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The Impact of Delivery Risk on Optimal Production and Futures Hedging ⋆

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Abstract. Multiple delivery specifications exist on nearly all commodity futures contracts. Sellers are typically allowed to choose among several grades of the underlying commodity. On the delivery day, the futures price converges to the spot price of the cheapest-to-deliver grade rather than to that of the par-delivery grade of the commodity, thereby imposing an additional delivery risk on hedgers. This paper derives the optimal production and futures hedging strategy for a risk-averse competitive firm facing delivery risk. We show that the option value of the multiple delivery specification induces the firm to produce more with than without the delivery risk if the firm gauges this value higher than the market. We further show that if the delivery risk is additively related to the commodity price risk, the firm optimally under-hedges its risk exposure. On the other hand, if the delivery risk is multiplicatively related to the commodity price risk, the firm may optimally choose an under- or over-hedge which we illustrate using a numerical example.

Key words: delivery risk, futures, production, risk management

JEL classification codes: G11, D21, D81

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1. Introduction

Many commodity futures contracts possess options as to what, where, when, and how much of the underlying commodity the seller of the futures contract can deliver. These multiple delivery specifications are known as quality options, location options, timing options, and quantity options. They are embedded in futures contracts in order to constrain the severity of market manipulation such as squeezes and corners.[1] This paper concentrates on the optimal production and futures hedging decisions of a commodity producer in the presence of a quality option.[2] If the seller of the futures contract exercises the quality option by deviating from the par-delivery grade, there will usually be a correction of the futures price he/she receives. Upward corrections are called premiums while downward corrections are called discounts. However, the realized price difference between the par-delivery grade and a non-par-delivery grade can deviate significantly from the premium or discount pre-determined for the latter. Therefore, the seller can minimize his/her cost by delivering the cheapest-to-deliver grade.[3] It follows that the (corrected) futures price on the delivery day converges to the spot price of the cheapest-to-deliver grade rather than that of the par-delivery grade.

Since it is uncertain ex ante which grade would be the cheapest to deliver, hedgers are exposed to an additional delivery risk vis-à-vis the underlying price risk. This delivery risk is not hedgeable, thereby impairing the effectiveness of using futures for hedging purposes (Garbade and Silber, 1983a).[4] Indeed, Kamara and Siegel (1987) show that the presence of delivery risk affects optimal futures positions when physical delivery is significant.[5] In this case, the regression approach of Benninga et al. (1983, 1984) fails to yield the minimum-variance hedge ratio.[6]

[1] Pirrong (1993) derives necessary and sufficient conditions for manipulating futures prices at contract expiry. Pirrong (2001) compares the probability of market manipulation for delivery-settled with that for cash-settled futures contracts.

[2] If a commodity at different locations is interpreted as different grades of the commodity, there is no difference between location options and quality options. For location options, see Garbade and Silber (1983a) and Pirrong et al. (1994). Financial futures contracts with delivery options are analyzed by Gay and Manaster (1986, 1991), Kane and Markus (1986), and Hemler (1990).

[3] The cheapest-to-deliver grade is the grade with the minimum delivery-adjusted spot price among all deliverable grades.

[4] Lien and Wong (2002) show that the presence of delivery risk calls for the use of options on futures for hedging purposes.

[5] Peck and Williams (1991, 1992) provide evidence for the significance of physical delivery. They document that deliveries on the Chicago Board of Trade (CBOT) wheat, corn, and soybean futures markets and the New York Commodity Exchange copper futures market are on average around 15% of the peak open interest in each delivery month in the 1970s and 1980s. On the delivery day, this figure is approximately 50%. Lien (1988, 1991) analyzes the welfare effects of introducing a delivery option into an existing futures contract. Garbade and Silber (1983b) compare the welfare effects of physical delivery and cash settlement.

[6] Empirically, Viswanath and Chatterjee (1992) document that the deviation due to mistakenly using the regression approach was not economically significant for the CBOT wheat contract between 1970 and 1981.

It also casts some doubts on the well-known separation theorem (Danthine, 1978; Holthausen, 1979; and Feder et al., 1980) that a firm's optimal production decision depends neither on the preferences of the firm nor on the underlying uncertainty when futures hedging is allowed.

The purpose of this paper is to examine the optimal production and futures hedging decisions of a risk-averse competitive firm encountering delivery risk. We show that the value of the delivery option enhances the firm's incentives to produce if the firm considers the delivery option to be more valuable than the market does.[7] The presence of the non-tradable delivery risk always renders the optimal output a function of the firm's preferences and of the underlying uncertainty. We further show that the firm's optimal futures position is an under-hedge if the delivery risk is additively related to the commodity price risk. This follows from the fact that reducing the commodity price risk via futures trading gives rise to an exposure to the delivery risk. However, if the delivery risk is multiplicatively related to the commodity price risk, the firm's optimal futures position may be an under- or over-hedge, depending on the firm's preferences and on the underlying uncertainty. This ambiguity is due to two opposing effects. First, as in the additive case, futures trading creates exposure to the delivery risk which induces the firm to use less futures hedging. Second, unlike in the additive case, the delivery risk in the multiplicative case is negatively correlated with the commodity price risk. This generates an additional incentive to hedge against the commodity price risk which induces the firm to sell more futures contracts. A priori, it is unclear which effect dominates. In a numerical example, we illustrate either possibility for the multiplicative case.

