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Towards a General Theory of Good-Deal Bounds *

TOMAS BJÖRK and IRINA SLINKO
Stockholm School of Economics

Abstract. We consider an incomplete market in the form of a multidimensional Markovian factor model, driven by a general marked point process (representing discrete jump events), as well as by a standard multidimensional Wiener process. Within this framework, we study arbitrage-free good-deal pricing bounds for derivative assets, thereby extending the results by Cochrane and Saá Requejo (2000) to the point process case, while, at the same time, obtaining a radical simplification of the theory. To illustrate, we present numerical results for the classic Merton jump-diffusion model. As a by-product of the general theory, we derive extended Hansen-Jagannathan bounds for the Sharpe Ratio process in the point process setting.

1. Introduction

The majority of existing financial markets are by nature highly incomplete, because there are not sufficiently many financial instruments on the market to hedge all risks. In some cases, the incompleteness can be considered negligible, but, in a large number of cases, the incompleteness is so severe that it leads to the need to study, on the theoretical level, market models that explicitly allow incompleteness.

Typical examples of incomplete market models are models for energy and weather derivatives, stochastic volatility models, models involving mortality risk, and models containing nontrivial jump components.

Among these models, the jump models are of particular interest. Firstly, the existence of jumps in asset prices is well documented in the empirical literature. Secondly, the existence of jumps is one of the major reasons why a market model can be incomplete, and thirdly, jumps are essential to all reduced-form credit risk models. The object of the present paper is to study derivative pricing in incomplete market models that are partially driven by jump processes.

Returning to general incomplete market models, suppose now that we would like to compute an arbitrage-free price process for a financial derivative within one of the model classes mentioned above. Then, we face the following well-known facts.

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- Since the underlying market is incomplete, there will not exist a unique martingale measure (or a unique stochastic discount factor). Thus there will exist infinitely many arbitrage free price processes for a given derivative.
- In incomplete market settings, the pricing bounds that are provided by merely requiring absence of arbitrage are extremely wide, and thus useless from a practical point of view.

There is evidently a need for "reasonable" pricing bounds for derivative assets. In the seminal paper by Cochrane and Saá Requejo (2000), the authors (henceforth CSR) introduce the new idea of ruling out, not only those prices which violate the no-arbitrage restriction, but also those prices which, in some sense, represent "deals which are too good" ("good deals"). CSR essentially formalize the idea of a good deal as an asset price process with a high Sharpe ratio and pose the problem of finding the upper and lower bound for all arbitrage-free price processes, given a bound on the Sharpe ratio of the derivative.

In Cochrane and Saá Requejo (2000), this question is analyzed in great detail for discrete time one-period and multi-period models. For continuous time models, the CSR setting is restricted to that of a purely Wiener-driven model, and it seems to be very hard to extend the CSR technique to allow for the existence of jumps (see below). The basic CSR approach is to work under the objective measure P using stochastic discount factors, and their basic technique is dynamic programming. Within this framework, Cochrane and Saá Requejo derive a pricing PDE, which is then studied in detail and, in some cases, solved numerically.

A similar, but not equivalent, approach to obtain asset price bounds, based on gains-loss-ratios, is presented in Bernardo and Ledoit (2000), for which Rodriguez (2000) provides an interesting connection to linear programming. A utility-based approach can be found in Černý (2003).

The principal object of the present paper is to extend the analysis of Cochrane and Saá Requejo (2000) to allow for the possibility of jumps in the random processes that describe the financial market under consideration. Thus, in the setup of the present paper, all processes are allowed to be driven, not only by a multidimensional standard Wiener process, but also by a general marked point process ("MPP"). In view of the above discussion of jump models, we think that this extension is quite important.

We wish to point out that the technique used in the present paper, as opposed to the one used in Cochrane and Saá Requejo (2000), is very much focused on martingale measures, rather than on stochastic discount factors. More precisely, while the authors in Cochrane and Saá Requejo (2000) choose to perform all calculations under the objective measure P using various SDFs, we instead work directly under the relevant equivalent martingale measures using the Girsanov Theorem. Of course, both approaches are logically equivalent, but, for a particular concrete problem, one approach may prove much more efficient than the other from a computational point of view. The general rule is that the method of working with SDFs under P is mainly employed in (empirical) asset pricing, whereas the direct use of martingale measures is the natural choice in derivatives pricing. For the class of problems studied in the present paper, the technique of working directly under the relevant martingale measures is, in our opinion, clearly to be preferred as a technical tool, since the problem at hand (including an MPP) cannot easily be addressed by the SDF approach.

To clarify the contributions – as well as the particular choice of technique – of the present paper, we start by summarizing some limitations of the CSR approach.

Under the SDF formulation in Cochrane and Saá Requejo (2000), the basic pricing problem is never precisely formulated as a standard stochastic control problem, in the sense that it there is no clear distinction between state variables and control variables. In addition, the dynamic programming argument is rather informal.

Cochrane and Saá Requejo make a number of *ad hoc* assumptions concerning the existence and structural properties of the upper and lower good-deal bound processes. Similar assumptions are made about the SDF process.

In the CSR model, the n-dimensional price process S for the underlying market is assumed to be driven by exactly n independent random factors (Wiener processes), and the volatility matrix for the price process S is assumed to have full rank. These constitute, in themselves, minor limitations.

However, the rank assumption is equivalent to assuming that the model is "conditionally complete", namely, incompleteness enters only through the factor process Y, and that, without the Y process, the market would be complete. This structural property is used crucially in the arguments and it is hard to see how the arguments could be extended to allow for a model that is not conditionally complete.

The main limitation of Cochrane and Saá Requejo (2000) is, however, the absence of jumps. Again, the existence of jumps is one of the major reasons why a market model can be incomplete, so this is a severe limitation from a modeling perspective. Furthermore, in a model without jumps, the SDF automatically remains positive, whereas in a model including jumps, the positivity constraint on the SDF becomes binding. This is already noted in Cochrane and Saá Requejo (2000), and it appears that the CSR technique cannot easily handle a binding positivity constraint on the SDF.

With respect to the specific contributions of the present paper, we begin by noting that, when jumps are introduced into the model, the above assumptions from Cochrane and Saá Requejo (2000) are violated. The implication is that a large and substantial part of the CSR argument leading up to the central pricing equation breaks down.

In contrast, the martingale measure approach of the present paper does not incur the above problems. In particular, it does not require assumptions about the number of driving Wiener processes, nor does it need rank assumptions for the volatility matrix. Furthermore, the technique used in the present paper leads to a drastic simplification of the entire theory, compared to Cochrane and Saá Requejo (2000).

The present paper's major contribution is, however, that it allows for jumps in the underlying and derivative processes; this is extremely important from a theoretical as well as from a practical point of view. In contrast to CSR, the market in our model is generically incomplete even *without* the presence of the factor process Y. This is of special importance when dealing with, for example, reduced-form credit risk models.

We also emphasize that, in the present paper, the good-deal pricing problem appears directly as a precise, well formulated standard stochastic control problem. Indeed, the relevant Bellman equation can be written down immediately, without need of a separate argument, and without need of any *ad hoc* assumptions.

As a by-product of our investigation, we obtain a generalization of the Hansen-Jagannathan (HJ) bounds for the case of jump diffusions.

In summary, the main contribution of the present paper is that we obtain a streamlined, rigorously developed, and surprisingly simple theory of good-deal pricing bounds for a very general class of jump diffusions.

The structure of our paper is as follows. Section 2 outlines the probabilistic framework for the rest of the paper. In Section 3.1, we present our basic factor market model. The pricing problem is formalized in Section 3.2 and, in Section 3.3, we derive the fundamental Dynamic Programming Equation for the upper and lower good-deal bounds. Section 3.4 discusses the special structure of this equation in some detail and connects the good-deal pricing bounds to the so-called "minimal martingale measure". Section 4 is a relatively detailed study of some concrete point process-driven models. The classical Merton jump-diffusion model in Merton (1976) falls within this category, and we present numerical results for that particular model, where we can compare the original Merton pricing formulas (for which the jump risk was assumed to be nonpriced), as well as the pricing formula obtained by using the minimal martingale measure, with our derived good-deal bounds.

In Appendix A, we extend the Hansen-Jagannathan bounds from Hansen and Jagannathan (1991) to the general setup of Section 2.

For completeness' sake, we conclude by studying, in Appendix B, the special case of a purely Wiener-driven model, and show how the Cochrane and Saá Requejo setup is nested within our framework.

2. General Setup

Our formal setup (see below for a more intuitive description) consists of a financial market model on a fixed time interval $[0, T]$, living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, P)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, and where the measure P is interpreted as the objective (or "physical") probability measure. The basis is assumed to carry a d-dimensional standard Wiener process W as well as a marked point process $\mu(dt, dx)$ on the mark space $(X, \mathcal{X})$. The space X can be fairly general, but, in most practical applications, it will be R^k. The filtration $\mathbf{F}$ is assumed to be the internal one, i.e., the filtration generated by W and μ. The predictable (see below for comment) σ-algebra is denoted by $\mathcal{P}$, and we make the definition $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{X}$. We assume that the predictable compensator $\nu(dt, dx)$ admits an intensity, i.e., that we can write $\nu(dt, dx) = \lambda_t(dx)dt$. The compensated point process $\mu(dt, dx) - \lambda_t(dx)dt$ is denoted by $\tilde{\mu}(dt, dx)$.

REMARK 2.1. The intuitive interpretation of the point process μ is that we are modeling events which occur at discrete points in time: a very concrete example could be the modeling of earthquakes (or stock market crashes). As opposed to a more standard counting process setting, these discrete events are not all of the same type. Instead, each event is identified by its "mark" $x \in X$. In the earthquake example, a natural mark would be the strength of the earthquake on the Richter scale; in this case, the mark space X is the positive real line. The informal interpretation of the point process μ is that μ is an integer-valued (random) measure such that μ has a unit point mass at the point $(t, x) \subseteq R_+ \times X$, if, at time t, there is an event of the type x. The interpretation of the intensity measure λ is, loosely speaking, that $\lambda_t(dx)$ is the expected number of events with marks in a "small set" dx, per unit time, conditional on the information in $\mathcal{F}_{t-}$. Thus, the compensated point process $\tilde{\mu}(dt, dx) := \mu(dt, dx) - \lambda_t(dx)dt$ is "detrended" and possesses a natural martingale property. The above concept of "predictability" is very important when dealing with jump processes. The formal definition is somewhat complicated but, in most concrete cases, the reader can informally substitute the term "predictable" by "adapted and left continuous".

We need some mild integrability conditions, which we summarize as follows.

ASSUMPTION 2.1. We assume the following:

- Defining the underlying counting process N by $N_t = \nu([0, t] \times X)$, we assume that $N_t < \infty$ P-a.s. for all finite t, i.e., μ is a multivariate point process in the terminology of Jacod and Shiryaev (1987). We note that the intensity process of N is given by $\lambda_t = \lambda_t(X)$.
- There exists a deterministic increasing real valued function C_t such that, for all $t \geq 0$, we have

$$\int_0^t \lambda_s ds < C_t, \quad P - a.s. \tag{1}$$

- We assume that the counting process N defined above has exponential moments of all orders, i.e., that for all $t \geq 0$ and all constants $c > 0$ we have

$$E^P \left[e^{cN_t} \right] < \infty. \tag{2}$$

3. A Factor Market Model

We now present our Markovian factor market model, and we also formalize our pricing problem.

