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Why is the Index Smile So Steep?*

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Abstract. Empirical evidence shows that the implied volatility smiles for index options are significantly steeper than those for individual options. We propose a model setup where we start from the joint dynamics of the stocks and where the index value is a weighted sum of individual stock prices. Then the differences between the index smile and the smiles for individual stocks are entirely determined by the dependence structure among the stocks. We illustrate our idea in a jump-diffusion framework where both the diffusion and the jumps are decomposed into common and idiosyncratic components. Empirical data for options on the German stock index DAX and on Deutsche Bank are used to show that the model can explain the stylized facts on implied volatility smiles.

1. Introduction and Motivation

There is growing empirical evidence that the implied volatility smile for index options is different from that for options on individual stocks. While implied volatility is in most cases a downward sloping function of the strike price, index smiles are usually significantly steeper than the corresponding curves for individual stocks, which are sometimes even flat. See, for example, Bakshi et al. (2003) for options on the S&P 100 index, and Bollen and Whaley (2001) for the S&P 500, who find that the slope of the function relating the implied volatility of options to their moneyness is around -0.308 for the typical stock in their sample, while for the index the curve is much steeper with a slope of -0.886 (the computational details behind these numbers are explained in Section 4).

Two questions arise in this context. The first is how to explain in general a downward sloping volatility smile, and the second is how to explain the differences between the implied volatility functions for indices and individual stocks. One hypothesis that has been put forward in the literature to explain downward sloping smiles is the so-called leverage effect. Interpreting the stock as a call option on the assets of the firm, a drop in firm value causes a drop in the stock price, which

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increases financial leverage. This, in turn, should result in a higher equity volatility. However, Figlewski and Wang (2000) strongly reject this conjecture in an empirical study of stock options on the U.S. market. The authors do not find any significant linkage between financial leverage and implied volatility in a variety of tests.

Another strand of the literature explains the smile by using more advanced stochastic processes for the underlying asset. For example, Bakshi et al. (1997) present a model incorporating stochastic volatility, stochastic jumps, and stochastic interest rates. While this very general model performs better than the simple Black and Scholes (1973) model in explaining market prices for index options, it has never been tested explicitly for options on individual stocks.

In addressing the second issue, Bakshi et al. (2003) argue that the different slopes of the curves are in part related to differences in the skewness of the risk-neutral distributions for the stocks and for the index. Assuming a return generating model with the market return as a factor, the differences in skewness can be explained by the properties of the idiosyncratic components. In contrast to these authors, Bollen and Whaley (2001) argue that the physical distributions of the stocks and the index are not significantly different; they try to explain the different slopes of the smiles not by the differences in the stochastic processes of the underlyings, but by buying pressure for certain out-of-the-money (OTM) index puts that serve as an insurance device against market crashes. This buying pressure increases option prices and therefore also the implied volatilities for options with low strike prices.

The objective of our paper is not to develop yet another pricing model for equity and index options. Instead, we want to propose a simple setup that is built on well-accepted theoretical option valuation models and that offers a straightforward economic explanation for the pricing differences between equity and index options. In this sense, our approach is similar in spirit to Berk (1995) who offers a simple economic rationale for the seemingly anomalous empirical relationship between expected stock returns and firm size.

The key feature of our approach is that it explicitly takes into account an additivity restriction for the index level, given individual stock prices. Our model is able to generate downward sloping smiles for both options on individual stocks and on the index, as well as a significantly steeper slope of the implied volatility curve for index options. We show that these goals can be achieved without resorting to extremely sophisticated stochastic models for returns and without dropping the assumption of perfect markets.

The main theoretical contribution of the paper is threefold. We first argue that differences in smiles must be explained by differences in risk-neutral distributions, not by differences in physical distributions. Indeed, in an incomplete market, the physical densities may well be the same for two assets while the risk-neutral densities are very different. This allows us to focus mainly on the distributions under the risk-neutral measure. Second, we explicitly take into account the aggregation restriction between an index and its component stocks. Because the index level is just the weighted sum of the individual stock prices, its distribution is completely determined by the joint distribution of the individual stocks. Thus, we do not model the index as an exogenous factor explaining the returns on the stocks, but we derive the behaviour of the index from the joint behaviour of the stocks. Third, we pay special attention to the behaviour of the stocks and the index in the case of large downward price moves. The greater the risk-neutral probability for such a large price drop, the steeper the resulting smile. Due to the aggregation restriction, a crash in the index can occur only if there is a downward move in the majority of the stocks, an issue that has been addressed in a different context by Nietert (1997). Therefore, the risk-neutral probability of a market crash depends critically on the risk-neutral probability that a downward move occurs for a large number of stocks simultaneously. This, however, implies that the slope of the index smile for low strike prices is determined by the dependence among stocks during a downward move. Indeed, there is empirical evidence that during a crash the stocks tend to move together more than during normal market periods. For example, Ang and Chen (2002) show that the dependence between stock returns and index returns during a downside move is greater than it would be under a multivariate normal distribution. Longin and Solnik (2001) as well as Campbell et al. (2002) show that for international markets correlations increase in bear markets.

In our model, we assume that each individual stock follows a jump-diffusion process with a rather simple specification. The option pricing formula for this framework is well-known and was developed by Merton (1976). In order to capture the dependence, we decompose both the diffusion and the jump term into a common and an idiosyncratic component. This decomposition can be performed under the physical as well as under the risk-neutral measure, and we explicitly show that it is the main determinant of the shape of the implied volatility function for the index. Of course, the specific approach presented in this paper can easily be generalized to include, e.g., stochastic volatility or jumps of a random size. However, we would not expect the basic characteristics of the model to change significantly as a result of such an extension.

It is important to note that the change of measure has quite different effects on the diffusion and the jump components. For diffusion processes neither volatilities nor correlations will change compared to the physical measure. For a jump process on the other hand, the change of measure will affect variances as well as correlations. Therefore, in a crash the dependence structure under the physical and under the risk-neutral measure can be very different. The inclusion of jumps therefore allows for differences between stocks and the index under the risk-neutral measure that do not exist under the physical measure.