The paper by Kamara and Siegel (1987) which also examines the optimal futures hedging decisions in the presence of delivery risk comes closest to ours. Nevertheless, we extend their work in three important aspects. First, Kamara and Siegel (1987) focus on the case of two deliverable grades whose spot prices are jointly normally distributed and have equal variance. We generalize their analysis to an arbitrary number of deliverable grades without making any distributional assumptions. Second, unlike Kamara and Siegel (1987) who rely on the mean-variance framework, we consider the problem within the expected utility paradigm. Last but not the least, Kamara and Siegel (1987) take the spot position of a hedger as given while we relate this position to the hedger's (i.e., the firm's) production decision.

The present paper is also related to that of Benninga and Oosterhof (2004) who investigate the behavior of a risk-averse competitive firm availing itself of

[7] Since delivery options can be interpreted as options to exchange one grade for another, one can apply valuation models for exchange options such as that of Margrabe (1978) which is employed by Gay and Manaster (1984) for the valuation of the quality option in the case of two deliverable grades. Boyle (1989) presents a theoretical pricing model for an arbitrary number of deliverable grades. See Chance and Hemler (1993) for an overview on the pricing issue. Even when delivery uncertainty is relatively small, Kamara (1990) demonstrates that the delivery structure still has crucial implications for equilibrium pricing and, hence, for market efficiency tests.

infinitely divisible forward and put option contracts for hedging purposes. Inter alia, they show that the firm optimally decreases its output whenever it uses put options. Their results are not at odds with ours since they are based on different assumptions. Although one can argue that a futures contract with a multiple delivery specification can be regarded as a forward contract with a put option attached (in a static two-date setting, forwards and futures are equivalent), there is no way to trade the forward contract and the put option separately as they are bundled together to form the futures contract. Unlike in the present paper, forward and put option contracts can be traded separately in the model of Benninga and Oosterhof (2004) such that different implications for the firm's production decision arise.

The rest of this paper is organized as follows. Section 2 delineates a two-date model of a risk-averse competitive firm facing both commodity price risk and delivery risk. Section 3 characterizes the firm's optimal production decision in general and the impact of the delivery risk on the optimal output in particular. Section 4 derives the firm's optimal futures hedging decision for both the additive and the multiplicative specification of the delivery risk and provides a numerical example for the multiplicative specification. The final section concludes.

2. The Model

Consider a one-period model with two dates, indexed by $t = 0$ and 1. There is a commodity which has several grades, labelled as grades $1, 2, \ldots, n$, where $n \geq 2$. At $t = 0$, a risk-averse firm operating in a competitive environment produces grade 1 of the commodity according to a cost function, $C(Q)$, where Q is the firm's output level to be sold at $t = 1$ and $C(Q)$ is compounded to $t = 1$. The cost function is strictly increasing and convex such that $C(0) \geq 0$, $C' > 0$, and $C'' > 0$.

When making its production decision at $t = 0$, the firm knows neither the spot price of grade 1 of the commodity at $t = 1$, denoted by $\tilde{P}_1$, nor the spot prices of the other grades, denoted by $\tilde{P}_2, \tilde{P}_3, \ldots, \tilde{P}_n$, where all $\tilde{P}_i$s are positive random variables.[8] Let $f(P_1, P_2, \ldots, P_n)$ be the firm's subjective joint density function of $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n$. Since $\tilde{P}_1$ is stochastic, the firm inevitably faces commodity price risk.

There is a competitive futures market where infinitely divisible futures contracts can be traded at $t = 0$. All contracts mature at $t = 1$ at which time sellers of the futures contracts have the right to choose among the deliverable grades $1, 2, \ldots, n$ of the commodity the particular grade they want to deliver for the fulfillment of their obligations. Since sellers exercise the delivery option by choosing the cheapest-to-deliver grade, the futures price at $t = 1$ is equal to the minimum of $P_1, P_2, \ldots, P_n$.[9] The futures price at $t = 0$, denoted by F, is determined according to the risk-

8 Throughout the paper, random variables have a tilde ($\tilde{}$) while their realizations do not.

9 For simplicity, it is assumed that there are no delivery adjustments in the form of premiums and discounts. Otherwise, the model had to be based on the delivery-adjusted prices of grades $2, 3, \ldots, n$.

neutral joint pricing density of $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n$, held by the market (or the marginal consumer). This pricing density is denoted by $q(P_1, P_2, \ldots, P_n)$. Succinctly, the futures price is given by

$$F = \mathrm{E}_q\Big[\min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\Big], \tag{1}$$

where $\mathrm{E}_q[\cdot]$ is the expectation operator with respect to q.

Let H be the number of futures contracts sold (purchased if H is negative) by the firm at $t = 0$. Whenever $H \neq 0$, the firm is exposed to an additional delivery risk vis-à-vis the commodity price risk. The firm's random profit at $t = 1$ is given by[10]

$$\tilde{\Pi} = \tilde{P}_1 Q - C(Q) + \Big[F - \min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\Big] H, \tag{2}$$

which is the sum of the operating cash flow and the gain/loss due to the futures position.