3.1. THE MODEL

We consider a financial market model built up by the following objects, where $\star$ denotes transpose.

- An n-dimensional price process $S = (S^1, \dots, S^n)^\star$
- A k-dimensional factor process $Y = (Y^1, \dots, Y^k)^\star$.

The interpretation of this is that $S^1, \dots, S^n$ are prices of underlying traded assets without dividends, whereas the components of Y are underlying nontraded factors.

REMARK 3.1. Note that the underlying asset in the price vector S above can be "standard" underlying assets, like common stocks, but that we also allow S to contain prices of derivatives of arbitrary types, for example, call options, bonds, or bond options. They are only "underlying" in the sense that they, within the model, are considered a priori given. Evidently, they do **not** include the derivatives for which we wish to derive the good-deal bounds.

The precise probabilistic specification of the market model is given by the following standing assumption. See the remark below for an interpretation of the point process integrals appearing in (3)–(4).

ASSUMPTION 3.1.

1. Under the objective measure P, we assume that (S, Y) satisfies the following stochastic differential equations (SDEs):

$$\begin{aligned} dS_t^i &= S_t^i \alpha_i\,(S_t, Y_t)\,dt + S_t^i \sigma_i(S_t, Y_t) dW_t \\ &\quad + S_{t-}^i \int_X \delta_i(S_{t-}, Y_{t-}, x)\mu(dt, dx), \quad i = 1, \dots, n \qquad (3) \\ dY_t^j &= a_j\,(S_t, Y_t)\,dt + b_j(S_t, Y_t) dW_t \\ &\quad + \int_X c_j(S_{t-}, Y_{t-}, x)\mu(dt, dx). \quad j = 1, \dots, k \qquad (4) \end{aligned}$$

2. We assume that, for each i and j, $\alpha_i(s, y)$ and $a_j(s, y)$ are deterministic scalar functions, $\sigma_i(s, y)$ and $b_j(s, y)$ are deterministic row vector functions, and $\delta_i(s, y, x)$ and $c_j(s, y, x)$ are deterministic scalar functions. To avoid negative asset prices, we also assume that $\delta_i(s, y, x) \geq -1$ for all i and all (s, y, x).

3. All functions above are assumed to be sufficiently regular to allow for the existence of a unique strong solution for the system of SDEs.
4. The point process μ has a predictable P-intensity measure λ. More precisely, we assume that the P-compensator $\nu(dt, dx)$ has the form

$$\nu(dt, dx) = \lambda(S_{t-}, Y_{t-}, dx)dt. \tag{5}$$

For brevity of notation, we often denote $\lambda(S_{t-}, Y_{t-}, dx)$ by $\lambda_t(dx)$. The compensated point process $\tilde{\mu}$ is defined by

$$\tilde{\mu}(dt, dx) = \mu(dt, dx) - \lambda(S_{t-}, X_{t-}, dx)dt, \tag{6}$$

i.e., we obtain $\tilde{\mu}$ by "detrending" μ with its mean.
5. We assume the existence of a short rate r of the form

$$r_t = r(S_t, Y_t).$$

6. We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk-neutral martingale measure Q.

REMARK 3.2. The interpretation of the point process integrals, which appear in (3)–(4), is very simple. If, at time t, there is an event of type x, then there will be a jump in S^i with relative jump size δ_i, and correspondingly for the Y processes. Note that if the relative jump size δ_i takes the value -1 at some point, then the stock price S^i immediately jumps to zero and stays there for ever. This property of the model is important when dealing with bankruptcy problems.

The present setup extends the one in Cochrane and Saá Requejo (2000) in two ways. In the CSR setting, the continuous time model is purely Wiener driven, whereas the main contribution of the present paper is that we extend the framework to also include a driving point process. Furthermore, even in the purely Wiener-driven case, our setup extends that of Cochrane and Saá Requejo (2000) because we do not make rank assumptions for the diffusion matrices σ and b, and because we allow the entire vector Wiener process W to drive the price vector S. (In the CSR setting, S is only allowed to be driven by an n-dimensional subset of the W vector process.)

For future use, we introduce some more compact notation. Note that all objects below are *functions* of s, y, and (in the case of δ) x, to spaces of vectors and matrices.

DEFINITION 3.1. The column vector functions, α, δ, a, and c are defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}. \tag{7}$$

The $n \times d$ matrix functions σ and the $k \times d$ matrix function b are defined by

$$\sigma = \begin{bmatrix} -\sigma_1- \\ \vdots \\ -\sigma_n- \end{bmatrix}, \quad b = \begin{bmatrix} -b_1- \\ \vdots \\ -b_k- \end{bmatrix}. \tag{8}$$

3.2. THE PROBLEM

On the market specified above, we consider an arbitrarily chosen contingent T-claim $\mathcal{Z}$ of the form

$$\mathcal{Z} = \Phi(S_T, Y_T). \tag{9}$$

The problem is to compute a reasonable price process $\Pi\,(t; \mathcal{Z})$ for the claim $\mathcal{Z}$.

From general theory, we know that each arbitrage-free price will be given by an expression of the form

$$\Pi\,(t; \mathcal{Z}) = E^Q\left[e^{-\int_t^T r_u du} \Phi\,(S_T, Y_T) \middle| \mathcal{F}_t \right], \tag{10}$$

where Q is a martingale measure for the underlying S.

Since the market in the general case is incomplete, the martingale measure Q (and the stochastic discount factor, see the appendix for details) will generically not be unique, so there will not be a uniquely determined arbitrage-free price for $\mathcal{Z}$. It is also well known that, in incomplete settings like this, the pricing bounds provided by merely requiring absence of arbitrage, i.e., varying Q in the above formula over the entire set of martingale measures, are extremely wide and thus useless from a practical point of view.

There is thus a clear need for reasonable pricing bounds for derivative assets. To this end, we follow the approach in Cochrane and Saá Requejo (2000) and pose the following informal problem.

PROBLEM 3.1. We wish to compute upper and lower bounds on the price of the derivative by ruling out, not only the prices which violate the no-arbitrage restriction, but also those prices which represent deals that are too good, in the sense that the Sharpe Ratio process is "too high".

This naive formulation of the pricing problem, in terms of a bound on the Sharpe Ratio process on the derivative under study, turns out to be mathematically intractable (see Cochrane and Saá Requejo, 2000), and does not rule out the existence of portfolios based on the derivative and the underlying with very high Sharpe ratios. What we therefore need is a formalization of the pricing problem that gives us a mathematically tractable problem, and that, at the same time, allows us complete control over the Sharpe ratio processes of *all portfolios*, based on the underlying assets and the derivative. Accordingly, we utilize the Hansen-Jagannathan inequality (see Appendix A) to replace the bound on the Sharpe ratio with a bound on the norm of the market price of risk vector.

To carry out this scheme, we start by determining the class of martingale measures for the underlying model, and we define, for each potential martingale measure Q equivalent to P, the corresponding likelihood process L through

$$\frac{dQ}{dP} = L_t, \quad \text{on } \mathcal{F}_t; \quad 0 \leq t \leq T. \tag{11}$$

Since L is always a P-martingale, and since every martingale within the present framework, where the filtration is the internal one, admits a stochastic integral representation (see Jacod and Shiryaev, 1987), we know that L must have dynamics of the form

$$\begin{cases} dL_t &= L_t h_t^\star dW_t + L_{t-} \int_X \varphi_t(x) \{\mu(dt, dx) - \lambda_t(dx)dt\}, \\ L_0 &= 1, \end{cases} \tag{12}$$

where the **Girsanov kernel** processes h and φ (where we view h as a column vector process, hence the transpose *) are predictable, suitably integrable (see Jacod and Shiryaev, 1987 for details), and where φ must satisfy the condition

$$\varphi_t(x) \geq -1, \quad \forall t, x \quad P - a.s. \tag{13}$$

in order to ensure the nonnegativity of the measure Q.

REMARK 3.3. We note that if we do have $\varphi_t(x) = -1$, then we may lose the equivalence between P and Q, but retain the property that Q is absolutely continuous with respect to P. Any ensuing dynamic arbitrage possibilities can be dealt with in two ways. One possibility is to replace the inequality $\varphi_t(x) \geq -1$ by $\varphi_t(x) \geq -1 + \epsilon$ for a very small value of ϵ. Another, more natural alternative is to view the good-deal bounds derived below with the constraint $\varphi_t(x) \geq -1$ as **open** intervals of good-deal bounds, since the constraint only becomes binding at the boundary.

The economic interpretation of h and φ is that $-h$ is "the market price vector of diffusion risk", and that $-\varphi_t(x)$ is the "market price of jump risk of type x" (see Appendix A for details). From the Girsanov Theorem, see Jacod and Shiryaev (1987), we also recall the following facts:

- We can write

$$dW = h_t dt + dW^Q, \tag{14}$$

  where W^Q is a Q-Wiener process.

- The point process μ will, under Q, have an intensity λ^Q, given by

$$\lambda_t^Q(dx) = \{1 + \varphi_t(x)\}\,\lambda_t^P(dx). \tag{15}$$

The immediate problem is to find out how the kernel processes h and φ above must be chosen in order to guarantee that Q actually is a martingale measure for S. To this end we use the Girsanov Theorem to obtain the Q-dynamics of S^i, $i = 1, \ldots, n$ as follows, where, for brevity of notation, we write σ_i as shorthand for $\sigma_i(S_t, Y_t)$, and similarly for other terms:

$$dS_t^i = S_t^i\{\alpha_i + \sigma_i h_t\}\,dt + S_t^i \sigma_i dW_t^Q + S_{t-}^i \int_X \delta_t(x)\mu(dt, dx).$$

We then compensate the point process μ under Q to obtain a representation of S as a Q martingale part, plus a drift part.

$$\begin{aligned} dS_t^i &= S_t\left\{\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\lambda_t^Q(dx)\right\}dt + S_t^i\sigma_i dW_t^Q \\ &\quad + S_{t-}^i \int_X \delta_i(x)\left\{\mu(dt, dx) - \lambda_t^Q(dx)dt\right\}. \end{aligned} \tag{16}$$

Using (15), this can be written as

$$\begin{aligned} dS_t^i &= S_t\left\{\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\{1 + \varphi_t(x)\}\,\lambda_t(dx)\right\}dt + S_t^i\sigma_i dW_t^Q \\ &\quad + S_{t-}^i \int_X \delta_i(x)\left\{\mu(dt, dx) - \lambda_t^Q(dx)dt\right\}. \end{aligned} \tag{17}$$

Recalling that the measure Q is a martingale measure if, and only if, the local rate of return of S under Q equals the short rate r, we thus obtain the **martingale condition**

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\{1 + \varphi_t(x)\}\,\lambda_t(dx) = r_t, \quad i = 1, \ldots, n. \tag{18}$$

A Girsanov kernel process (h, φ), which satisfies the positivity constraint (13) and the martingale condition (18), is referred to as an **admissible** Girsanov kernel.

We can now state our extension of the Hansen-Jagannathan inequality. See Appendix A for the proof and for a precise definition of the Sharpe ratio process SR.