From an economic point of view, it can be argued that the dependence is higher under the risk-neutral measure. A simultaneous crash in all stocks will reduce the wealth of the representative investor by much more than a downward move in only a few stocks that can partially be diversified away. The fear of jumps induces investors to demand a much higher risk premium for market crashes than for downward moves in only one or a small number of stocks. In our model, this effect is captured by a risk-neutral probability of a common downward move that is higher than the probability of the same event under the physical measure. This setup generates downward sloping volatility curves for individual stocks as well as for the index that can exhibit different slopes. The issue of physical versus risk-neutral distributions is also discussed theoretically by Bates (2001), who relates their differences to investors' risk aversion and to potential heterogeneity across investors.

We illustrate the properties of our model using data for options on the German stock market index DAX and on Deutsche Bank, and indeed our model is able to reproduce the qualitative features of the data quite well.

The remainder of the paper is organized as follows. In Section 2 we discuss the aggregation restriction on the index. The model for individual stocks is introduced in Section 3, where we also show how to incorporate the dependence structure into the model. The implications of our setup and the impact of the dependence structure on the slope of the index smile are shown in Section 4. Here we also perform some sensitivity analyses of our model. Our empirical study is presented in Section 5, and Section 6 contains some concluding remarks.

2. The Aggregation Restriction for the Index

In our analysis we consider options on individual stocks as well as options on the index composed of these stocks. In empirical studies researchers usually find that for both types of options the smile is downward sloping, but that the smile is much steeper for index options than for individual stock options.

When comparing the pricing of equity options to the pricing of index options, the index must obey what we call the *aggregation restriction*, since the index level is equal to a weighted sum of the stock prices. A general formula for the index level I_t at time t is given by

$$I_t = \sum_{i=1}^{m} w_t^i S_t^i, \tag{1}$$

where S_t^i denotes the price of stock i at time t and the weights w_t^i $(i = 1, \dots, m)$ are determined by the respective index formula. For example, the weights may be related to the number of shares outstanding or to the total market capitalization of a stock. In our analysis we will set the weights identically equal to $1/m$ for the sake of simplicity. This assumption is not crucial, since changing the weights will have only a marginal impact on our results. Equation (1) explicitly shows that the behaviour of the index is completely determined by the joint behaviour of the stocks.

The first step in setting up a model is to choose a stochastic process for each individual stock that is able to match the smile of equity options as it is observed in real data. The second step is then to model the dependence between the stocks. Indeed, we will show that the shape of the index smile, compared to the equity smile, is almost entirely determined by this dependence.

The slope of the smile is related to the shape of the risk-neutral distribution. The more left- (right-)skewed the distribution, the greater the implied volatilities for low (high) strike prices, and the fatter the tails of the distribution, the higher the implied volatilities for both deep in-the-money and out-of-the-money options. Recent models, like Bakshi et al. (1997), incorporate both stochastic volatility and jumps. Both features are able to generate risk-neutral distributions which are more left- or right-skewed than the lognormal distribution, and which exhibit excess kurtosis. The exact direction and size of the effects depend on the ultimate process and the actual parameters. Among other things, Das and Sundaram (1999) discuss how the effects of stochastic volatility and jumps depend on time to maturity. They show that stochastic volatility has a greater impact for longer maturities, while jumps are necessary for short-term smiles to exhibit significantly negative slopes (given that the average size of a jump is negative).

Our objective is to explain a downward sloping smile, so the risk-neutral distribution of returns must be left-skewed. In order to achieve this for both individual stock options and the index, we rely on downward jumps for both the individual stocks and the index. The steepness of the smile curves then depends on the risk-neutral probabilities of these (large) downward moves. Given the jump parameters for the stocks, the jump characteristics of the index depend on the joint behaviour of the stocks due to the aggregation restriction. In particular, there can be a crash in the index only when there is a downward move in the majority of the stocks (see also Nietert (1997)). The probability of a market crash, therefore, depends on the probability that a downward jump in a stock is a common jump affecting most of the stocks simultaneously. The greater this probability under the risk-neutral measure, the steeper the smile for index options, while individual smiles will not be affected.

To get the intuition for the importance of the dependence structure consider two extreme scenarios. In the first case, the jumps in the stocks all occur independently, i.e., the probability of a market crash is extremely low. Here we expect the impact of jumps in the individual stocks on the prices of index options to be small because of the diversification effect. In the second case, assume that downward jumps in the stocks all occur at the same time. In this case, the jump component has a high impact on the prices of OTM index puts, and the index smile will be quite steep.

3. Pricing Individual Stock Options

As discussed above, the stochastic behaviour of the index is *perfectly* determined by the stochastic processes of the individual stocks and their dependence. Our main focus is on this joint behaviour of the stocks: How much do the stock price movements depend on each other? We use a model setup that is as simple as possible and that allows us to concentrate on the analysis of the dependence structure. Each individual stock follows a jump-diffusion process, where the parameters are assumed to be identical across stocks. Furthermore, the measures of dependence are assumed to be equal for any two pairs of stocks. The correlation matrix thus has equal entries for all elements off the main diagonal.

Each stock is driven by a Wiener process and a Poisson process, where the Poisson process captures the jumps in the stock price. Both the Wiener process and the Poisson process have a common and an idiosyncratic component. The common component affects all stocks in the index simultaneously, whereas the idiosyncratic components are mutually independent. Formally we assume the following dynamics for each stock i $(i = 1, \dots, m)$

$$\begin{aligned} dS_t^i &= \mu^i S_{t-}^i dt + \sigma S_{t-}^i (\sqrt{\rho_W} dW_t^c + \sqrt{1-\rho_W} dW_t^i) \\ &\quad - \gamma S_{t-}^i (dN_t^c + dN_t^i). \end{aligned} \tag{2}$$

Here W^c is the Wiener process that is common to all stocks, and W^i is the purely idiosyncratic diffusion component of asset i independent of W^c. This independence property implies that $\sqrt{\rho_W} W^c + \sqrt{1-\rho_W} W^i$ is again a Wiener process for every stock i. Furthermore, we assume a constant correlation of the diffusion parts between any two stocks equal to $\rho_W \geq 0$.