The firm has a von Neumann-Morgenstern utility function, $U(\Pi)$, defined over its profit at $t = 1$, with $U' > 0$ and $U'' < 0$, indicating risk aversion. The firm's decision problem at $t = 0$ is to choose an output level, Q, and a futures position, H, so as to maximize the expected utility of its profit at $t = 1$:

$$\max_{Q,H} \; \mathrm{E}_f\Big[U(\tilde{\Pi})\Big] \tag{3}$$

subject to equations (1) and (2), where $\mathrm{E}_f[\cdot]$ is the expectation operator with respect to f.

The first-order conditions for program (3) are given by

$$\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\Big(\tilde{P}_1 - C'(Q^*)\Big)\Big] = 0, \tag{4}$$

$$\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\Big(F - \min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\Big)\Big] = 0, \tag{5}$$

where an asterisk (*) indicates an optimal level. Given risk aversion and the strict convexity of the cost function, the second-order conditions for program (3) are satisfied and the maximum solution, (Q^*, H^*), is unique.

3. Optimal Production Decisions

In this section, the firm's optimal production decision and its reaction to the introduction of delivery risk is examined. As a benchmark, we consider first the

[10] Since transaction costs in the delivery process are neglected, there is no difference between cash settlement and physical delivery.

hypothetical case wherein only grade 1 of the commodity is deliverable. In this case, program (3) reduces to

$$\max_{Q_a, H_a} \mathrm{E}_f\Big[U\Big(\tilde{P}_1 Q_a - C(Q_a) + (F_a - \tilde{P}_1)H_a\Big)\Big] \tag{6}$$

subject to $F_a = \mathrm{E}_q[\tilde{P}_1]$, where subscript a indicates the absence of delivery options. The well-known separation theorem applies to the solution of program (6) in that the firm's optimal output level, Q_a^*, solves $C'(Q_a^*) = F_a$. The intuition is that the firm can always sell the marginal unit of its output via the futures market at the known price, F_a. Hence, the firm optimally produces at the point where the marginal cost of production equals F_a. This is the case even when $F_a \neq \mathrm{E}_f[\tilde{P}_1]$, i.e., when the firm and the market disagree on the valuation of the futures price at $t = 0$.[11]

Now, we return to the firm's decision problem in the presence of delivery risk. Equation (5) can be written as

$$\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\Big(F - \tilde{P}_1 + \max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big)\Big] = 0\,. \tag{7}$$

$\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)$ is the cash flow from exercising the delivery option at $t = 1$. Substituting equation (7) into equation (4) and rearranging terms yields

$$\begin{aligned} C'(Q^*) &= F + \frac{\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\ \max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big]}{\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\Big]} \\ &= F_a - \mathrm{E}_q\Big[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big] \\ &\quad + \frac{\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\ \max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big]}{\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\Big]}\,, \end{aligned} \tag{8}$$

where the second equality follows from equation (1) and $F_a = \mathrm{E}_q[\tilde{P}_1]$.

Consider the following function:

$$g(P_1, P_2, \ldots, P_n) = \frac{U'(\Pi^*)}{\mathrm{E}_f\Big[U'(\tilde{\Pi}^*)\Big]}\, f(P_1, P_2, \ldots, P_n)\,. \tag{9}$$

[11] When $F_a \neq \mathrm{E}_f[\tilde{P}_1]$, the full-hedging theorem does not hold, i.e., $H_a^* \neq Q_a^*$. As shown by Holthausen (1979), the existence of a futures market ensures that the separation theorem holds if there is no delivery risk. He has also shown that the full-hedging theorem requires the futures market to be unbiased from the perspective of the firm, given the current assumptions. In the slightly different setting of Benninga and Oosterhof (2004), the firm fully hedges even if it perceives the futures market as biased as long as the market regards the futures market as unbiased.

It follows from equation (9) that $g \geq 0$ since $U' > 0$ and $f \geq 0$. Equation (9) also implies that the integral of g over all values of its arguments is unity. Thus, g can be treated as a joint density function of $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n$. Since it stems from the firm's preferences and its subjective joint density function, g can be interpreted as the firm's joint pricing density. Using equation (9), we can write equation (8) as

$$C'(Q^*) = F_a - \mathrm{E}_q\Big[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big] + \mathrm{E}_g\Big[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big], \tag{10}$$

where $\mathrm{E}_g[\cdot]$ is the expectation operator with respect to g. Since $C'(Q_a^*) = F_a$, the following proposition is immediately invoked by using equation (10) and the strict convexity of the cost function.

PROPOSITION 1. If the risk-averse competitive firm attaches a higher risk-adjusted value to the delivery option than the market such that $\mathrm{E}_g[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)] > \mathrm{E}_q[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)]$, then introducing a delivery option to the futures contracts induces the firm to raise its optimal output, i.e., $Q^* > Q_a^*$. If $\mathrm{E}_g[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)] < [=]\, \mathrm{E}_q[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)]$, then $Q^* < [=]\, Q_a^*$.