THEOREM 3.1 (Extended Hansen-Jagannathan Bounds). For every arbitrage-free price process, derivative or underlying, and for every admissible Girsanov kernel (market price of risk) process (h, φ), the following inequality holds

$$|SR_t|^2 \le \|h_t\|_{R^d}^2 + \int_X \varphi_t^2(x)\lambda_t(dx), \tag{19}$$

where SR denotes the Sharpe Ratio Process.

REMARK 3.4. It is important to note that the HJ inequality not only holds for the given underlying asset prices. Specifically, suppose that we choose a fixed pair of Girsanov kernels (h, φ), and that we use the martingale measure induced by these to price various derivatives. Then, the inequality holds for all underlying assets, for all derivatives, and for all self-financing portfolios based on the underlying and the derivatives. In other words, for a given choice of (h, φ), the HJ inequality gives us a uniform upper bound of Sharpe ratios for the entire economy. Note that the Sharpe ratio is not a number but a full fledged random process.

Since the right hand side of the HJ inequality also trivially bounds the underlying market, we need an assumption of uniform boundedness of the underlying market in order to have a nontrivial problem.

ASSUMPTION 3.2. We assume that there exists a real number B such that

$$\|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx) \leq B, \tag{20}$$

with probability one for all t, and for all Girsanov kernels that are admissible for the underlying market. Henceforth, B_0 will denote the smallest B with this property.

We now express our pricing problem as the problem of finding upper and lower arbitrage-free pricing bounds subject to an upper bound on the norm of the Girsanov kernel (h, φ), or, equivalently, on the norm of market price of risk vector $(-h, -\varphi)$.

DEFINITION 3.2. Given a bound B for the market prices of risk, the **upper good-deal price process** is defined as the optimal value process for the following optimal control problem:

$$\max_{h,\varphi} \quad E^Q\left[e^{-\int_t^T r_u du} \Phi\left(S_T, Y_T\right) \Big| \mathcal{F}_t \right] \tag{21}$$

with Q dynamics

$$\begin{aligned} dS_t^i &= S_t^i \left\{ r_t - \int_X \delta_i(x)\left\{1 + \varphi_t(x)\right\}\lambda_t(dx) \right\} dt + S_t^i \sigma_i dW_t^Q \\ &\quad + S_{t-}^i \int_X \delta_i(x)\mu(dt, dx), \quad i = 1, \ldots, n \end{aligned} \tag{22}$$

$$\begin{aligned} dY_t^j &= \left\{a_j + b_j h_t\right\} dt + b_j dW_t^Q \\ &\quad + \int_X c_j(x)\mu(dt, dx). \quad j = 1, \ldots, k. \end{aligned} \tag{23}$$

The predictable processes h and φ are subject to constraints

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\{1 + \varphi_t(x)\}\lambda_t(dx) = r_t, \quad i = 1, \dots, n \tag{24}$$

$$\|h_t\|_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx) \le B^2, \tag{25}$$

$$\varphi_t(x) \ge -1, \quad \forall t, x. \tag{26}$$

The following comments are in order:

- The expected value in (21) is the standard risk-neutral valuation formula for contingent claims.
- In (24), we have the martingale conditions from (18) on h and φ, guaranteeing that the induced measure Q is indeed a martingale measure for $S^1, \dots, S^n$.
- The induced Q dynamics of $S^1, \dots, S^n$ are given in (22).
- The induced Q dynamics of $Y^1, \dots, Y^k$ are given in (23).
- The constraint (26) is the constraint that rules out good deals. To obtain a nondegenerate problem, we only consider the case when $B \ge B_0$, where B_0 is given in Assumption 3.2.
- The constraint (26) is needed to ensure that Q is a nonnegative measure.
- The lower pricing bound is defined by the corresponding minimum problem.

REMARK 3.5. We also need to check that the likelihood process induced by a pair of admissible Girsanov kernels above is a true martingale and not merely a local martingale. In our case, this is automatically ensured by the good-deal bound constraints. For the Wiener part, the standard Novikov condition is trivially satisfied. For the point process part, the result follows from the good-deal bound constraint, the integrability properties ensured by Assumption 2.1, and from Theorem 11 in Chapter VIII of Bremaud (1981).

3.3. THE PRICING EQUATION

To treat the optimal control problem above with dynamic programming methods, we have to make an additional assumption that ensures that the Markovian structure is preserved under the martingale measure Q.

ASSUMPTION 3.3. We assume henceforth that the Girsanov kernel processes h and φ are of the restricted form

$$h_t = h(t, S_t, Y_t), \tag{27}$$

$$\varphi_t(x) = \varphi(t, S_{t-}, Y_{t-}, x). \tag{28}$$

Here, with a slight abuse of notation, the right hand side occurences of h and φ denote deterministic functions of the form $h : R_+ \times R^n \times R^k \to R^n$, and $\varphi : R_+ \times R^n \times R^k \times X \to R$, respectively.

Next, we present the basic pricing equation for the upper and lower good-deal bounds – in the present setting, this is quite straightforward. Under Assumption 3.3, the optimal expected value in (21) can be written as $V(t, S_t, Y_t)$, where the deterministic mapping $V : R_+ \times R^n \times R^k \to R$ is known as the **optimal value function**. Since we are in a standard setting for dynamic programming (DynP), we know from general DynP theory that the optimal value function will satisfy the following Bellman-Hamilton-Jacobi equation on the time interval $[0, T]$:

$$\frac{\partial V}{\partial t} + \sup_{h,\varphi} \mathbf{A}^{h,\varphi} V - rV = 0, \tag{29}$$

$$V(T, s, y) = \Phi(s, y), \tag{30}$$

where the sup is subject to constraints of the form (24)–(26), and where $\mathbf{A}^{h,\varphi}$ denotes the infinitesimal operator for the process (S, Y), under the measure Q defined by h and φ. We recall that, from an operational point of view, the infinitesimal operator $\mathbf{A}^{h,\varphi}$ is merely the integro-differential operator that appears in the dt term in the stochastic differential $dV(t, S_t, Y_t)$ (when the point process increment has been compensated). A standard application of the Itô formula for semimartingales will, in fact, give us the following result.

PROPOSITION 3.1. The infinitesimal operator $\mathbf{A}^{h,\varphi}$ is given by

$$\begin{aligned}
&\mathbf{A}^{h,\varphi} V(t, s, y) \\
&\quad = \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}(t, s, y) s_i \left\{ r - \int_X \delta_i(s, y, x)\{1 + \varphi(t, s, y, x)\} \lambda_t(s, y, dx) \right\} \\
&\quad + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t, s, y) \left\{ a_j(s, y) + b_j(s, y) h(t, s, y) \right\} \\
&\quad + \int_X \Delta V(t, s, y, x)\{1 + \varphi(t, s, y, x)\} \lambda_t(s, y, dx) \\
&\quad + \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l}(t, s, y) s_i s_l \sigma_i^\star(s, y) \sigma_l(s, y) \\
&\quad + \frac{1}{2} \sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l}(t, s, y) b_j^\star(s, y) b_l(s, y) \\
&\quad + \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j}(t, s, y) s_i \sigma_i^\star(s, y) b_j(s, y).
\end{aligned} \tag{31}$$

Here, ΔV is defined by

$$\Delta V(t,s,y,x) = V\,(t, s(1+\delta(s,y,x)), y + c(s,y,x)) - V(t,s,y), \qquad (32)$$

where addition and multiplication in $s(1+\delta(s,y,x))$ and $y + c(s,y,x)$ are interpreted componentwise.

Proof. An easy application of the Itô formula. ■

Collecting the facts above, we can finally present the basic equation for the upper good-deal bound.

THEOREM 3.2. The upper good-deal bound function is the solution V to the following boundary value problem:

$$\frac{\partial V}{\partial t}(t,s,y) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi} V(t,s,y)\right\} - r(s,y)V(t,s,y) \;=\; 0, \qquad (33)$$

$$V(T,s,y) \;=\; \Phi(s,y). \qquad (34)$$

Here, $\mathbf{A}^{h,\varphi} V$ is given by (31), and the supremum in (33) is over all functions $h(t,s,y)$ and $\varphi(t,s,y,x)$, satisfying, for all (t,s,y), the constraints

$$\alpha_i + \sigma_i h + \int_X \delta_i(x)\,\{1+\varphi(x)\}\,\lambda_t(dx) \;=\; r, \quad i = 1,\ldots,n \qquad (35)$$

$$\|h\|_{R^d} + \int_X \varphi^2(x)\lambda_t(dx) \;\leq\; B^2, \qquad (36)$$

$$\varphi(x) \;\geq\; -1. \qquad (37)$$

The lower bound price function satisfies the same equation with the supremum operator replaced by $\inf_{h,\varphi}$.

3.4. ON THE STRUCTURE OF THE PRICING EQUATION

The pricing equation (33)–(34) is a partial integro-differential equation (PIDE,) which generally cannot be solved analytically. However, the equation possesses particular features which we wish to highlight.

As in all applications of stochastic dynamic programming, we note that the stochastic intertemporal optimal control problem (21)–(26) is reduced to the following two purely deterministic problems:

1. The static optimization problem of finding, for each fixed (t, s, y), the optimal h and φ in the constrained maximization problem

$$\sup_{h,\varphi} \left\{ \mathbf{A}^{h,\varphi} V(t, s, y) \right\}, \tag{38}$$

as appears in (33).

2. Having solved the static problem above, and denoted the optimal h, φ by $\hat{h}, \hat{\varphi}$, we have to solve the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\hat{h},\hat{\varphi}} V - rV = 0, \tag{39}$$

$$V(T, s, y) = \Phi(s, y). \tag{40}$$

Obviously, if we wish to solve the PIDE in step 2 above, we first have to solve the static optimization problem in step 1; it is of great importance to understand the structure of the static problem. We then note that this is, in fact, an infinite-dimensional problem.

More precisely, problem (38) has to be solved for every fixed choice of (t, s, y), and the control variables are h and φ. However, whereas the diffusion kernel $h(t, s, y)$ (for fixed t, s and y) is merely a d-dimensional vector, the point process kernel $\varphi(t, s, y, \cdot)$ has to be determined as a function of x and, hence, $\varphi(t, s, y, \cdot)$ is an infinite-dimensional control variable. We thus see that the static optimization problem is not a standard finite-dimensional mathematical programming problem, but a full fledged variational problem. See Section 4.6 for a more detailed discussion of how a random jump size affects the good-deal bounds.

The infinite dimensionality of the static optimization problem is intimately connected to the cardinality of the mark space X (or rather, to the cardinality of the support of the measure $\lambda_t(dx)$). If the mark space has an infinite number of elements, then the static problem is infinite dimensional. If, on the other hand, X has a finite number of elements, then the static problem is a finite-dimensional problem. From a modeling point of view, this basically means that if we wish to model a situation with an infinite number of possible jump sizes, then the static problem becomes a variational problem.

Even if the static problem is infinite dimensional, it has a very particular structure. When we examine the expression (31) for the infinitesimal operator $\mathbf{A}^{h,\varphi}$, we see that only three terms involve the control variables h and φ and that, in fact, the control variables enter linearly. With notation as in Definition 3.1, we formalize this observation in a lemma.