In case of a jump the stock price decreases by the constant multiplicative factor γ. Jumps have intensity h so that for some small Δt the probability of a jump occuring in the interval $[t, t + \Delta t]$ is approximately equal to $h\Delta t$. A jump is a common jump with probability ρ_N and an idiosyncratic jump with probability $1 - \rho_N$. This is modelled by defining the common Poisson process N^c with intensity $h\rho_N$ and the purely idiosyncratic Poisson process N^i with intensity $h(1 - \rho_N)$, where N^c and N^i are independent. The common Poisson process captures market crashes, and the greater ρ_N, the greater the probability of such a crash. $N_t^c + N_t^i$ is again a Poisson process with intensity h.[1]

In a first step, we consider the pricing of options on individual stocks. For this analysis the distinction between idiosyncratic components and common components does not matter. Only the total risk of the stock is important, which is determined by the sum of the diffusion components and by the sum of the Poisson processes. For the purpose of pricing options on individual stocks we can therefore rewrite the dynamics from Equation (2) as[2]

$$dS_t = \mu S_{t-} dt + \sigma S_{t-} dW_t - \gamma S_{t-} dN_t. \tag{3}$$

[1] Our approach is not restricted to the simple framework proposed in this paper. Concerning the dependence structure, one could start from an arbitrary correlation structure and include additional components that only affect certain subsets of the universe of all stocks. Concerning the stochastic processes of the stocks we can integrate our approach into a more general model setup allowing for stochastic jumps, stochastic volatility, and stochastic interest rates, like the models suggested by Bakshi et al. (1997) or Pan (2002).

[2] When going from Equation (2) to Equation (3), we set $dW_t = \sqrt{\rho}\, dW_t^c + \sqrt{1-\rho}\, dW_t^i$ and $dN_t = dN_t^c + dN_t^i$ and drop, for ease of notation, the superscript i.

Here W is a Wiener process, and N is a Poisson process with intensity h. Under the risk-neutral measure the stock price process becomes

$$dS_t = (r + \widehat{h}\gamma) S_{t-} dt + \sigma S_{t-} d\widehat{W}_t - \gamma S_{t-} dN_t, \tag{4}$$

where $\widehat{W}$ is again a Wiener process. The intensity of the Poisson process changes from h to $\widehat{h} = \phi h$, where $\phi > 0$ is the market price of jump risk. The variance of the continuous return is given by

$$\widehat{Var}\,[\ln S_t - \ln S_0] = \left(\sigma^2 + \widehat{h}\,(\ln(1-\gamma))^2\right) t.$$

This variance depends on the jump intensity, so that it will in general be different under the physical and the risk-neutral measure. Note that this is a special feature of models incorporating jump components. For a pure diffusion model with constant volatility, like in the case when we restrict our model to have zero jump intensity, we would have $\widehat{Var}\,[\ln S_t - \ln S_0] = Var\,[\ln S_t - \ln S_0]$, i.e., the variance would not be affected by the change of measure.

As shown by Merton (1976) the price of a European call option on the stock in the framework of Equations (3) and (4) is given by

$$\begin{aligned} &C^{JD}(S_0, K, r, T, \sigma, \widehat{h}, \gamma) \\ &\quad = \sum_{n=0}^{N} \frac{\left(\widehat{h}(1-\gamma)T\right)^n}{n!}\, e^{-\widehat{h}(1-\gamma)T} C^{BS}\,(S_0, K, r_n, T, \sigma) \end{aligned}$$

where

$$r_n = r + \widehat{h}\gamma + \frac{n \ln(1-\gamma)}{T},$$

and $C^{BS}\,(S, K, r, T, \sigma)$ is the price of a European call option in the Black-Scholes model.

In our model with $\gamma > 0$, the risk-neutral distribution has a higher variance and is more left-skewed than a lognormal distribution. Therefore, the implied volatility function is in general decreasing in the strike price. Implied volatilities for options with low strikes will increase (and, consequently, the smile will become steeper) with an increase in the intensity of jumps, in the market price of jump risk, and in the jump size.

We now return to the question of how to model the joint behaviour of the stocks under the risk-neutral measure. Under the physical measure, Equation (2) shows that the joint behaviour is governed by ρ_W and by ρ_N. Under the risk-neutral measure, the stock price process is

$$dS_t^i = (r + \widehat{h}\gamma) S_{t-}^i dt + \sigma S_{t-}^i (\sqrt{\rho_W} d\widehat{W}_t^c + \sqrt{1-\rho_W} d\widehat{W}_t^i)$$

$$- \gamma S^i_{t-}(dN^c_t + dN^i_t). \tag{5}$$

Here $\widehat{W}^c$ and $\widehat{W}^i$ are again Wiener processes. If we consider only the diffusion components, the correlation of any two stocks under the risk-neutral measure is again ρ_W. However, as in (4), the intensity of the Poisson process $N^c + N^i$ changes from h to $\widehat{h} = h\phi$, and also the probability that a jump is common changes from ρ_N to $\widehat{\rho}_N$. As the risk-neutral measure and the physical measure must be equivalent, i.e., assign a zero probability to identical sets of events, $\rho_N = 0$ implies $\widehat{\rho}_N = 0$, and $\rho_N = 1$ implies $\widehat{\rho}_N = 1$. Note that this is all we can say about the relationship between ρ_N and $\widehat{\rho}_N$ so that for $\rho_N \in (0, 1)$ the only restriction is $\widehat{\rho}_N \in (0, 1)$.

As already shown, the variance of the jump components is different under the physical and under the risk-neutral measure. Furthermore, the correlation of the jump components $N^c + N^i$ and $N^c + N^j$ of stocks i and j is different under the physical and under the risk-neutral measure as well. Under the physical measure,

$$corr[N^c_t + N^i_t, N^c_t + N^j_t] = \rho_N,$$

whereas under the risk-neutral measure

$$\widehat{corr}[N^c_t + N^i_t, N^c_t + N^j_t] = \widehat{\rho}_N.$$

This fact may in part explain why physical distributions do not seem very helpful in explaining the shapes of smiles observed in the options markets, as documented by Bollen and Whaley (2001). The lower tail of the distribution for the index level can be quite different under the physical and under the risk-neutral distribution, since correlations can change almost arbitrarily. In particular, we can no longer use the dependence under the physical probability measure to assess the dependence under the risk-neutral distribution.