To see the intuition behind Proposition 1, we write equation (2) as

$$\tilde{\Pi} = \Big[F + \max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)\Big] Q - C(Q) + \Big[F - \min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\Big] (H - Q)\,. \tag{11}$$

Consider the firm's marginal revenue under the assumption that the firm always hedges its marginal unit of output in the futures market such that $dH = dQ$.[12] Then, it is evident from equation (11) that the marginal unit of output generates a stochastic revenue of $F + \max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)$.[13] Hence, at $t = 0$ the firm optimally chooses the level of output such that the marginal cost of production equals the futures price, F, plus the risk-adjusted value of the delivery option, based on the firm's joint pricing density, g.

Suppose for a moment that the firm ignores the delivery option such that it always delivers grade 1. In this case, the selling price of the marginal unit of output at $t = 1$ is fixed at F via futures hedging. It then follows from $F = F_a - \mathrm{E}_q[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)] < F_a$ and the strict convexity of the cost function that $Q^* < Q_a^*$. Taken in isolation, ignoring the delivery

[12] This assumption does not result in a loss of generality since the firm can always choose to sell the marginal unit of output in the futures market.

[13] Marginal output also affects the final term on the right-hand side of equation (11) but leaves marginal revenue unaffected since the marginal unit of output is hedged, $dH = dQ$.

option leads to a negative effect of delivery risk on the firm's output. This effect is larger the higher the value attached to the delivery option by the market, $\mathrm{E}_q[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)]$.

Now, suppose that the firm takes the delivery option into account: If $P_1 > \min(P_2, \ldots, P_n)$, the firm optimally sells its entire output (which is in grade 1) at the spot price, P_1, and uses part of the proceeds to purchase the cheapest-to-deliver grade at the spot price, $\min(P_2, \ldots, P_n)$, for the fulfillment of the futures contracts. Doing so generates an additional marginal revenue of $\max(0, P_1 - P_2, \ldots, P_1 - P_n)$ at $t = 1$. This additional marginal revenue, albeit stochastic at $t = 0$, is never negative and must be strictly positive for a non-degenerate set of states, lest the multiple delivery specification becomes trivial. The value of this additional marginal revenue at $t = 0$ is the risk-adjusted value of the delivery option from the firm's point of view, given by $\mathrm{E}_g[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)] > 0$. Taken in isolation, this results in a positive effect of delivery risk on the firm's output.

Proposition 1 claims that the net effect is positive, $Q^* > Q_a^*$, if the latter effect is larger than the former effect such that $\mathrm{E}_g[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)] > \mathrm{E}_q[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)]$, and vice versa.[14]

It is worth mentioning that even when $\mathrm{E}_g[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)] = \mathrm{E}_q[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)]$ so that $Q^* = Q_a^*$, the separation theorem does not hold. This follows from equation (9) since any change in either U' or f affects g and thereby $\mathrm{E}_g[\max(0, \tilde{P}_1 - \tilde{P}_2, \ldots, \tilde{P}_1 - \tilde{P}_n)]$. In other words, the separation theorem never holds when there is delivery risk. This is not surprising since there is no way to make the marginal revenue non-stochastic in the presence of non-tradable delivery risk.[15]

4. Optimal Hedging Decisions

In order to focus on the hedging role of futures contracts, we assume in the sequel that the joint pricing density function held by the firm, f, and the joint pricing density function held by the market, q, are such that $F = \mathrm{E}_f[\min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)] = \mathrm{E}_q[\min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)]$.[16] In other words, the firm considers the futures market as being unbiased.[17] Consequently, any speculative positions can be precluded and hedging is the only motive for using futures contracts. In the rest of the paper, all

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[14] Suppose that the market and the firm attach the same (objective) probabilities to all possible states (as assumed in Benninga and Oosterhof, 2004). Then, the firm attaches a higher [lower] value to the delivery option than the market does if the firm's risk aversion is smaller [higher] than that of the market. Hence, the firm raises [reduces] output when a delivery option is introduced.

[15] It is usually the case that the separation theorem fails to hold when there is non-hedgeable risk. See, e.g., Broll et al. (1995), Adam-Müller (1997), and Wong (2003).

[16] In the previous section, it was not necessary to restrict the relation between the market's and the firm's pricing density function in order to derive Proposition 1.

[17] See Benninga and Oosterhof (2004) for the necessary and sufficient conditions that ensure the unbiasedness of forward and put option prices.

expectations, covariances etc. are taken with respect to f such that the subindex is suppressed for notational simplicity.

In order to derive clear-cut statements on the firm's optimal hedging position, it is necessary to impose some structure on the joint probability distribution of $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n$. For tractability, we shall consider the following two alternative specifications.