LEMMA 3.1. The static optimization problem in (33) and (38) can be written as

$$\max_{h,\varphi} \quad \langle \Delta V, \varphi \rangle_{\lambda_t} - V_s D(s) \langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t} + V_y b h, \tag{41}$$

subject to the constraints

$$\alpha + \sigma h + \langle \delta, \mathbf{1} \rangle_{\lambda_t} + \langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t} = r\mathbf{1}, \tag{42}$$

$$\|h\|^2_{R^d} + \|\varphi\|^2_{\lambda_t} \leq B^2, \tag{43}$$

$$\varphi \geq -1. \tag{44}$$

Here, V_s and V_y denote the gradients of V wrt, the vector variables s, and y, respectively, $D(s)$ denotes the diagonal matrix with the components of s on the diagonal, $\mathbf{1}$ denotes the column vector with 1 in all components, and the inner products $\langle \delta, \varphi \mathbf{1} \rangle_{\lambda_t}$ and $\langle \delta, \mathbf{1} \rangle_{\lambda_t}$ are interpreted componentwise.

When we write the static problem in this form, we see very clearly that we have a linear objective function, an infinite-dimensional linear equality constraint, a scalar quadratic inequality constraint, and an infinite-dimensional linear inequality constraint. Since the set of admissible points is convex, the linearity of the objective function implies that the optimal point is an extremal point of the admissible set. It is also clear that at least one of the inequality constraints has to be binding.

3.5. POSITIVITY AND THE MINIMAL MARTINGALE MEASURE

The static problem (41) is, apart from the positivity constraint (44), a fairly standard linear quadratic problem in the space $L^2[X, \lambda(dx)]$. The really problematic part is the generically infinite-dimensional positivity constraint (44). The problem with this constraint is that, although the standard finite-dimensional Kuhn-Tucker theory can, to some extent, be transferred into an infinite-dimensional setting, it requires some nontrivial topological assumptions to be satisfied. These conditions appear only in the truly infinite-dimensional cases, and in those cases the conditions are vital. In our case, the technical condition needed is that the positive cone has a nonempty interior but, unfortunately for us, the positive cone in L^2 does not contain any interior points, which effectively prohibits us from using standard infinite-dimensional Kuhn-Tucker methodology. Below, we discuss the positivity constraint in more detail.

As a first approach to solving Problem (41), one hopes (perhaps) that the positivity constraint is not binding in the optimal solution. It is thus natural to solve a relaxed version of Problem (41) where the positivity constraint is not present and, having found the optimal solution, to then check whether the positivity constraint is binding or not. If the positivity constraint turns out not to be binding at the relaxed optimal point, then all is well and we have found our optimal solution. If the constraint is violated at the optimal point of the relaxed problem, then the induced measure is not a positive measure. If we use this measure for pricing, it still gives us pricing bounds, but these are wider than the optimal ones. We now formalize these ideas, and we also relate them to the concept of the minimal martingale measure from the theory of local risk minimization.

DEFINITION 3.3.

- Denote the optimal upper and lower bound Girsanov kernels from Theorem 3.2 by (h^s, φ^s) and (h^i, φ^i), denote the corresponding **optimal martingale measures** by Q^s, Q^i, and define the pricing functions V^s and V^i correspondingly. Here, "s" stands for "sup" and "i" stands for "inf".
- Denote by $(\bar{h}^s, \bar{\varphi}^s)$ the optimal kernels for the relaxed static problem

$$\sup_{h,\varphi} \quad \mathbf{A}^{h,\varphi} V(t, s, y), \tag{45}$$

  subject to the constraints

$$\alpha_i + \sigma_i h + \int_X \delta_i(x) \{1 + \varphi(x)\} \lambda_t(dx) \ = \ r, \quad i = 1, \ldots, n \tag{46}$$

$$\|h\|_{R^d} + \int_X \varphi^2(x) \lambda_t(dx) \ \leq \ B^2, \tag{47}$$

  and denote by $(\bar{h}^i, \bar{\varphi}^i)$ the optimal solutions to the corresponding minimization problem. Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\bar{h}^s, \bar{\varphi}^s} V - rV \ = \ 0, \tag{48}$$

$$V(T, s, y) \ = \ \Phi(s, y), \tag{49}$$

  by $\bar{V}^s$, and define $\bar{V}^i$ in the same way. The **relaxed martingale measures** induced by $(\bar{h}^s, \bar{\varphi}^s)$ and $(\bar{h}^i, \bar{\varphi}^i)$ are denoted by $\bar{Q}^s$, and $\bar{Q}^i$, respectively.
- Denote by (h^m, φ^m) the Girsanov kernels obtained by solving the problem

$$\min_{h,\varphi} \quad \|h\|^2_{R^d} + \|\varphi\|^2_{\lambda_t} \tag{50}$$

  subject to

$$\alpha_i + \sigma_i h + \int_X \delta_i(x) \{1 + \varphi(x)\} \lambda_t(dx) = r, \quad i = 1, \ldots, n. \tag{51}$$

  Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{h^m, \varphi^m} V - rV \ = \ 0, \tag{52}$$

$$V(T, s, y) \ = \ \Phi(s, y), \tag{53}$$

  by V^m. The measure Q^m, henceforth referred to as the **Minimal Martingale Measure** ("MMM"), is defined as the (possibly signed) measure induced by (h^m, φ^m).

For the measures defined above, the first four depend on the choice of derivative to be priced, whereas the minimal martingale measure Q^m is independent of the choice of derivative. We see that the MMM kernels (h^m, φ^m) are obtained by minimizing the right hand side of the HJ inequality, subject to the "martingale condition" (51), but without the positivity constraint, so the MMM is the P-equivalent measure with pointwise minimal L^2 norm satisfying the martingale constraint. It may happen that the MMM is a signed measure, thus assigning negative "probability" to some events. The problem with a signed MMM only occurs in the presence of jumps, so in a pure diffusion setting the MMM will always be a bona fide probability measure. The minimal martingale measure was first defined in connection with local risk minimization (see Schweizer, 1995), where it plays a fundamental role. The original definition of the MMM in Schweizer (1995) is not the one given above, but it is fairly easy to see that, in the present context, the two definitions coincide. We now derive the following easy result.

PROPOSITION 3.2.

- We always have the relations

$$\bar{V}^i \leq V^i \leq V^s \leq \bar{V}^s, \tag{54}$$

  and

$$\bar{V}^i \leq V^m \leq \bar{V}^s. \tag{55}$$

- If the positivity constraint is satisfied by $\bar{h}^i$, $\bar{h}^s$ and h^m, then $\bar{Q}^i$, $\bar{Q}^s$ and Q^m are probability measures (and not just signed measures), and we have

$$\bar{V}^i \leq V^i \leq V^m \leq V^s \leq \bar{V}^s. \tag{56}$$

Proof. Obvious from the definitions. ■

The moral of this can be summarized as follows:

- The minimal martingale measure provides us with a canonical benchmark for pricing any derivative. Furthermore, since the MMM is the solution to a standard minimum norm problem in L^2, it can easily be computed.
- The relaxed martingale measures $\bar{Q}^i$ and $\bar{Q}^s$ are much easier to compute than the optimal measures Q^i and Q^s. The reason for this is that, for the relaxed measures, we disregard the positivity constraints which are highly complex and impossible to handle with Kuhn-Tucker methodology. The corresponding relaxed static problems can be solved explicitly using vector space methods.

- The pricing bounds provided by the relaxed measures $\bar{Q}^i$ and $\bar{Q}^s$ are generically not optimal but, for reasonable values of the Sharpe ratio constraint B, they turn out to be much tighter than the no-arbitrage bounds.
- The bounds obtained by the harder-to-compute optimal measures Q^i and Q^s are, in their turn, considerably tighter than those obtained from $\bar{Q}^i$ and $\bar{Q}^s$.
- Both the minimal martingale measure Q^m and the relaxed measures $\bar{Q}^i$ and $\bar{Q}^s$ can be computed explicitly in terms of input data. Due to the complex nature of the the general formulas, we have chosen not to include them. See Section 4.4 for a concrete, worked-out example.

4. Point Process Examples

In this section, we study a number of illustrative concrete examples. We restrict ourselves to models which include jumps, in line with the main focus of our paper. See Appendix B for the purely Wiener-driven case.

As opposed to a purely Wiener-driven model, the introduction of a driving point process (together with a Wiener process) produces a nontrivial incomplete market model even without including the factor model Y. For this reason, but also for tractability reasons, we only investigate pure jump-diffusion stock price models without any external factors. Specifically, all models studied in this section will be assumed to have the following structure.

ASSUMPTION 4.1. We consider a financial market and a scalar price process S satisfying the SDE

$$dS_t = S_t\alpha dt + S_t\sigma dW_t + S_{t-}\int_X \delta(x)\mu(dt, dx). \tag{57}$$

For this model, we furthermore assume that:

1. The Wiener process W is one-dimensional.
2. The drift α and diffusion volatility σ are deterministic constants.
3. The jump function δ is a time invariant deterministic function of x only, i.e., δ is a mapping $\delta : X \to R$.
4. The point process μ has a P-compensator of the form

   $$\nu^P(dt, dx) = \lambda(dx)dt,$$

   where λ is a time-invariant deterministic finite nonnegative measure on $(X, \mathcal{X})$.
5. The short rate r is constant.

Under this assumption, the model parameters α, σ, δ, and λ are deterministic objects that do not depend on the stock price S. In particular, the assumption about λ implies that the point process μ has the following properties under P:

- The jump events (disregarding the mark) will occur according to a standard Poisson process with the constant intensity $\lambda(X)$.
- If X_n denotes the mark of event number n, then the sequence $X_1, X_2, \dots$ is i.i.d. with the common probability distribution

$$\frac{1}{\lambda(X)}\lambda(dx). \tag{58}$$

The sequence above is also independent of the interarrival times of the events.

To obtain a feeling for the techniques used, we start with a very simple example and then proceed to consider more complicated cases.

4.1. THE POISSON-WIENER MODEL

The simplest special case in the jump-diffusion setting above is when we define the point process μ as a standard Poisson process with constant intensity. In terms of the notation above, this means that the mark space X contains a single point, denoted by x_0. Hence, $X = \{x_0\}$, the measure $\lambda(dx)$ is merely a point mass $\lambda(x_0)$ at x_0, and the jump function δ a real number $\delta(x_0)$. For brevity, we denote $\lambda(x_0)$ by λ and $\delta(x_0)$ by δ. Hence, we obtain the following P dynamics of S:

$$dS_t = S_t\alpha dt + S_t\sigma dW_t + S_{t-}\delta dN_t \tag{59}$$

where N is Poisson with constant intensity λ.