4. Pricing Index Options

Having modelled the joint behaviour of the stocks, we now consider the characteristics of the index. Assuming $S^i_0 = 1$ for $i = 1, \ldots, m$ and weights that are identically equal to $1/m$ for each stock the index level at time t is given by

$$\begin{aligned} I_t &= \frac{1}{m} \underbrace{\exp\left\{(r - 0.5\sigma^2 + \gamma\widehat{h})t + \sqrt{\rho_W}\,\widehat{W}^c_t\right\}(1-\gamma)^{N^c_t}}_{\text{common component}} \\ &\quad \times \underbrace{\sum_{i=1}^{m} \exp\left\{\sqrt{1-\rho_W}\,\widehat{W}^i_t\right\}(1-\gamma)^{N^i_t}}_{\text{idiosyncratic components}}. \end{aligned}$$

The first term represents the common component, since the respective impact of $\widehat{W}^c$ and N^c is the same for each stock and for the index. The second term is the sum of m idiosyncratic components. The lower the correlation coefficients ρ_W and $\widehat{\rho}_N$, the more diversification effects will show up in this term.

In the following analysis, we are primarily interested in the influence of ρ_W and of $\widehat{\rho}_N$ on the slope of the index smile.[3] As we will show the impact of these two correlation measures on the shape of the smile for index options is distinctively different. Note further that these parameters have no impact on the prices of options on individual stocks. As argued above, prices of individual options depend only on the total diffusion component and the total jump component, but not on the decomposition into common and idiosyncratic parts. Taken together, changing the dependence between the stocks via ρ_W and $\widehat{\rho}_N$ changes the smile of the index without changing the smile of individual stocks. These parameters can therefore be used to achieve a given difference in the slopes of the smile of stocks and the smile of the index.

There is no closed-form solution for index option prices in our setting. Since the index is some weighted arithmetic average of individual stock prices, such a formula would not exist even if each stock followed a geometric Brownian motion and all the stocks were independent. We therefore use a Monte Carlo simulation to calculate the prices of index call options. From these prices, we infer the implied volatilities and compare them to the implied volatilities of the stocks.

Our base case has the following parameters (note that the parameters are identical for all the individual stocks and any pair of stocks): $S_0 = 1$, $r = 0$, $T = 1/12$, $\sigma = 0.4$, $\gamma = 0.2$, $\widehat{h} = 1.25$, with varying correlation parameters ρ_W and $\widehat{\rho}_N$. To calculate the option prices we use 1,000,000 simulation runs to generate the distribution of the terminal prices of $m = 30$ stocks. The number of common and of idiosyncratic jumps is drawn from independent Poisson distributions with intensities $\widehat{\rho}_N \widehat{h} T$ and $(1 - \widehat{\rho}_N)\widehat{h}T$, respectively. Of course, more than one jump may occur over the interval $[0, T]$.

Figure 1 shows the impact of ρ_W on the implied volatilities of index options for $\widehat{\rho}_N = 0.05, 0.2, 0.8$. Because of the forces of diversification the implied volatility curve of the index is always below the one for the typical stock over the full range of strikes. For low strike prices, the influence of ρ_W on implied volatilities can almost be neglected, whereas for higher exercise prices, index-implied volatility increases with the correlation ρ_W. This holds independently of $\widehat{\rho}_N$. Note that some of the curves do not show values for high strikes, which is due to the fact that our simulations produce very few realizations of the terminal index value in this range. Option prices for such high strikes are therefore practically equal to zero, so that implied volatilities are not well-defined.

[3] The different slopes of the smiles for equity and index options are not due to the simple and well-known fact that the sum of lognormal random variables is itself not lognormally distributed (except in the extreme case of perfect correlation). Extensive simulations show that the deviation of the index from lognormality generates only marginal differences between the slopes of the smile curves (which are flat for individual stocks here).