ASSUMPTION A. The spot price of grade i ($i = 2, 3, \ldots, n$) of the commodity at $t = 1$ is related to that of grade 1 in the following additive manner:

$$\tilde{P}_i = \tilde{P}_1 + \tilde{\varepsilon}_i \,, \tag{12}$$

where $\tilde{\varepsilon}_i$ is a zero-mean random variable independent of $\tilde{P}_1$.[18]

ASSUMPTION M. The spot price of grade i ($i = 2, 3, \ldots, n$) of the commodity at $t = 1$ is related to that of grade 1 in the following multiplicative manner:

$$\tilde{P}_i = (1 + \tilde{\gamma}_i)\tilde{P}_1 \,, \tag{13}$$

where $\tilde{\gamma}_i$ is a zero-mean random variable independent of $\tilde{P}_1$.[19]

As will be shown shortly, the firm's optimal futures position depends on which of these two assumptions holds. Whether Assumption A or Assumption M is a better description for the behavior of the random spot prices of different deliverable grades of a commodity is certainly an empirical question. From an economic point of view, however, one can argue that the nature of price changes in the economy should provide a clue to the choice between these two assumptions. For instance, suppose that the expected real production costs for grades 1 and i ($i = 2, \ldots, n$) differ by a constant amount.[20] If the general price level in the economy is rather stable, the price difference, $\tilde{P}_i - \tilde{P}_1$, should reflect the monetary equivalent of the difference in real production costs plus any deviation from the expected level. Furthermore, this price difference is likely to capture any erratic changes in the prices of grades 1 and i that are not related to the cost difference. In this regard, Assumption A seems to dominate Assumption M. If, in contrast, the general price level in the economy is highly volatile, the monetary equivalent of the (constant) real production costs as such varies positively with the general price level. The same argument applies to the prices of grades 1 and i and, thus, $\tilde{P}_i - \tilde{P}_1$ varies positively with the general price level as well. In this case, Assumption M dominates Assumption A.

[18] The distribution of $\tilde{\varepsilon}_i$ is assumed to be such that $\tilde{P}_i$ has positive realizations only. It is not necessary to assume that $\tilde{\varepsilon}_i$s are independent of each other.

[19] To ensure positive realizations of $\tilde{P}_i$ under Assumption M, it is assumed that $\gamma_i > -1$ in all states. The $\tilde{\gamma}_i$s do not have to be independent of each other.

[20] Taking an agricultural commodity such as soybean oil as an example, farming conditions or processing costs at oil mills may be more favorable for grade 1 relative to grade i.

For the ease of exposition, we reformulate the firm's decision problem by fixing its output level at Q^*. Let $\mathrm{E}U = \mathrm{E}[U(\tilde{\Pi})]$, where $\tilde{\Pi}$ is defined in equation (2) with $Q = Q^*$. Partially differentiating $\mathrm{E}U$ with respect to H yields

$$\begin{aligned}\frac{\partial \mathrm{E}U}{\partial H} &= \mathrm{E}\Big[U'(\tilde{\Pi})\Big(F - \min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\Big)\Big] \\ &= -\operatorname{cov}\Big[U'(\tilde{\Pi}), \min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)\Big],\end{aligned} \tag{14}$$

where the second equality follows from the fact that $F = \mathrm{E}[\min(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)]$, and $\operatorname{cov}[\cdot, \cdot]$ is the covariance operator. When H is optimally chosen, $H = H^*$, the right-hand side of equation (4) equals zero.

4.1. OPTIMAL FUTURES HEDGING UNDER ASSUMPTION A

Evaluating the right-hand side of equation (4) at $H = 0$ and using equation (12) yields

$$\begin{aligned}\left.\frac{\partial \mathrm{E}U}{\partial H}\right|_{H=0} &= -\operatorname{cov}\Big[U'\Big(\tilde{P}_1 Q^* - C(Q^*)\Big), \tilde{P}_1 + \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)\Big] \\ &= -\operatorname{cov}\Big[U'\Big(\tilde{P}_1 Q^* - C(Q^*)\Big), \tilde{P}_1\Big],\end{aligned} \tag{15}$$

where the second equality follows from the fact that all $\tilde{\varepsilon}_i$s are independent of $\tilde{P}_1$.[21] It follows from $U'' < 0$ that the right-hand side of equation (4.1) is positive. The strict concavity of $\mathrm{E}U$ then implies that $H^* > 0$.

Now, evaluating the right-hand side of equation (4) at $H = Q^*$ and using equation (12) again results in

$$\begin{aligned}\left.\frac{\partial \mathrm{E}U}{\partial H}\right|_{H=Q^*} &= -\operatorname{cov}\Big[U'\Big(FQ^* - \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)Q^* - C(Q^*)\Big), \\ &\qquad \tilde{P}_1 + \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)\Big] \\ &= -\operatorname{cov}\Big[U'\Big(FQ^* - \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)Q^* - C(Q^*)\Big), \\ &\qquad \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)\Big],\end{aligned} \tag{16}$$

[21] Two random variables, $\tilde{X}$ and $\tilde{Y}$, are independent if, and only if, $\operatorname{cov}[F(\tilde{X}), G(\tilde{Y})] = 0$ for all functions, $F(\cdot)$ and $G(\cdot)$. See Ingersoll (1987, p. 15), for example.

where the second equality follows from the fact that all $\tilde{\varepsilon}_i$s are independent of $\tilde{P}_1$. Since $U'' < 0$, the right-hand side of equation (16) is negative. It then follows from the strict concavity of EU that $H^* < Q^*$.

We summarize the above results in the following proposition.

PROPOSITION 2. Suppose that Assumption A holds. If the futures market is unbiased, then the firm's optimal futures position is a short under-hedge, i.e., $0 < H^* < Q^*$.