In this case, the kernel function $h(t, s)$ is scalar, and the kernel $\varphi(t, s)$ does not depend upon x. The upper good-deal bound function $V(t, s)$ is the solution to the following boundary value problem.

$$\frac{\partial V}{\partial t}(t, s) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}V(t, s)\right\} - rV(t, s) = 0, \tag{60}$$

$$V(T, s) = \Phi(s), \tag{61}$$

where we – for the moment – suppress the constraints, and where

$$\begin{aligned}\mathbf{A}^{h,\varphi}V(t, s) &= \frac{\partial V}{\partial s}s\{r - \delta\lambda(1 + \varphi)\} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} \\ &\quad + \{V(t, s(1 + \delta)) - V(t, s)\}\lambda(1 + \varphi).\end{aligned} \tag{62}$$

The static optimization problem in Lemma 3.1 then becomes

PROBLEM 4.1.

$$\max_{h,\varphi} \quad \lambda\{V(t, s(1 + \delta)) - V(t, s) - V_s(t, s)s\delta\}\varphi \tag{63}$$

subject to the constraints

$$\begin{aligned} \alpha + \sigma h + \delta\lambda \{1 + \varphi\} &= r, &(64)\\ h^2 + \varphi^2\lambda &\leq B^2, &(65)\\ \varphi &\geq -1. &(66) \end{aligned}$$

To study the static problem in more detail, we need some notation.

DEFINITION 4.1. Define $(h_{\max}, \varphi_{\max})$ as the optimal solution to the programming problem

$$\max_{h,\varphi} \varphi, \tag{67}$$

subject to the constraints (64)–(66), and $(h_{\min}, \varphi_{\min})$ as the optimal solution to the problem

$$\min_{h,\varphi} \varphi, \tag{68}$$

subject to the same constraints.

We will need $h_{\max}$, $\varphi_{\max}$, $h_{\min}$, and $\varphi_{\min}$ below, so we should describe these constants in terms of the given model parameters. This is a simple exercise in constrained optimization theory, but somewhat messy. The result is as follows.

LEMMA 4.1. Denote the excess return $\alpha + \delta\lambda - r$ by R. Then, the following hold.

- The constants $h_{\max}$ and $\varphi_{\max}$ are given by

$$h_{\max} = -\frac{\sigma R}{(\sigma^2 + \delta^2\lambda)\lambda} - \frac{\delta\sqrt{B^2\left(\sigma^2 + \delta^2\lambda\right) - R^2)}}{\left(\sigma^2 + \delta^2\lambda\right)\sqrt{\lambda}} \tag{69}$$

$$\varphi_{\max} = -\frac{\delta R}{\sigma^2 + \delta^2\lambda} + \frac{\sigma\sqrt{B^2\left(\sigma^2 + \delta^2\lambda\right) - R^2)}}{\left(\sigma^2 + \delta^2\lambda\right)\sqrt{\lambda}} \tag{70}$$

- The constants $h_{\min}$ and $\varphi_{\min}$ are given by the following expressions:
  1. If

$$-\frac{\delta R}{\sigma^2 + \delta^2\lambda} - \frac{\sigma\sqrt{B^2\left(\sigma^2 + \delta^2\lambda\right) - R^2)}}{\left(\sigma^2 + \delta^2\lambda\right)\sqrt{\lambda}} > -1, \tag{71}$$

then

$$h_{\min} = -\frac{\sigma R}{(\sigma^2 + \delta^2\lambda)\lambda} + \frac{\delta\sqrt{B^2\left(\sigma^2 + \delta^2\lambda\right) - R^2)}}{\left(\sigma^2 + \delta^2\lambda\right)\sqrt{\lambda}} \tag{72}$$

$$\varphi_{\min} = -\frac{\delta R}{\sigma^2 + \delta^2 \lambda} - \frac{\sigma \sqrt{B^2 \left(\sigma^2 + \delta^2 \lambda\right) - R^2)}}{\left(\sigma^2 + \delta^2 \lambda\right) \sqrt{\lambda}.} \tag{73}$$

2. If

$$-\frac{\delta R}{\sigma^2 + \delta^2 \lambda} - \frac{\sigma \sqrt{B^2 \left(\sigma^2 + \delta^2 \lambda\right) - R^2)}}{\left(\sigma^2 + \delta^2 \lambda\right) \sqrt{\lambda}} \leq -1, \tag{74}$$

then

$$h_{\min} = \frac{r - \alpha}{\sigma}, \tag{75}$$

$$\varphi_{\min} = -1. \tag{76}$$

Proof. A direct application of Kuhn-Tucker. ■

Now we can present a preliminary description of the optimal kernels.

PROPOSITION 4.1. The optimal kernels $(\hat{h}, \hat{\varphi})$ for the static problem (63)–(66) have the following structure:

1. For all (t, s) such that

$$V(t, s(1 + \delta)) - V(t, s) - V_s(t, s)s\delta \geq 0, \tag{77}$$

the optimal kernels $(\hat{h}, \hat{\varphi})$ are given by

$$\hat{h}(t, s) = h_{\max}, \quad \hat{\varphi}(t, s) = h_{\max}. \tag{78}$$

2. For all (t, s) such that

$$V(t, s(1 + \delta)) - V(t, s) - V_s(t, s)s\delta < 0, \tag{79}$$

the optimal kernels $(\hat{h}, \hat{\varphi})$ are given by

$$\hat{h}(t, s) = h_{\min}, \quad \hat{\varphi}(t, s) = h_{\min}. \tag{80}$$

Proof. Obvious from the arguments above. ■

We thus see that the optimal kernels have a so-called bang-bang structure, i.e., they switch between the extremal choices $(h_{\max}, \varphi_{\max})$ and $(h_{\min}, \varphi_{\min})$. For an arbitrarily chosen problem, switches do indeed occur, and the number of switches depend of course upon the optimal value function V through the conditions (77) and (79). An interesting special case arises when there are no switches and when the optimal kernels remain constant. Before proving the main result in this direction, we need some preliminary lemmas.

LEMMA 4.2.

1. If the optimal value function $V(t, s)$ is convex in the s-variable for all fixed values of t, then

$$\hat{h}(t, s) = h_{\max}, \quad \hat{\varphi}(t, s) = \varphi_{\max}, \quad \forall t, s. \tag{81}$$

2. If the optimal value function $V(t, s)$ is concave in the s-variable for all fixed values of t, then

$$\hat{h}(t, s) = h_{\min}, \quad \hat{\varphi}(t, s) = \varphi_{\min}, \quad \forall t, s. \tag{82}$$

Proof. If V is convex in s, then (as a function of s) the tangent of V lies below the graph at each point, which implies condition (77). The concave case is similar. ∎

LEMMA 4.3. Consider the Poisson-Wiener model in (59), a fixed martingale measure Q generated by some choice of kernel functions (h, φ), and a contract function $\Phi(s)$. Assume the following:

1. The contract function Φ is convex (concave).
2. The kernel functions h and φ are deterministic functions of time, i.e., they are of the form

$$h(t, s) = h(t), \quad \varphi(t, s) = \varphi(t). \tag{83}$$

Then, the arbitrage-free pricing function $F(t, s)$ defined by

$$F(t, s) = e^{-r(T-t)} E_{ts}^{Q} \left[\Phi(S_T) \right], \tag{84}$$

which is convex (concave) in the s-variable, and where E_{ts}^{Q} denotes expectation under Q, given $S_t = s$.

Proof. Using the Itô formula it is easy to see that, given $S_t = s$, the SDE (59) has the solution

$$S_T = s(1 + \delta)^{N_T - N_t} e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}, \tag{85}$$

which we will write as

$$S_T = s \cdot Z, \tag{86}$$

with the random variable Z defined in the obvious way above. We thus have

$$F(t, s) = e^{-r(T-t)} E_{ts}^{Q} \left[\Phi(s \cdot Z) \right], \tag{87}$$

and, from the assumptions on h and φ, it follows that the Q-distribution of Z does not depend on the value of s. The assumed convexity of Φ now immediately implies the convexity of F. ■

We can now state and prove the main theoretical result concerning the Poisson-Wiener model.

PROPOSITION 4.2. Assume that the contract function Φ is convex. Then, the following hold:

1. The optimal upper bound value function V is convex.
2. The optimal kernels $\hat{h}$ and $\hat{\varphi}$ are constant and given by

$$\hat{h} = h_{\max}, \quad \hat{\varphi} = \varphi_{\max}. \tag{88}$$

3. V satisfies the PIDE

$$\frac{\partial V}{\partial t}(t,s) + \mathbf{A}^{\hat{h},\hat{\varphi}} V(t,s) - rV(t,s) = 0, \tag{89}$$

$$V(T,s) = \Phi(s), \tag{90}$$

where $\hat{h}$, $\hat{\varphi}$ are defined by (88), and where

$$\begin{aligned} \mathbf{A}^{\hat{h},\hat{\varphi}} V &= \frac{\partial V}{\partial s} s \left\{ r - \delta\lambda(1+\hat{\varphi}) \right\} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2} \\ &\quad + \{ V(t, s(1+\delta)) - V(t,s) \} \lambda(1+\hat{\varphi}). \end{aligned} \tag{91}$$

Instead, if the contract function Φ is concave, then V is concave, and items 2–3 above still hold, with the only change that $h_{\max}$ and $\varphi_{\max}$ are replaced by $h_{\min}$ and $\varphi_{\min}$.

Proof. Define the function F as the solution of

$$\frac{\partial F}{\partial t}(t,s) + \mathbf{A}^{\hat{h},\hat{\varphi}} F(t,s) - rF(t,s) = 0, \tag{92}$$

$$F(T,s) = \Phi(s), \tag{93}$$

with $\mathbf{A}^{\hat{h},\hat{\varphi}}$ defined as above. We now want to show that $F = V$, i.e., that F is in fact equal to the optimal value function of the control problem for the upper bound. To do this, we first apply a Feynman-Kac Representation Theorem to deduce that we can write F as

$$F(t,s) = e^{-r(T-t)} E_{ts}^{\hat{Q}} \left[\Phi(S_T) \right], \tag{94}$$

where $\hat{Q}$ is generated by $\hat{h}$ and $\hat{\varphi}$. From Lemma 4.3 we then deduce that F is convex in the s-variable, and this implies – as in the Proof of Lemma 4.2 – that F also satisfies the PIDE

$$\frac{\partial F}{\partial t}(t,s) + \sup_{h,\varphi}\left\{\mathbf{A}^{h,\varphi}F(t,s)\right\} - rF(t,s) \;=\; 0, \tag{95}$$

$$F(T,s) \;=\; \Phi(s), \tag{96}$$

with obvious notation and standard constraints on h and φ. We have thus shown that F, defined by (92)–(93), satisfies the Hamilton-Jacobi-Bellman equation for the optimal control problem for the upper good-deal bound, and we now apply a standard verification theorem (see Björk, 2004, Ch. 19) to deduce that $F = V$. ■

The moral of this result is that, for convex contract functions like European puts and calls, we derive a well-behaved standard pricing equation for the upper bound, and a corresponding lower bound equation, without an embedded supremum operator. For nonconvex contract functions like that of a digital option, the situation is much more complicated and we must solve the full Hamilton-Jacobi-Bellman equation numerically.

4.2. THE COMPOUND POISSON-WIENER MODEL

We now turn to the general Compound Poisson-Wiener Model specified by Assumption 4.1. For this model, the static problem of Lemma 3.1 has the following form:

PROBLEM 4.2,

$$\max_{h,\varphi} \quad \int_X \Delta V(t,s,x)\varphi(t,s,x)\lambda(dx) - sV_s(t,s)\int_X \delta(x)\varphi(t,s,x)\lambda(dx), \tag{97}$$

subject to

$$\alpha + \sigma h + \int_X \delta(x)\lambda(dx) + \int_X \delta(x)\varphi(x)\lambda(dx) \;=\; r, \tag{98}$$

$$h^2 + \int_X \varphi^2(x)\lambda(dx) \;\le\; B^2, \tag{99}$$

$$\varphi(x) \;\ge\; -1, \tag{100}$$

where, as before,

$$\Delta V(t,s,x) = V(t,s(1+\delta(x))) - V(t,s). \tag{101}$$

Using, as above, the notation R for the risk premium

$$R = \alpha + \int_X \delta(x)\lambda(dx) - r, \tag{102}$$

we express the problem in functional analytical terms, as follows.