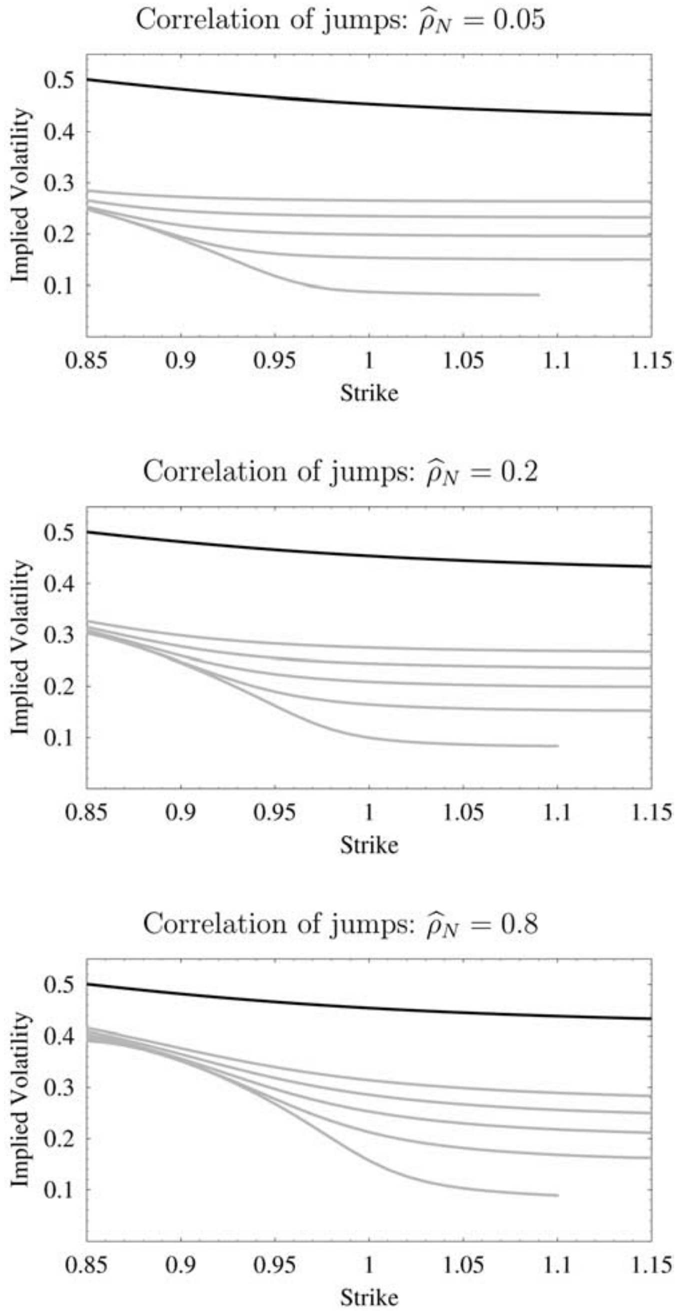

Figure 1. Jump-diffusion model: Impact of the correlation ρ_W of the Wiener processes. Implied volatility curves for the representative stock (black line) and for the index (grey lines) are shown as a function of the strike price. The implied volatilities for the index increase in the correlation ρ_W of the Wiener processes. The five grey curves in each picture correspond to (from bottom to top) $\rho_W = 0.0, 0.1, 0.2, 0.3, 0.4$. The other parameters are $S_0 = 1$, $r = 0$, $T = 1/12$, $\sigma = 0.4$, $\gamma = 0.2$, and $\widehat{h} = 1.25$, the number of stocks in the index is 30.

The effect on the implied volatilities for high strike prices can be explained via the variance of the index. The greater the correlation of the individual Wiener processes, the greater the variance of the index, since there is less diversification. This increases the prices of index options, resulting in a higher implied volatility for the index. To get the intuition for what happens for low strike prices, note that low realizations of the index are reached either by low realizations of the Wiener process, or by downward jumps. Below a certain level, the probability that a price in this range is reached by the Wiener process is nearly zero, while the probability that it is reached by downward jumps is relatively high. The implied volatility for options with low strike prices is thus mainly determined by the characteristics of the jump component. This explains that implied volatilities for low strike prices are not affected significantly by the correlation ρ_W of the Wiener processes.

The graphs in Figure 2 show implied volatility curves for varying values of $\widehat{\rho}_N$, the correlation coefficient for the Poisson processes under the risk-neutral measure. The Wiener correlation ρ_W takes on the values 0, 0.1, and 0.3. Similar to the above situation some very high and very low index levels are not reached in the simulation, so that implied volatilities do not exist for the corresponding strike prices. Compared to the previous figure the picture is exactly reversed. Again, implied volatilities are increasing in the correlation measure. However, we now observe a much larger impact for options with low strike prices, whereas the effect of an increase in $\widehat{\rho}_N$ is almost negligible for calls with high exercise prices.

Again the explanation is found by looking at the total variance of the index. As mentioned above the greater the correlation, the less diversification and the greater the variance. Therefore, the overall level of implied volatilities increases in $\widehat{\rho}_N$. The different impact of $\widehat{\rho}_N$ on options with low and high strike prices can be explained by the fact that we consider only downward jumps. The greater $\widehat{\rho}_N$, the greater the probability that there is a crash in the index, and the greater the price of an OTM put on the index. The price of an OTM call, on the other hand, is hardly affected, because high levels of the index are not reached by jumps, but only due to large positive shocks in the diffusion part.

Taken together, implied volatilities for low strike prices are increasing in $\widehat{\rho}_N$. Implied volatilities for high strike prices are increasing in ρ_W. *The absolute amount of the slope is therefore increasing in* $\widehat{\rho}_N$ *and decreasing in* ρ_W. This shows that there is basically no need to introduce more complex stochastic processes for the individual stocks to generate the differences between implied volatility curves for equity and index options observed in real data. In the case of index options the dependence structure among the stocks in the index is the key variable in the model, and it is mainly this dependence that generates steeper slopes for the index, based on a model which also generates negatively sloped smile curves for individual stocks.[4]

[4] The dependence structure can be different under the physical and under the risk-neutral measure. While the correlation of the Wiener processes remains the same, both the jump intensity and the probability that a jump is a common jump can change. Therefore, differences in the physical