The intuition of Proposition 2 is as follows. Under Assumption A, the futures price at $t = 1$ is given by $\tilde{P}_1 + \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)$ which can be interpreted as a package of the underlying commodity price risk, $\tilde{P}_1$, and the delivery risk, $\min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)$. Using equation (12), we can rewrite equation (2) as

$$\tilde{\Pi} = FQ - C(Q) + (F - \tilde{P}_1)(H - Q) - \min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)\, H\,. \tag{17}$$

The last term on the right-hand side of equation (17) is always non-negative when $H > 0$ which unambiguously raises the firm's random profit at $t = 1$. Inspection of equation (17) also reveals that hedging against the commodity price risk using futures contracts always generates the same amount of exposure to the delivery risk. Complete elimination of the commodity price risk calls for full-hedging, i.e., $H = Q$, whereas complete elimination of the delivery risk calls for no hedging, i.e., $H = 0$. Since these two sources of risk are independent, the firm can never enter into a riskless position. Indeed, the firm's optimal futures position as stated in Proposition 2 is simply a compromise between full-hedging and no hedging. This can be interpreted as diversification between the independent risks $\tilde{P}_1$ and $\min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)$. The smaller the delivery risk relative to the commodity price risk, the more closely the optimal futures position approaches a full-hedge.

4.2. OPTIMAL FUTURES HEDGING UNDER ASSUMPTION M

Evaluating the right-hand side of equation (14) at $H = 0$ and using equation (13), we have

$$\begin{aligned}\left.\frac{\partial \mathrm{E}U}{\partial H}\right|_{H=0} &= -\operatorname{cov}\Big[U'\Big(\tilde{P}_1 Q^* - C(Q^*)\Big), \tilde{P}_1\Big(1 + \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big)\Big] \\ &= -\operatorname{cov}\Big[U'\Big(\tilde{P}_1 Q^* - C(Q^*)\Big)\Big(1 + \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big), \tilde{P}_1\Big],\end{aligned} \tag{18}$$

where the second equality follows from the fact that all $\tilde{\gamma}_i$s are independent of $\tilde{P}_1$.[22] It follows from $U'' < 0$ that the right-hand side of equation (4.2) is positive. The strict concavity of EU then implies that $H^* > 0$.

[22] For any three random variables, $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$, where $\tilde{X}$ and $\tilde{Y}$ are independent of $\tilde{Z}$, $\operatorname{cov}[\tilde{X}, \tilde{Y}\tilde{Z}] = \mathrm{E}[\tilde{X}\tilde{Y}\tilde{Z}] - \mathrm{E}[\tilde{X}]\mathrm{E}[\tilde{Y}\tilde{Z}] = \operatorname{cov}[\tilde{X}\tilde{Z}, \tilde{Y}] + \mathrm{E}[\tilde{X}\tilde{Z}]\mathrm{E}[\tilde{Y}] - \mathrm{E}[\tilde{X}]\mathrm{E}[\tilde{Y}]\mathrm{E}[\tilde{Z}] = \operatorname{cov}[\tilde{X}\tilde{Z}, \tilde{Y}]$.

Now, we evaluate the right-hand side of equation (4) at $H = Q^*$ by using equation (13) to obtain

$$\begin{aligned}\left.\frac{\partial \mathrm{E}U}{\partial H}\right|_{H=Q^*} = &-\mathrm{cov}\Big[U'\Big(FQ^* - C(Q^*) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, Q^*\Big), \tilde{P}_1\Big] \\ &-\mathrm{cov}\Big[U'\Big(FQ^* - C(Q^*) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, Q^*\Big), \\ &\tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big].\end{aligned} \tag{19}$$

To sign the first covariance on the right-hand side of equation (19), we use the law of iterated expectations to obtain

$$\begin{aligned}&\frac{\partial\, \mathrm{E}\Big[U'\Big(FQ^* - C(Q^*) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, Q^*\Big)\Big|\, P_1\Big]}{\partial P_1} \\ &= -\,Q^*\, \mathrm{E}\Big[U''\Big(FQ^* - C(Q^*) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, Q^*\Big) \\ &\quad \times \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big|\, P_1\Big],\end{aligned} \tag{20}$$

which is negative since $U'' < 0$ and the fact that $\min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)$ is non-positive in all states and negative in at least one state. Thus, the first covariance on the right-hand side of equation (19) is negative. For the same reason, the second covariance on the right-hand side of equation (19) is positive. Proposition 3 follows immediately from the strict concavity of EU.

PROPOSITION 3. Suppose that Assumption M holds. If the futures market is unbiased, then the firm's optimal futures position is a short position, i.e., $H^* > 0$. Furthermore, the firm optimally opts for an under-hedge [over-hedge], i.e., $H^* <$ [$>$] Q^*, if, and only if,

$$\begin{aligned}&\mathrm{cov}\Big[U'\Big(FQ^* - C(Q^*) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, Q^*\Big),\ \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big] \\ &\quad > [<]\ -\mathrm{cov}\Big[U'\Big(FQ^* - C(Q^*) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, Q^*\Big), \tilde{P}_1\Big].\end{aligned} \tag{21}$$

The intuition of Proposition 3 is as follows. Under Assumption M, the futures price at $t = 1$ is given by $\tilde{P}_1 + \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)$ which can again be interpreted as a package of the underlying commodity price risk, $\tilde{P}_1$, and the delivery risk, $\tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)$. Using equation (13), we can write equation (2) as

$$\tilde{\Pi} = FH - C(Q) - \tilde{P}_1(H - Q) - \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\, H\,. \tag{22}$$

As in the additive case, the last term on the right-hand side of equation (22) is always non-negative when $H > 0$ which raises profit at $t = 1$. Whether the optimal futures position is an under- or over-hedge hinges on two opposing effects. As in the additive case, shown in equation (17), inspection of equation (22) reveals that hedging against the commodity price risk using futures contracts creates an exposure to the delivery risk, thereby inducing the firm to use less futures hedging. This negative effect is captured by the first covariance in (21) which measures the co-variability of the firm's marginal utility and the delivery risk, $\tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)$.