PROBLEM 4.3.

$$\max_{h,\varphi} \quad \langle H, \varphi \rangle_\lambda, \tag{103}$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R \ = \ 0, \tag{104}$$

$$h^2 + \|\varphi\|_\lambda \ \leq \ B^2, \tag{105}$$

$$\varphi \ \geq \ -1, \tag{106}$$

where,

$$H(t,s,x) = \Delta V(t,s,x) - V_s(t,s)s\delta(x). \tag{107}$$

We now proceed to study this problem in detail.

4.3. THE MINIMAL MARTINGALE MEASURE

We begin by computing the minimal martingale measure Q^m for this model. The measure Q^m is generated by the optimal kernels (h^m, φ^m) for the following programming problem.

PROBLEM 4.4.

$$\min_{h,\varphi} \quad h^2 + \|\varphi\|_\lambda^2, \tag{108}$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R = 0, \tag{109}$$

This is a standard minimum norm problem in L^2, so, from general theory, we know that the optimal (h, φ) has to be of the form

$$(h, \varphi) = c \cdot (\sigma, \delta),$$

for some scalar c. When we substitute this into the constraint (109), it yields the following result.

PROPOSITION 4.3. The minimal martingale measure Q^m is generated by the kernels (h^m, φ^m), given by

$$h^m = -\frac{R \cdot \sigma}{h^2 + \|\varphi\|_\lambda^2}, \tag{110}$$

$$\varphi^m = -\frac{R \cdot \delta}{h^2 + \|\varphi\|_\lambda^2}. \tag{111}$$

4.4. THE POSITIVITY CONSTRAINT

As we noted in Section 3.5, Problem 4.3 above is, apart from the constraint (106), a fairly standard linear quadratic problem in the space $L^2[X, \lambda(dx)]$. Following the arguments of Section 4.4, we are led to study Problem 4.3 without the constraint (106) in order to compute the "relaxed measures" $\bar{Q}^s$ and $\bar{Q}^i$. The problem for determining $\bar{Q}^s$ is thus as follows.

PROBLEM 4.5.

$$\max_{h,\varphi} \quad \langle H, \varphi \rangle_\lambda, \tag{112}$$

subject to

$$\sigma h + \langle \delta, \varphi \rangle_\lambda + R \ = \ 0, \tag{113}$$

$$h^2 + \|\varphi\|_\lambda^2 \ \leq \ B^2. \tag{114}$$

This is now a standard optimization problem in $L^2[X, \lambda(dx)]$, which we can solve by finding the extremal points of the associated Lagrangian. See Luenberger (1997) for details and for all unexplained terminology from functional analysis below. First, however, we simplify the problem by using (104) to eliminate h. Since the remaining constraint has to be binding, we face the following problem.

PROBLEM 4.6.

$$\max_{\varphi} \quad \langle H, \varphi \rangle, \tag{115}$$

subject to

$$2R\langle \delta, \varphi \rangle + \langle \delta, \varphi \rangle^2 + \sigma^2 \|\varphi\|^2 + R^2 - \sigma^2 B^2 = 0, \tag{116}$$

where we have suppressed λ in $\|\cdot\|_\lambda$ and $\langle \cdot, \cdot \rangle_\lambda$.

This is a standard programming problem in L^2, and the Lagrangian function $\mathcal{L}$ is given by

$$\mathcal{L}(\varphi,\gamma) = \langle H,\varphi\rangle + \gamma\left\{2R\langle\delta,\varphi\rangle + \langle\delta,\varphi\rangle^2 + \sigma^2\|\varphi\|^2 + R^2 - \sigma^2 B^2\right\}. \quad (117)$$

From the Kuhn-Tucker Theorem (see Luenberger, 1997) we know that the optimal solution $\hat{\varphi}$ is an extremal point of $\mathcal{L}$, i.e., a point where the Frechet derivative vanishes. Denoting the Frechet derivative of $\mathcal{L}$ w.r.t. φ by $\mathcal{L}_\varphi$, we easily obtain

$$\mathcal{L}_\varphi(\varphi,\gamma) = H + 2\gamma\left\{R\delta + \langle\delta,\varphi\rangle\delta + \sigma^2\varphi\right\}. \quad (118)$$

Thus, the first-order conditions for Problem 4.6 are

$$H + 2\gamma\left\{R\delta + \langle\delta,\varphi\rangle\delta + \sigma^2\varphi\right\} = 0. \quad (119)$$

When we take the inner product with δ in (119), it gives us the relation

$$\langle H,\delta\rangle + 2\gamma\left\{R\|\delta\|^2 + \langle\delta,\varphi\rangle\|\delta\|^2 + \sigma^2\langle\delta,\varphi\rangle\right\} = 0, \quad (120)$$

and, solving for $\langle\delta,\varphi\rangle$, we obtain

$$\langle\delta,\varphi\rangle = -\frac{\langle H,\delta\rangle + 2\gamma R\|\delta\|^2}{2\gamma\left\{\|\delta\|^2 + \sigma^2\right\}}. \quad (121)$$

Inserting this expression into (119) gives us the optimal φ as

$$\varphi = \frac{\langle H,\delta\rangle\delta}{2\gamma\sigma^2\left\{\|\delta\|^2+\sigma^2\right\}} - \frac{R\delta}{\|\delta\|^2+\sigma^2} - \frac{H}{2\gamma\sigma^2}. \quad (122)$$

To determine the Lagrange multiplier γ, we substitute (121)–(122) into the constraint (116). After some calculation, we obtain the following quadratic equation for γ:

$$\gamma^2 = \frac{\sigma^2\langle H,\delta\rangle^2 + \langle H,\delta\rangle\|\delta\|^2 + \|H\|^2K^2 - 2\langle H,\delta\rangle^2K^2}{4\sigma^4K\left\{B^2K - R^2\right\}}, \quad (123)$$

where

$$K = \|\delta\|^2 + \sigma^2. \quad (124)$$

The Lagrange multiplier is the positive root of this equation. We thus have the following pricing result.

THEOREM 4.1. The upper and lower relaxed good-deal bound pricing functions, $\bar{V}^s(t,s)$ and $\bar{V}^i(t,s)$ are given as solutions of the PIDE

$$\frac{\partial V}{\partial t} + \mathbf{A}^{\varphi} V - rV = 0, \tag{125}$$

$$V(T, s) = \Phi(s, y), \tag{126}$$

where $\mathbf{A}^{\varphi}$ is given by

$$\mathbf{A}^{\varphi} V(t,s) = \frac{\partial V}{\partial s}(t,s)s \left\{ r - \int_X \delta(s,x) \{1 + \varphi(t,s,x)\} \lambda(dx) \right\}$$
$$+ \int_X \Delta V(t,s,y,x) \{1 + \varphi(t,s,x)\} \lambda(dx) + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2}(t,s),$$

with $\varphi = \bar{\varphi}^s$ and $\varphi = \bar{\varphi}^i$, respectively. The optimal relaxed kernels $\bar{\varphi}^s$ and $\bar{\varphi}^i$ are given by (122), where γ is the positive, respectively negative, root of (123).

The question now arises whether – at the optimal point – the positivity constraint is binding or not. We have no general theoretical results concerning this question, but our numerical experience (see Section 4.5 below) indicates strongly that the constraint is indeed binding in the generic case. The implication of this negative fact is that, in the generic case, the static problem has to be solved numerically. In the next section, we study a concrete numerical example where we discuss the numerical solution of the static problem in more detail.

4.5. A NUMERICAL EXAMPLE

In the graphs below, we provide the numerical results for a special case of the Wiener compound Poisson model described above. The model under consideration is the Merton jump-diffusion stock price model of Merton (1976), where the relative jump size has a shifted lognormal distribution, i.e., it is of the form $e^Z - 1$, where Z is Gaussian. In terms of the notation above this means that $X = [-1, \infty)$, $\delta(x) = x$ and

$$\lambda(dx) = \lambda_0 f(x) dx, \tag{127}$$

where λ_0 is the intensity of the underlying Poisson process, and f is the density of the shifted lognormal distribution. The derivatives under considertion are European call options, and in all graphs we have the stock price on the horisontal axis and the option price on the vertical axis. In the first graph, we see the upper and lower relaxed bounds. In between these, we find the optimal bounds, and in the middle, we have the price generated by the minimal martingale measure. In the second figure, we show how the MMM price curve relates to the price curve generated by the Merton option pricing formula (where, by assumption, the market price of jump risk equals zero).

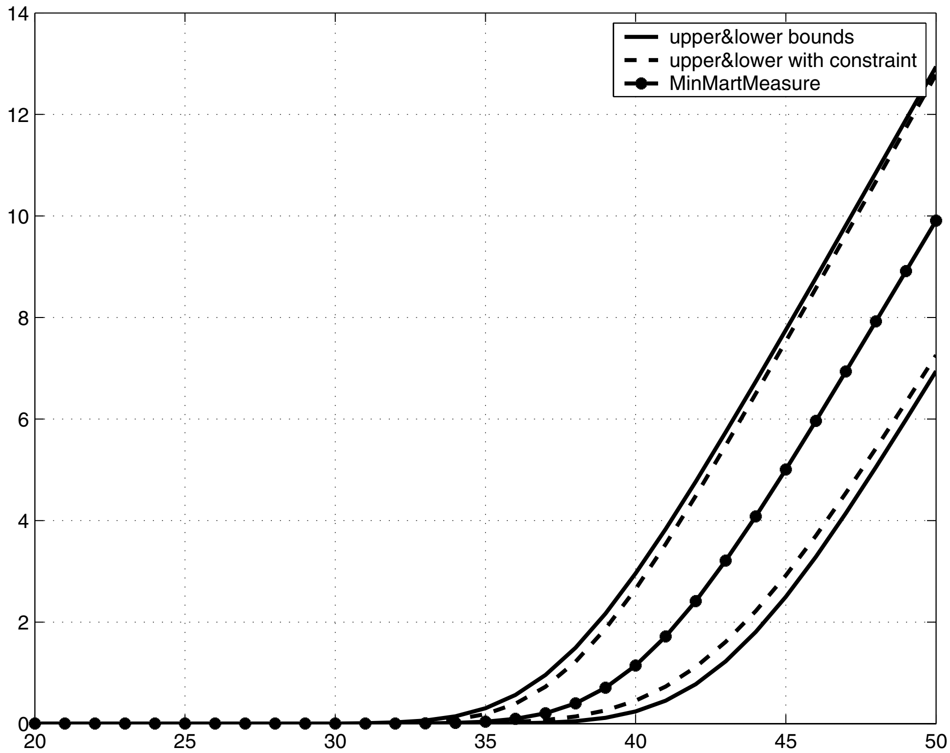

Figure 1. Good deal pricing bounds.