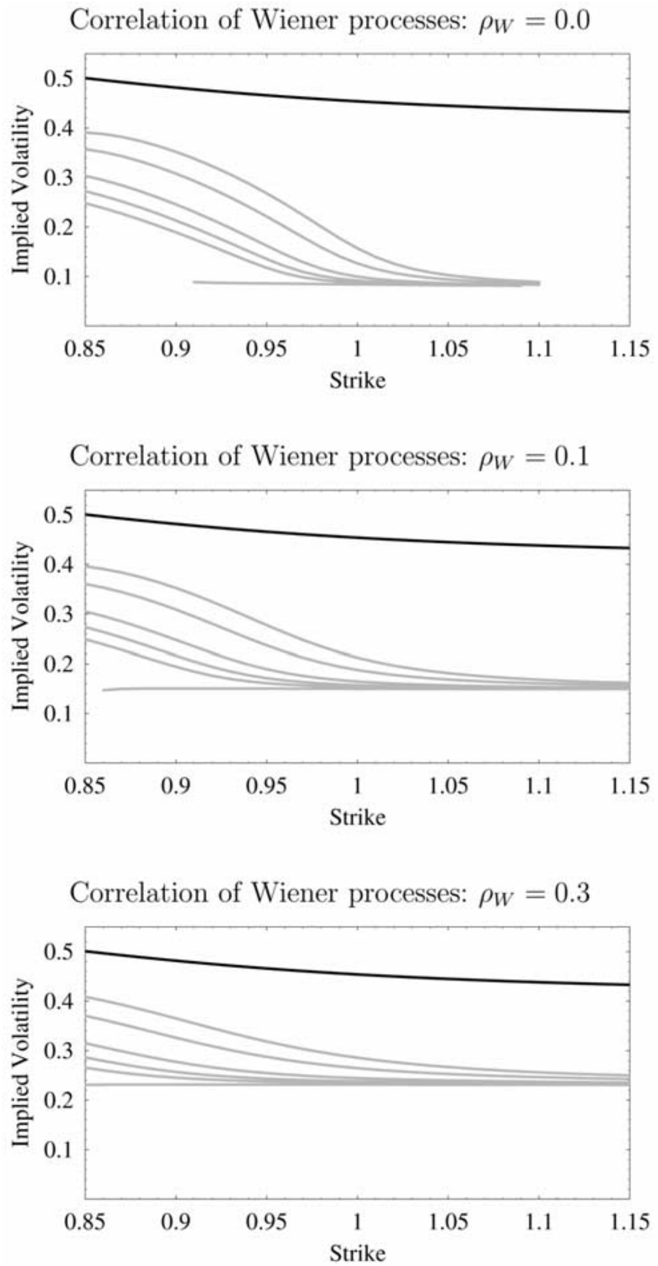

Figure 2. Jump-diffusion model: Impact of the correlation $\widehat{\rho}_N$ of the Poisson processes. Implied volatility curves for the representative stock (black line) and for the index (grey lines) are shown as a function of the strike price. The implied volatilities for the index increase in the correlation $\widehat{\rho}_N$ of the Poisson processes under the risk-neutral measure. The six grey curves in each picture correspond to (from bottom to top) $\widehat{\rho}_N = 0, 0.05, 0.1, 0.2, 0.5, 0.8$. The other parameters are $S_0 = 1$, $r = 0$, $T = 1/12$, $\sigma = 0.4$, $\gamma = 0.2$, and $\widehat{h} = 1.25$, the number of stocks in the index is 30.

To illustrate this point we compute the slopes of the implied volatility curve for the index for a variety of values for ρ_W and $\widehat{\rho}_N$. To quantify the slope of the smile one needs a metric. Several alternatives have been suggested in the literature, and we chose the measure suggested by Bollen and Whaley (2001). For the options on a certain stock or on the index they run the regression

$$\ln \sigma_i^{IV} = \beta_0 + \beta_1 \ln M_i + \varepsilon_i, \tag{6}$$

where σ_i^{IV} and M_i denote the implied volatility and the moneyness of option i on the given underlying, respectively. The parameters to be estimated are β_0 and β_1, and ε_i is an error term. Bollen and Whaley (2001) then compute the level of the implied volatility curve as

$$\text{level} = e^{\widehat{\beta}_0} \tag{7}$$

and the slope as

$$\text{slope} = e^{\widehat{\beta}_0}\, \widehat{\beta}_1\, 0.9^{\widehat{\beta}_1 - 1}, \tag{8}$$

where the hats denote estimated values. For our simulated data we use the implied volatilities for $M = 1$ and $M = 0.9$ to infer the parameters β_0 and β_1 which then completely characterize the implied volatility curves according to (7) and (8).

Figure 3 shows the values of the slope obtained for different combinations of the correlation parameters ρ_W and $\widehat{\rho}_N$. First of all one can observe that, for ρ_W held constant, the smile curve becomes steeper with increasing $\widehat{\rho}_N$, since the slope becomes more negative. Second, for a given value of $\widehat{\rho}_N$, the smile tends to flatten out for an increasing correlation of the Wiener processes. For $\widehat{\rho}_N$ near zero the slope approaches zero rather quickly when ρ_W becomes larger.

The analysis up to now has been conducted exclusively for options with a maturity of one month. However, the impact of maturity on the shape of the smile curves is certainly of interest. Figure 4 shows smile graphs for both individual and index options for option maturities of one, three, and six months ($T = 1/12, 1/4, 1/2$), Table I gives the corresponding numbers for level and slope. Both for the stock and the index the curves become flatter with increasing time to maturity, with a much more pronounced effect for the index. This can be explained along the lines of Das and Sundaram (1999) who show that smile curves in general tend to become flatter with increasing maturity, since the risk-neutral distribution of the underlying approaches the lognormal case. This in turn implies that the decrease in the slope of the index smile has to be sharper than that for the stock, which is just what we observe in Figure 4.

Indices on the major stock markets around the world differ not only with respect to computational details, but also with respect to the number of stocks included. It

distributions are certainly one potential source for differences in the smiles, but they are by no means the only one, and differences in smiles can even exist if the physical distributions are quite similar.

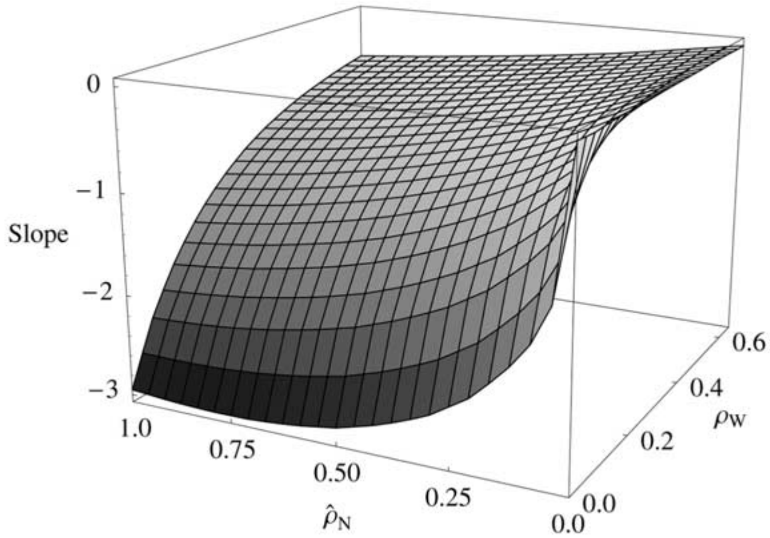

Figure 3. Jump-diffusion model: Slope of the index smile as a function of ρ_W and $\widehat{\rho}_N$. The slope of the implied volatility functions for the the index is shown as a function of the correlation ρ_W of the Wiener processes and the correlation $\widehat{\rho}_N$ of the Poisson processes under the risk-neutral measure. The other parameters are $S_0 = 1$, $r = 0$, $T = 1/12$, $\sigma = 0.4$, $\gamma = 0.2$, and $\widehat{h} = 1.25$, the number of stocks in the index is 30.

