-1} + \left(1 + \frac{\gamma_t}{1 + k_1} \right) \frac{\max[(1 + \gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t - 1)]}{1 - k_2} \right\} \right.$$

$$\left. \times \frac{S_t}{S_{t-1}}, S_t, t \right) \mid S_{t-1} \Big]$$

$$\geq E \left[V \left(x'_{t-1} R + \frac{y'_{t-1} \gamma_t S_t}{S_{t-1}}, \right. \right.$$

$$\left\{ y'_{t-1} + \left(1 + \frac{\gamma_t}{1+k_1} \right) \frac{\max[(1+\gamma_{t-1})S_{t-1} - K, N(S_{t-1}, t-1)]}{1-k_2} \right\} \frac{S_t}{S_{t-1}} \\ - \frac{\max[(1+\gamma_t)S_t - K, N(S_t, t)]}{1-k_2}, t \Big| S_{t-1} \Big]$$

(by the fact that Equation (30) holds at t by induction)

$$\geq E \left[V \left(x'_{t-1} R + \frac{y'_{t-1} \gamma_t S_t}{S_{t-1}}, y'_{t-1} \frac{S_t}{S_{t-1}} + \frac{h(S_{t-1}, S_t, t)}{1+k_1}, t \right) \Big| S_{t-1} \right] \\ \geq V(x_{t-1}, y_{t-1}, t-1).$$

(by following the same steps as in the proof in part (iii))

Therefore, Equation (30) holds at $t-1$ in this case. This completes the proof.

Appendix D: Proof of Proposition 4

For the proof of Proposition 4 we use the following result for a European put option, proven in Constantinides and Perrakis (2002, Proposition 6). When $T' = T$, a lower bound on a European put option is equal to

$$\underline{p}(S_t, t) = \frac{E \left[[K - S_T]^+ \Big| S_t \right]}{\prod_{s=t}^{T-1} \hat{R}_s}, t \leq T. \quad (\text{D.1})$$

Suppose that the trader with horizon $T = T'$ can purchase at time t a European or American call at price C , $C > 0$, where C is equal to the right-hand side of Equation (31). We demonstrate that the trader can purchase the call and enter into a position at date t with net cost equal to the lower bound on the reservation purchase price of a European put, $\underline{p}(S_t, t)$, as given in (D.1). We also demonstrate that the cash inflow at date T equals the cash inflow associated with the exercise of a put at its expiration date. We then invoke Proposition 6 of CP (2002) and claim that the trader may increase his/her expected utility if the trader may purchase at time t a European or American call at price C . We conclude that C is a lower bound on the reservation purchase price of an American or European call option.

The trader can take the following set of actions at time t : (1) buy the call at price C ; (2) sell n_t shares of the stock, where

$$n_t = \frac{1}{1 - k_2} \frac{1}{\prod_{s=t+1}^T \left(1 + \frac{\gamma_s}{1 - k_2}\right)},$$

and receive an amount

$$\frac{S_t}{\prod_{s=t+1}^T \left(1 + \frac{\gamma_s}{1 - k_2}\right)}$$

in cash; and (3) lend $R^{-(T-t)}K$. The net cash outflow equals

$$\frac{E[\max[K - S_T, 0] \mid S_t]}{\prod_{s=t}^{T-1} \hat{R}_s},$$

the lower bound on the reservation purchase price of a European put, $\underline{p}(S_t, t)$, as given in (D.1).

Suppose that the trader refrains from exercising the call early, even if it is American. At time $t + 1$, the trader replenishes the missing dividend income $n_t \gamma_{t+1} S_{t+1}$ (since the trader sold n_t shares earlier) by selling $n_t \frac{\gamma_{t+1}}{1 - k_2}$ additional shares. The cumulative number of shares sold is

$$n_{t+1} = n_t \left(1 + \frac{\gamma_{t+1}}{1 - k_2}\right).$$

By repeating this argument at dates $t + 2, t + 3, \dots, T$, we find that the cumulative number of shares sold by time T is

$$n_T = n_t \prod_{s=t+1}^T \left(1 + \frac{\gamma_{s+1}}{1 - k_2}\right) = (1 - k_2)^{-1}.$$

Since the expiration date T coincides with the trader's end of the horizon, at date T the proceeds from the sale of all of the trader's shares are lower by the amount $(1 - k_2)(1 - k_2)^{-1} S_T = S_T$. Then the cash inflow at the expiration date is $[S_T - K]^+ - S_T + K = [K - S_T]^+$ and equals the cash inflow associated with the exercise of a put at its expiration. This completes the proof.

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