The MMM price and the relaxed pricing bounds are obtained by inserting the relevant kernels from the previous sections into the pricing PIDE, and then solving the PIDE numerically. To obtain the optimal pricing bounds (including the positivity constraint), we solve the static problem numerically (for each point in the discretized state space), using an interior point algorithm that was kindly provided to us by Mathias Stolpe (Stolpe, 1997). The ensuing kernels are then fed into the PIDE, which has been solved numerically using finite differences.

We use the following parameter values: Maximum grid size $M = 60$, grid stock price step $\delta S = 1$, grid time step $\delta t = 0, 0003125$, time to maturity $TT = 0, 25$, number of steps $T = 800$, interest rate $r = 0, 05$, strike price $K = 40$, volatility $\sigma = 0, 15$, Poisson intensity $\lambda_0 = 0, 1$, $\alpha = -0, 1$, $B = 1$, and the parameters for the normal distribution generating the lognormal jump distribution were: mean 0, 89, and standard deviation 0, 45.

4.6. THE IMPACT OF A STOCHASTIC JUMP SIZE

With respect to the examples above, one natural question to ask is how the introduction of a stochastic jump size into a purely Poisson model impacts the good-deal

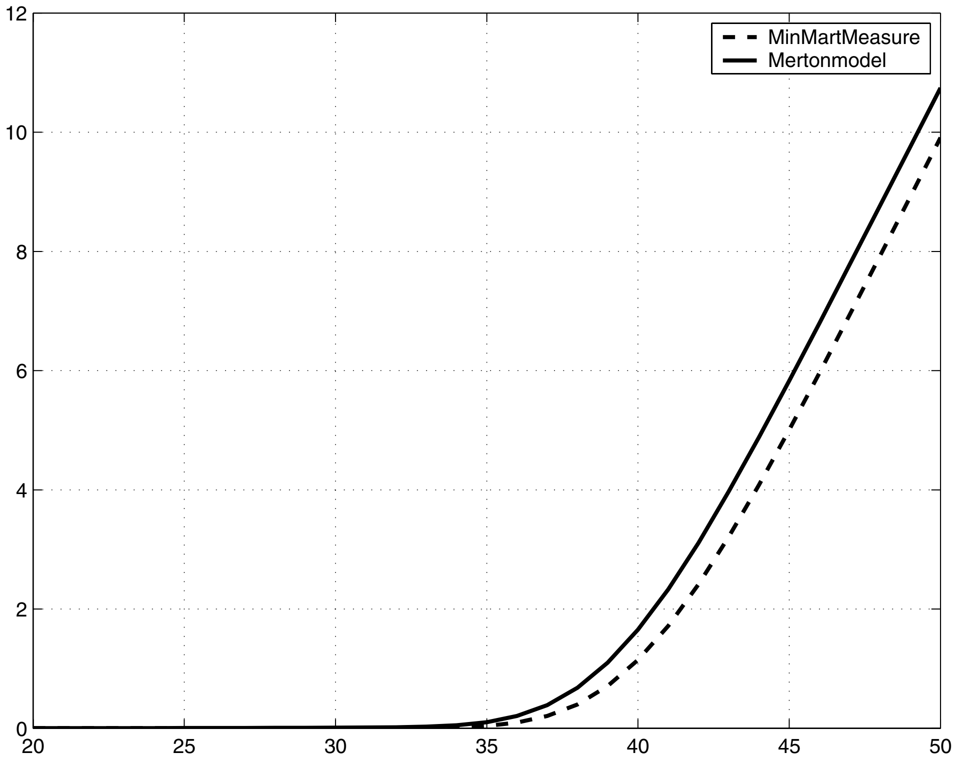

Figure 2. The minimal martingale measure and the Merton model.

bounds. In order to understand this question, let us consider two models: a compound Poisson model with a stochastic jump size, and a pure Poisson model with a deterministic jump size. We assume that the intensity of the underlying event process is the same for both models, and that the jump size of the pure Poisson model equals the expected value of the stochastic jump size of the compound Poisson model. The problem is understanding how the randomness of the jump size affects the length of the good-deal bound interval. The intuition for this problem is as follows.

Since the good-deal bound constraint (i.e., the right hand side of the HJ bound) is quadratic in the control variables h and φ, it is natural to interpret it as a constraint on the control energy (per unit time). Disregarding h for the moment, a given amount of admissible control energy – corresponding to a given good-deal bound B – will, in the pure Poisson case, be used exclusively for changing the intensity process of the underlying Poisson process. In the compound Poisson case, the control energy can be employed to change the intensity as well as the distribution of the jump size. It is thus clear that, from a control perspective, the stochastic jump size case provides more flexibility than does the pure Poisson case. Consequently, we may expect wider good-deal bounds for the compound Poisson

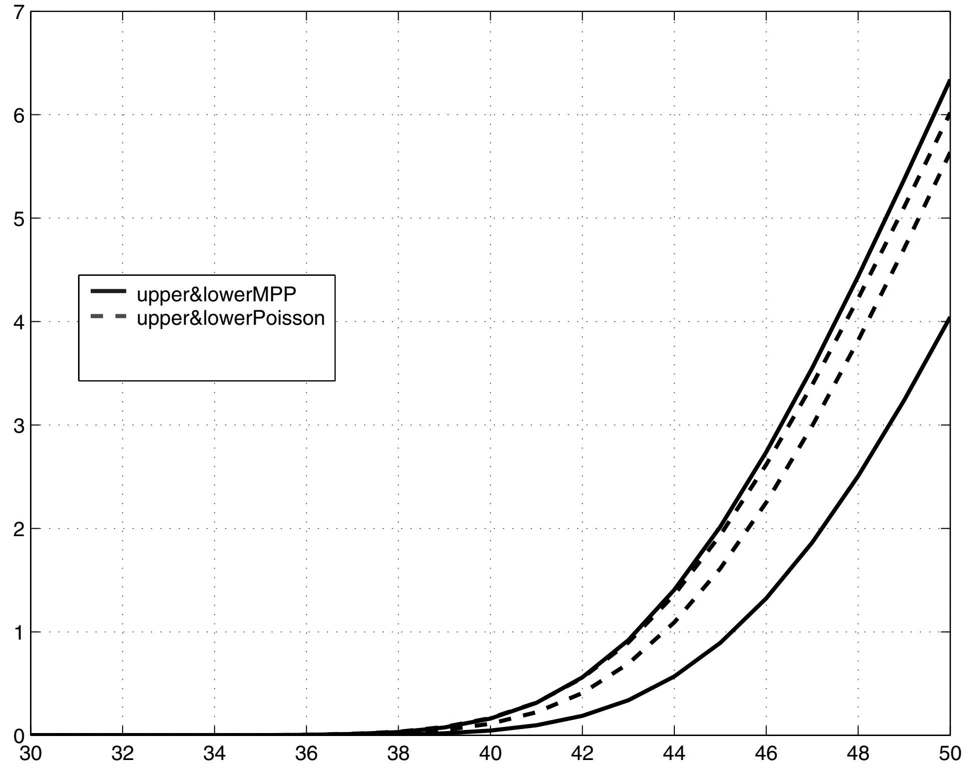

Figure 3. The impact of a stochastic jump size.

model, compared to the pure Poisson model. In the graph below, this intuition is very clearly validated.

We use the following parameter values: Maximum grid size $M = 60$, grid stock price step $\delta S = 1$, grid time step $\delta t = 0,0025$, time to maturity $TT = 0,25$, number of steps $T = 100$, interest rate $r = 0,05$, strike price $K = 45$, volatility $\sigma = 0,15$, Poisson intensity $\lambda_0 = 0,1$, $\alpha = -0,1$, $B = 1$, and the parameters for the normal distribution generating the lognormal jump distribution were: mean 0, 89, and standard deviation 0, 45. The jump size for the Poisson process is taken as the mean of the lognormal distribution associated with the normal distribution with the above parameters.

5. Conclusions

In this paper, we extend the good-deal bound pricing theory from Cochrane and Saá Requejo (2000) to allow for jumps in the underlying processes. While performing this extension we use a somewhat different technique than the one used in the original CSR paper. The result is a streamlined, rigorously developed, and simple theory of good-deal pricing bounds for a very general class of jump diffusions.

A number of intriguing open problems remain for future research. Firstly, on the theoretical side, we note that the good-deal bound theory is a pure pricing theory. Since the problem of arbitrage pricing is dual to the problem of hedging, one would therefore expect that it should be possible to develop a dual "good-deal hedging theory". In our view, the task of developing such a theory constitutes a highly challenging open problem. Secondly, the problem of numerically determining good-deal bounds is relatively complex in terms of CPU time, so there is a very clear need to develop fast, approximative good-deal pricing algorithms. Thirdly, the theory is now ready for application to real-life problems, and it would be very interesting to analyze, say, a reduced-form credit risk model from a good-deal perspective.

Appendix A. Extended Hansen-Jagannathan Bounds

In this appendix, we derive an extension of the Hansen-Jagannathan bounds (see Hansen and Jagannathan, 1991) to the present point process setting. The HJ bounds provide an inequality for the Sharpe ratio process of any traded asset (underlying or derivative) in terms of the market prices of risk of the driving random sources, and, to make this idea precise, we consider the arbitrage-free price process π of an arbitrary asset (derivative or underlying) with P-dynamics given by

$$d\pi_t = \pi_t \alpha_t dt + \pi_t \sigma_t dW_t + \pi_{t-} \int_X \delta_t(x) \mu(dt, dx). \tag{128}$$

Here, α and σ are optional processes, whereas δ is predictable. In order to avoid negative asset prices we must also assume that $\delta_t(x) > -1$.

Compensating the point process μ in (128), we obtain the P-semimartingale dynamics of S as

$$d\pi_t = \pi_t \left\{ \alpha_t + \int_X \delta_t(x) \lambda_t(dx) \right\} dt + \pi_t \sigma_t dW_t + \pi_{t-} \int_X \delta_t(x) \tilde{\mu}(dt, dx). \tag{129}$$

Since the last two terms in this equation are martingale differentials, we see that the local mean rate of return under P is given by the expression

$$\alpha_t + \int_X \delta_t(x) \lambda_t(dx),$$

so, denoting the possibly stochastic short rate process by r, the **risk premium** process R is given by the formula

$$R_t = \alpha_t + \int_X \delta_t(x) \lambda_t(dx) - r_t. \tag{130}$$

We now proceed to define the predictable (total) **volatility** process v, which intuitively should equal the conditional variance of the return of the stock price, i.e., it should roughly be given by the expression

$$v_t^2 dt = Var^P \left[\frac{d\pi_t}{\pi_{t-}} \middle| \mathcal{F}_{t-} \right]. \tag{131}$$

We need to make this notion mathematically precise and this is achieved by formally defining v through the relation

$$d\langle \pi, \pi \rangle_t = \pi_{t-}^2 v_t^2 dt, \tag{132}$$

where $\langle , \rangle$ denotes the usual predictable bracket process (see Jacod and Shiryaev, 1987). From (129), it is not hard to obtain

$$d\langle \pi, \pi \rangle_t = \pi_{t-}^2 \left\{ \|\sigma_t\|_{R^d}^2 + \int_X \delta_t^2(x) \lambda_t(dx) \right\} dt, \tag{133}$$

so, by comparing (133) with (132), we see that the squared volatility process is given by

$$v_t^2 = \|\sigma_t\|_{R^d}^2 + \|\delta_t\|_{\lambda_t}^2, \tag{134}$$

where $\|\cdot\|_{\lambda_t}$ denotes the norm in the Hilbert space $L^2[X, \lambda_t(dx)]$. For future use, we can also express the volatility v as

$$v_t = \|(\sigma_t, \delta_t)\|_{\mathcal{H}}, \tag{135}$$

where we view (σ_t, δ_t) as a vector in the Hilbert space $\mathcal{H} = R^d \times L^2[X, \lambda_t(dx)]$. We finally define the **Sharpe Ratio Process** SR by

$$SR_t = \frac{R_t}{v_t}. \tag{136}$$

Our goal is to derive an inequality for the Sharpe Ratio process in terms of the set of market prices of risk that turn up in connection with the class of risk-neutral martingale measures Q. Note again, that the Sharpe ratio process SR is not a number, but an entire random process.