However, unlike in the additive case, the amount of delivery risk in this multiplicative case is no longer independent of the commodity price risk but negatively correlated with it since[23]

$$\mathrm{cov}\Big[\tilde{P}_1,\ \tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big] = \mathrm{E}\Big[\min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)\Big]\mathrm{var}[\tilde{P}_1] < 0\,, \quad (23)$$

where var[·] is the variance operator. In other words, if the realization of the commodity price, $\tilde{P}_1$, is low, the realization of the delivery risk variable, $\tilde{P}_1 \min(0, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n)$, is expected to be high. The firm can make use of this negative correlation to reduce the variability of its profit by choosing the futures position in such a way that the weights given to each risk have the same sign. As follows from (22), the firm would therefore have to choose a positive weight for the commodity price risk, $(H^* - Q^*) > 0$, since the weight for the delivery risk, H^*, is always positive as claimed in the first part of Proposition 3. Hence, there is an incentive to raise the futures position to a level above Q^*. This positive effect is captured by the second covariance in (21) which represents the co-variability of marginal utility and the commodity price risk, $\tilde{P}_1$. A priori, it is unclear which effect dominates without knowing the specific information about U and f so as to sign inequality (21).

4.3. A NUMERICAL EXAMPLE

To show that both under-hedging and over-hedging can be optimal under Assumption M, we consider the following numerical example. There is only one alternative grade, say grade x, of the commodity whose spot price at $t = 1$ is given by $\tilde{P}_x = \tilde{P}_1(1 + \tilde{\gamma})$. The marginal probability density function of $\tilde{P}_1$ has a three-point support: $P_1 = 50$ with probability 40%, $P_1 = 50 - \delta$ with probability 30%, and $P_1 = 50 + \delta$ with probability 30%. Thus, $\tilde{P}_1$ is symmetrically distributed with $\mathrm{E}[\tilde{P}_1] = 50$ and $\mathrm{var}(\tilde{P}_1) = 0.6\,\delta^2$. The marginal probability density function of $\tilde{\gamma}$ has a two-point support: $\gamma = -0.2$ and $\gamma = 0.2$ with equal probability so that $\mathrm{E}[\tilde{\gamma}] = 0$, $\mathrm{E}[\min(0, \tilde{\gamma})] = -0.1$, and $\mathrm{var}(\tilde{\gamma}) = 0.04$. There are six possible states in the joint probability density function of $\tilde{P}_1$ and $\tilde{\gamma}$. Due to the independence of

[23] For any two independent random variables, $\tilde{X}$ and $\tilde{Y}$, $\mathrm{cov}[\tilde{X}, \tilde{X}F(\tilde{Y})] = \mathrm{cov}[\tilde{X}, \mathrm{E}[\tilde{X}F(\tilde{Y}) \mid \tilde{X}]] = \mathrm{E}[F(\tilde{Y})]\mathrm{var}[\tilde{X}]$ for all functions $F(\cdot)$.

Table I. The joint probability distribution

State	Probability	P_1	γ	P_x
1	15%	$50+\delta$	$+0.2$	$60+1.2\delta$
2	15%	$50+\delta$	-0.2	$40+0.8\delta$
3	20%	50	$+0.2$	60
4	20%	50	-0.2	40
5	15%	$50-\delta$	$+0.2$	$60-1.2\delta$
6	15%	$50-\delta$	-0.2	$40-0.8\delta$

$\tilde{P}_1$ and $\tilde{\gamma}$, four of the six states occur with probability 15% each, and the remaining two occur with probability 20% each. In all states, P_x is either 20% higher or 20% lower than P_1. Hence, $\tilde{P}_x$ is not symmetrically distributed. These assumptions are summarized in Table I. Furthermore, $F = \mathrm{E}[\tilde{P}_1]\,(1 + \mathrm{E}[\min(0, \tilde{\gamma})]) = 45$. The cost function is $C(Q) = 0.025\,Q^2$.

Table II presents the optimal production and hedging decisions of the firm in eleven scenarios that differ in the value of δ, i.e., the spread parameter for $\tilde{P}_1$. In each scenario, the joint probability density function is characterized by the standard deviations and correlation coefficients (columns 2 to 5). The remaining columns present the firm's optimal decisions under the assumption of constant relative risk aversion (CRRA) such that wealth effects are precluded. The level of CRRA increases from CRRA = 1 (logarithmic utility) in integral steps to CRRA = 4 in the last three columns in Table II. CRRA utility functions are commonly used in the literature because they have the desirable properties of decreasing absolute risk aversion and prudence *à la* Kimball (1990, 1993). In the numerical example, relative prudence as given by $-\Pi\, U'''(\Pi)\,/\,U''(\Pi)$ increases from 2 to 5.[24] Positive prudence is a reasonable property of utility functions since it leads to plausible behavior in various decision problems under risk (see Gollier, 2001, Ch. 16). It also implies that the firm is particularly sensitive to low realizations of its profit (more than under quadratic utility) such that there is a precautionary motive to avoid such realizations.