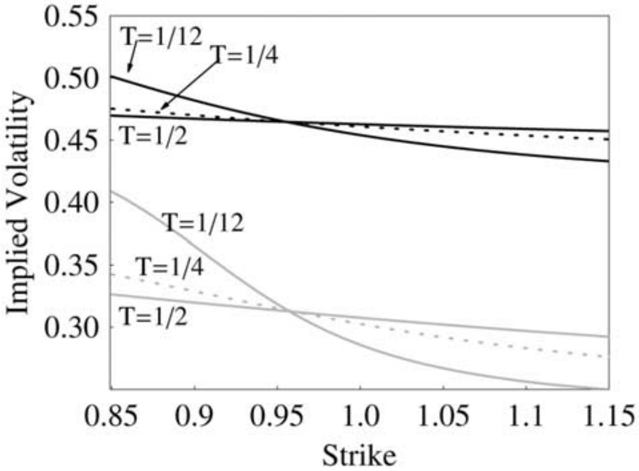

Figure 4. Jump-diffusion model: Impact of time to maturity. Implied volatility functions for a representative stock (black lines) and for the index (grey lines) are shown as a function of the strike price. The time to maturity is equal to $T = 1/12$, $T = 1/4$ (dashed line) and $T = 1/2$. The other parameters are $S_0 = 1$, $r = 0$, $\sigma = 0.4$, $\gamma = 0.2$, $\widehat{h} = 1.25$, $\rho_W = 0.3$, and $\widehat{\rho}_N = 0.8$, the number of stocks in the index is 30.

Table I. Model level and slope for index and stock

		Maturity			
		1 month	2 months	3 month	6 month
Index	Level	0.286	0.297	0.303	0.308
	Slope	−0.944	−0.453	−0.287	−0.131
	Change in slope	–	0.491	0.657	0.813
Stock	Level	0.454	0.459	0.461	0.463
	Slope	−0.302	−0.146	−0.096	−0.047
	Change in slope	–	0.156	0.206	0.255

Level and slope obtained from our model for $S_0 = 1$, $r = 0.0$, $T = 1/12$, $\sigma = 0.4$, $\gamma = 0.2$, $\widehat{h} = 1.25$, $\rho_W = 0.3$, and $\widehat{\rho}_N = 0.8$. The number of stocks in the index is 30. The change in slope is computed as the difference between the slope for the respective maturity and the slope for options with one month to maturity.

is therefore interesting to investigate the properties of our model when the number of component stocks increases. In Figure 5 we provide index smile curves for indices consisting of 5, 10, 30, and 100 stocks, respectively. The main finding here is that the curves look the steeper the more stocks are contained in the index. On the other hand the marginal effect of adding stocks to the index decreases rather quickly. Whereas the difference between five and ten stocks still seems significant, going from 30 to 100 stocks does not change the smile curves very much anymore. The additional diversification which can be achieved by adding stocks to the index portfolio is obviously not so important once a certain threshold for the number of stocks has been reached.

5. Empirical Evidence

So far we have mainly focused on the analytic properties of our model. To see whether it offers a suitable explanation of real world phenomena we will compare it to smile curves observed on EUREX, the fully electronic German exchange for equity and index options. We consider options written on the German stock market index DAX and on Deutsche Bank (DBK), which is included in this index. DBK options are among the most frequently traded equity contracts on EUREX, which makes them a suitable choice for this empirical exercise, since price distortions by a potential lack of liquidity are minimized. Options on the DAX are European, so one can directly use the Black-Scholes formula to infer implied volatilities from option prices. Equity options on EUREX are American, so that there are cases when the Black-Scholes formula cannot be applied. In order to compute implied volatilities for those DBK options with potential early exercise we used an iterative algorithm where the option price is computed using a tree-based method for the case of known cash dividends (Hull, 2002, p. 404).

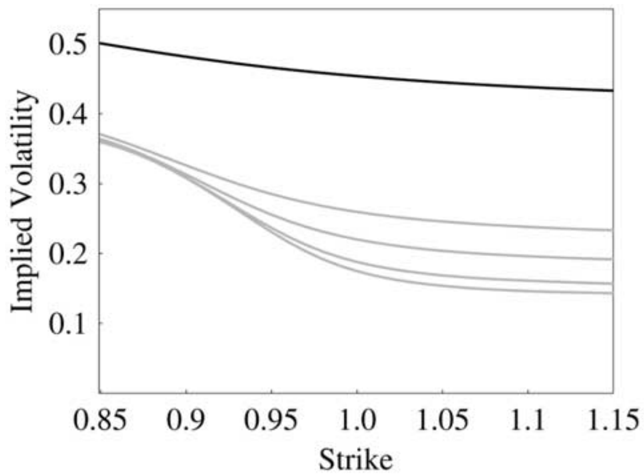

Figure 5. Jump-diffusion model: Impact of the number of stocks in the index. Implied volatility functions for a representative stock (black line) and for the index (grey lines) are shown as as a function of the strike price. The implied volatilities decrease in the number of stocks in the index which is equal to 5, 10, 30, and 100. The other parameters are $S_0 = 1$, $r = 0$, $\sigma = 0.4$, $\gamma = 0.2$, $\widehat{h} = 1.25$, $\rho_W = 0.3$, $\widehat{\rho}_N = 0.8$, and $T = 1/12$.

The graphs in Figure 6 represent the fitted smile curves for DAX and DBK options for the months of March and September 1998. To obtain the curves we estimated the regression in Equation (6) for monthly pooled observations. The individual observations (not shown in the figure) for DBK implied volatility would fluctuate randomly around the fitted curve due to a relatively low liquidity, whereas the values for DAX options are located very close to the curve. Moneyness is defined as the present value of option's strike price divided by the current value of the underlying minus the present value of any dividends paid over the life of the option. Note that the DAX is a performance index so that dividends are reinvested into the index portfolio which makes such an adjustment for dividends unnecessary. The options in our sample are then sorted into two maturity categories. The first category contains options which expire in at most 45 days, whereas options in maturity class 2 have a time to maturity between 46 and 90 days.