Turning to martingale measures, we now copy the arguments in Section 3.1 and see that if the measure Q is generated by the Girsanov kernels (h, φ), then Q is a martingale measure if, and only if, the following conditions are satisfied:

$$\alpha_t + \sigma_t h_t + \int_X \delta_t(x) \{1 + \varphi_t(x)\} \lambda_t(dx) = r_t. \tag{137}$$

A Girsanov kernel process (h, φ) for which the induced measure Q is a martingale measure, i.e., a kernel process satisfying the martingale condition (137) and the positivity condition

$$\varphi_t(x) \geq -1, \quad \text{for all } t, x \tag{138}$$

is referred to as an **admissible** Girsanov kernel.

We now derive an inequality for the Sharpe ratio process SR, and we start by noting that we can rewrite the martingale condition (137) as

$$\alpha_t + \int_X \delta_t(x)\lambda_t(dx) - r_t = -\sigma_t h_t - \int_X \varphi_t(x)\delta_t(x)\lambda_t(dx). \tag{139}$$

From (130), we recognize the risk premium R in the left hand side of this equation so we can write R as

$$R_t = -\sigma_t h_t - \int_X \varphi_t(x)\delta_t(x)\lambda_t(dx). \tag{140}$$

From this expression we see that the Girsanov kernel process (h, φ) has a natural economic interpretation. The component $-h^i$ can be interpreted as the market price of risk for the i:th Wiener process, and $-\varphi(x)$ is the market price of risk for a jump event of type x. Using (140), we may also state and prove the main result of this appendix.

THEOREM A.1. (Extended Hansen-Jagannathan Bounds). For any arbitrage-free price processes π and for every admissible Girsanov kernel (market price of risk) process (h, φ), the following inequality holds:

$$|SR_t| \leq \|(h_t, \varphi_t)\|_{\mathcal{H}}. \tag{141}$$

Specifically, this inequality can be written as

$$|SR_t|^2 \leq \|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx). \tag{142}$$

We note that the inequality (141) is an inequality between random processes, not between numbers.

Proof. A closer look at (140) reveals that the right hand side can be viewed as an inner product in the Hilbert space $\mathcal{H}$. Denoting this inner product by $\langle, \rangle_{\mathcal{H}}$, we can thus write

$$R_t = \langle (h_t, \varphi_t), (\sigma_t, \delta_{St}) \rangle_{\mathcal{H}}, \tag{143}$$

and, from the Schwartz inequality, we obtain

$$|R_t| \leq \|(h_t, \varphi_t)\|_{\mathcal{H}} \cdot \|(\sigma_t, \delta_t)\|_{\mathcal{H}}. \tag{144}$$

The inequality (141) now follows immediately from (135), (136), and (144). ■

Appendix B. Purely Wiener-Driven Models

Although the main object of the present paper is to study good-deal bounds in the presence of a marked point process, for completeness' sake, we also study pricing in our model without a driving point process, i.e, we study the special case of a purely Wiener-driven model. This is an extension, although a very modest one, of the model originally considered in Cochrane and Saá Requejo (2000); the main difference is that we do not need a certain rank condition assumed by Cochrane and Saá Requejo. We derive the general HJB equation for the upper (lower) price bounds and, given the appropriate rank condition, we derive the pricing PDE presented by Cochrane and Saá Requejo.

B.1. THE GENERAL CASE

We recall our model for the purely Wiener-driven situation.

ASSUMPTION B.1.

1. The price and factor dynamics under objective probability measure P are given by

$$dS_t^i = S_t^i \alpha_i (S_t, Y_t)\, dt + S_t^i \sigma_i (S_t, Y_t) dW_t, \quad i = 1, \ldots, n$$

$$dY_t^j = a_j (S_t, Y_t)\, dt + b_j (S_t, Y_t) dW_t, \quad j = 1, \ldots, k.$$

2. We assume that for each i and j, $\alpha_i(s, y)$ and $a_j(s, y)$ are deterministic scalar functions, and $\sigma_i(s, y)$ and $b_j(s, y)$ are deterministic row vector functions.
3. All functions above are assumed to be sufficiently regular to allow for the existence of a unique strong solution for the system of SDEs.
4. We assume the existence of a short rate r of the form

$$r_t = r(S_t, Y_t).$$

5. We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk-neutral martingale measure Q.

Theorem 3.2 shows that the upper good-deal bound function $V(t, s, y)$ satisfies the following boundary value problem

$$\frac{\partial V}{\partial t}(t,s,y) + \sup_h \left\{ \mathbf{A}^h V(t,s,y) \right\} - rV(t,s,y) \;=\; 0, \tag{145}$$

$$V(T,s,y) \;=\; \Phi(s,y), \tag{146}$$

where we momentarily suppress all the constraints, and where the infinitesimal operator $A^{h,\varphi}$ is given by

$$\begin{aligned}
\mathbf{A}^h V(t,s,y) \;=\; & \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}(t,s,y)s_i r \\
& + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t,s,y) \left\{ a_j(s,y) + b_j(s,y)h(t,s,y) \right\} \\
& + \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l}(t,s,y)s_i s_l \sigma_i^{\star}(s,y)\sigma_l(s,y) \\
& + \frac{1}{2} \sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l}(t,s,y)b_j^{\star}(s,y)b_l(s,y) \\
& + \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j}(t,s,y)s_i \sigma_i^{\star}(s,y)b_j(s,y).
\end{aligned} \tag{147}$$

Thus, we have the following static optimization problem.

PROBLEM B.1.

$$\max_h \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t,s,y) \left\{ b_j(s,y)h(t,s,y) \right\}, \tag{148}$$

subject to the constraints

$$\alpha_i + \sigma_i h \;=\; r, \quad i = 1, \ldots, n \tag{149}$$

$$\|h\|_{R^d}^2 \;\leq\; A^2. \tag{150}$$

This is a very simple finite-dimensional optimization problem and, using standard Kuhn-Tucker techniques, we obtain the following result.

PROPOSITION B.1. Denote the excess return $\alpha - r$ by R.

- The upper (lower) good-deal bound function $V(t, s, y)$ satisfies the following boundary value problem:

$$\frac{\partial V}{\partial t} + r\sum_{i=1}^{n}\frac{\partial V}{\partial s_i}s_i + \sum_{j=1}^{k}\frac{\partial V}{\partial y_j}\left\{a_j + b_j\hat{h}\right\} + \frac{1}{2}\sum_{j,l=1}^{k}\frac{\partial^2 V}{\partial s_i \partial s_l}s_i s_l \sigma_i^{\star}\sigma_l$$

$$+\frac{1}{2}\sum_{j,l=1}^{k}\frac{\partial^2 V}{\partial y_j \partial y_l}b_j^{\star}b_l + \sum_{i,j=1}^{k}\frac{\partial^2 V}{\partial s_i \partial y_j}s_i\sigma_i^{\star}b_j - rV = 0 \tag{151}$$

$$V(T, s, y) = \Phi(s, y). \tag{152}$$

- For the upper bound, the kernel $\hat{h} = \hat{h}_{\max}$ is given by

$$\hat{h}_{\max} = b'V_y - \sigma'(\sigma\sigma')^{-1}\sigma b'V_y + \frac{\sigma'(\sigma\sigma')^{-1}R\sqrt{V_y'b\left\{I - \sigma'(\sigma\sigma')^{-1}\sigma\right\}b'V_y}}{\sqrt{A^2 - R'(\sigma\sigma')^{-1}R}}. \tag{153}$$

- For the lower bound, the kernel $\hat{h} = \hat{h}_{\min}$ is given by

$$\hat{h}_{\min} = b'V_y - \sigma'(\sigma\sigma')^{-1}\sigma b'V_y - \frac{\sigma'(\sigma\sigma')^{-1}R\sqrt{V_y'b\left\{I - \sigma'(\sigma\sigma')^{-1}\sigma\right\}b'V_y}}{\sqrt{A^2 - R'(\sigma\sigma')^{-1}R}}. \tag{154}$$

Here, we use the notation $V_y = \left(\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_k}\right)^{\star}$

B.2. THE COCHRANE AND SAÁ REQUEJO MODEL

The pricing PDE of Cochrane and Saá Requejo (2000) can now be derived as a particular case of our slightly more general model above. In terms of our notation, the CSR model is specified by the following set of assumptions.

ASSUMPTION B.2.

- The price dynamics are assumed to be of the form

$$dS_t^i = S_t^i\alpha_i(S_t, Y_t, t)dt + S_t^i\sigma_i(S_t, Y_t, t)dZ_t, \quad i = 1, \ldots, n, \tag{155}$$

  where Z is an n-dimensional standard Wiener process.

- The factor dynamics are assumed to be of the form

$$dY_t^j = a_j(S_t, Y_t, t)dt + b_j^z(S_t, Y_t, t)dZ + b_j^w(S_t, Y_t, t)dW_t, \quad j = 1, \dots, k, \tag{156}$$

  where W is a k-dimensional standard Wiener process orthogonal to Z.
- The $n \times n$ volatility matrix σ is assumed to be invertible.

Given this assumption, we have a d-dimensional driving Wiener process (Z, W), where $d = n + k$, so the kernel process h is now $(n + k)$-dimensional and can be decomposed as $h = (h_z, h_w)$. The martingale condition (149) will take the form

$$\alpha_i + \sigma_i h^z = r, \quad i = 1, \dots, n, \tag{157}$$

and the invertibility assumption above allows us to solve for h^z to obtain

$$h_z = \sigma^{-1}(r - \alpha). \tag{158}$$

Thus, the static problem is simplified as follows.

PROBLEM B.2.

$$\max_h \sum_{j=1}^{k} \frac{\partial V}{\partial y_j}(t, s, y) \left\{ b_j(s, y) h_w(t, s, y) \right\} \tag{159}$$

subject to the constraints

$$h_w' h_w \leq A^2 - R'R.$$

This is a trivial linear-quadratic optimization problem and the optimal h_w is easily found as

$$h_w = \frac{\sqrt{A^2 - R'R}}{\sqrt{b'V_y V_y' b}} \cdot b'V_y.$$

We have thus determined the entire optimal vector $h = (h_z, h_w)$, and substituting this into the pricing equation of Proposition, one obtains the pricing PDE of Cochrane and Saá Requejo (2000).

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