In general the empirical analysis shows that our model is able to explain the stylized facts about equity and index option smiles. Note that the smiles for the index are in all cases below the DBK curve, which is obviously due to diversification effects as discussed above. Furthermore, one can clearly see that the curve for the index is much steeper than the one for the stock. As predicted by our model the smile for both underlyings are flatter for options with longer time to maturity. This effect is especially pronounced for the March observations, where for DBK the slope decreases in absolute terms from -0.37 to -0.15, whereas for the DAX it decreases from -1.06 to -0.49. In September the slope for DBK also decreases

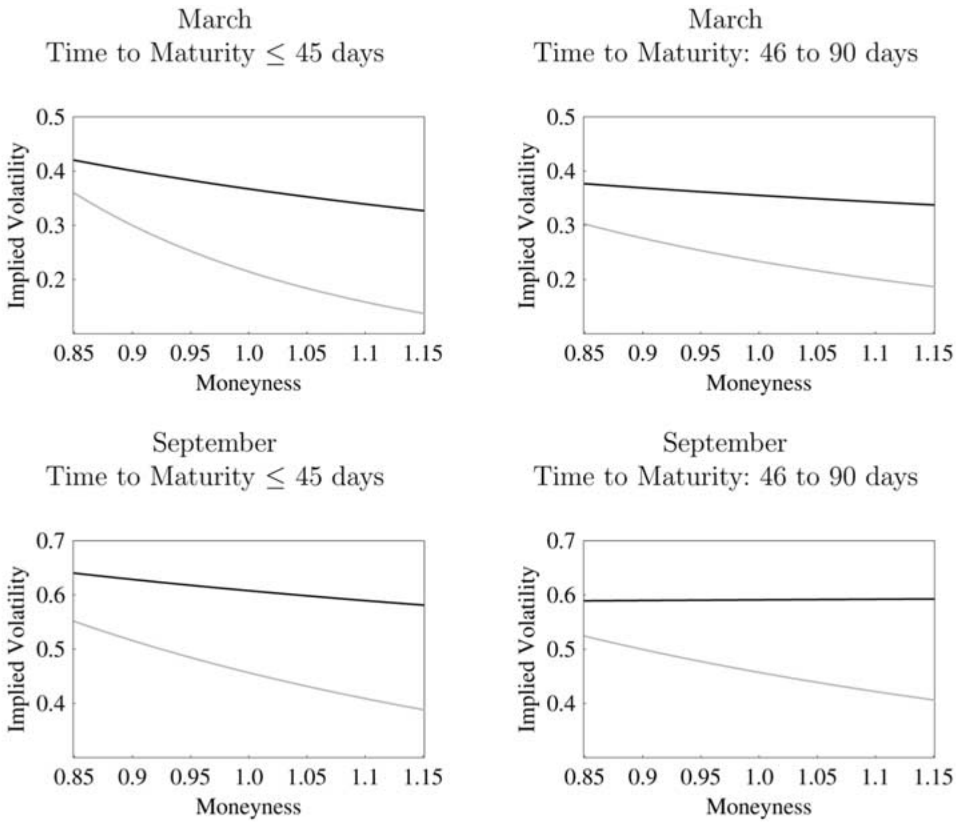

Figure 6. Smile for DAX and DBK. Implied volatility functions for options on the DAX (grey line) and on Deutsche Bank (black line) are shown as a funtion of option moneyness. The curves represent the fitted values of a regression of the logarithm of implied volatilities on the logarithm of moneyness for March and September 1998.

significantly from -0.22 to 0.01, while there is not such a sharp decline in slope for DAX options (from -0.66 to -0.47).

In addition to this more qualitative analysis it is also important to check if the model is able to generate numerical output for the level and the slope of the smile which is comparable to what is observed empirically. We sort the options into two maturity categories where the first category contains options which expire in at most 45 days and the second category contains options with time to maturity between 46 and 90 days. Every week for the year of 1998 we estimate a regression of the natural log of implied volatility on the natural log of moneyness. We then compute level and slope for the respective week according to Equations (7) and (8) and average these values across the 52 weeks of the year. Averaging the estimates for level and slope across the complete year of 1998, we obtain the values for DAX and DBK shown in Table II. Based on the parameters $S_0 = 1$, $r = 0.0$, $T = 1/12$, $\sigma = 0.4$, $\gamma = 0.2$, $\widehat{h} = 1.25$, $\rho_W = 0.3$, and $\widehat{\rho}_N = 0.8$ our model generates the

Table II. Average level and slope for DAX and DBK (1998)

		Maturity category	
		1	2
DAX	Level	0.304	0.312
	Slope	−0.774	−0.499
DBK	Level	0.443	0.430
	Slope	−0.259	0.019

Level and slope are measured as in Bollen and Whaley (2001).

numbers for level and slope shown in Table I. The ratio of index to equity slopes produced by our model is similar to the empirical data, with a factor of roughly 3 for the shorter maturity category. For the longer maturity the similarities are not as pronounced anymore, since the empirical smile for the stock is nearly flat, whereas in our model the slope only decreases by half, from -0.3 for one month to -0.15 for two months. However, despite its simplicity (i.e., with deterministic downward jumps and without stochastic volatility) our model is able to explain a significant part of the empirical phenomena.

6. Conclusion

The stochastic behaviour of an index is completely determined by the joint behaviour of the stocks, i.e., by the individual stochastic processes of the stocks and by their dependence structure. This paper explicitly takes this aggregation restriction into account when comparing the implied volatility smiles of options on individual stocks and on the index.

We show that the dependence structure among the stocks in the index is able to explain the differential pricing of stock and index options. Assuming for simplicity that each jump is a downward jump, we show that the skew is the steeper the greater the correlation of the jumps and the lower the correlation of the Wiener processes.

In our analysis we highlight the fundamental differences between diffusions and jumps. For diffusions, variances and correlations are identical under the physical and under the risk-neutral measure. On the contrary, for jumps these moments can be different under the physical and under the risk-neutral measure. We are thus free to choose the dependence structure under the risk-neutral measure, the only restriction being the equivalence to the physical measure. Therefore, the inclusion of jump components is critical in our model.

In summary, our analysis demonstrates that the empirical phenomenon of different smile slopes for equity and index options can be well explained within the class of standard derivatives models without resorting to more sophisticated approaches.

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