The first five columns in Table II exhibit the assumptions on δ and some characteristics of the joint probability density function of $\tilde{P}_1$ and $\tilde{P}_x$. Column 1 gives the value of δ. For example, $\delta = 20$ means that P_1 is either 30, 50 or 70. Both the volatility of $\tilde{P}_1$ and the one of $\tilde{P}_x$ increase in δ as columns 2 and 3 show. Columns 4 and 5 show the correlation coefficients, $\rho(\cdot)$, between $\tilde{P}_1$ and $\tilde{P}_x$ and between $\tilde{P}_1$ and $\min(\tilde{P}_1, \tilde{P}_x)$. Both correlation coefficients are high, indicating that the deliv-

[24] More precisely, the case of CRRA = 1 is based on $U(\Pi) = \ln(\Pi)$. The other cases use the power utility function, $U(\Pi) = -\Pi^{-r}\,/\,r$, where relative risk aversion equals $1 + r$ while relative prudence amounts to $2 + r$. For example, choosing $r = 2$ leads to CRRA of 3 and constant relative prudence of 4.

Table II. Optimal decisions for different levels of the spread parameter δ

	Joint probability distribution of $\tilde{P}_1$ and $\tilde{P}_x$				Optimal decisions											
					for CRRA = 1 (log utility)			for CRRA = 2			for CRRA = 3			for CRRA = 4		
δ	std($\tilde{P}_1$)	std($\tilde{P}_x$)	$\rho(\tilde{P}_1, \tilde{P}_x)$	$\rho(\tilde{P}_1, \min(\tilde{P}_1, \tilde{P}_x))$	Q^*	$(H^* - Q^*)$	$(H^{mv} - Q^*)$	Q^*	$(H^* - Q^*)$	$(H^{mv} - Q^*)$	Q^*	$(H^* - Q^*)$	$(H^{mv} - Q^*)$	Q^*	$(H^* - Q^*)$	$(H^{mv} - Q^*)$
20	15.5	18.7	0.829	0.936	978.84	−41.76	−25.60	961.31	−44.57	−25.14	947.26	−45.61	−24.77	936.21	−45.48	−24.48
21	16.3	19.4	0.840	0.941	978.51	−33.27	−15.50	960.82	−36.86	−15.22	946.74	−38.57	−14.99	935.74	−39.02	−14.82
22	17.0	20.1	0.850	0.945	978.19	−25.82	−6.57	960.35	−30.10	−6.45	946.24	−32.40	−6.36	935.29	−33.38	−6.28
23	17.8	20.7	0.859	0.949	977.88	−19.24	1.34	959.89	−24.14	1.32	945.75	−26.98	1.30	934.86	−28.42	1.28
24	18.6	21.4	0.867	0.953	977.57	−13.40	8.39	959.44	−18.87	8.24	945.28	−22.17	8.12	934.44	−24.03	8.02
25	19.4	22.1	0.875	0.956	977.27	−8.20	14.70	959.00	−14.17	14.42	944.81	−17.90	14.21	934.02	−20.12	14.05
26	20.1	22.8	0.882	0.959	976.96	−3.55	20.35	958.55	−9.97	19.96	944.34	−14.08	19.67	933.60	−16.62	19.45
27	20.9	23.6	0.888	0.961	976.66	0.62	25.44	958.10	−6.21	24.95	943.88	−10.65	24.58	933.18	−13.48	24.30
28	21.7	24.3	0.894	0.963	976.35	4.39	30.03	957.66	−2.81	29.45	943.41	−7.55	29.01	932.76	−10.63	28.69
29	22.5	25.0	0.899	0.965	976.05	7.79	34.19	957.20	0.27	33.53	942.93	−4.74	33.03	932.34	−8.05	32.66
30	23.2	25.7	0.903	0.967	975.74	10.87	37.96	956.75	3.06	37.22	942.45	−2.18	36.67	931.90	−5.70	36.26

ery risk is rather small in this example. The coefficients increase in δ because an increase in δ causes the covariances to grow at a higher rate than the products of the standard deviations.[25]

Columns 6 and 7 exhibit the optimal production and hedging decisions of the firm for CRRA = 1 (logarithmic utility). In Column 6, the optimal output slightly decreases in the volatility of the commodity price. This indicates that futures hedging has eliminated most of the risk exposure given that the delivery risk is rather small. The optimal output as such is not too sensitive to the volatility of the commodity price.

Before analyzing the optimal futures position, H^*, it is useful to consider the futures position that minimizes the variance of the firm's profit at $t = 1$, assuming that the firm's risk exposure is fixed at Q^*:

$$\begin{aligned}\min_H \ \mathrm{var}[\tilde{\Pi}] \ = \ &(Q^* - H)^2 \,\mathrm{var}[\tilde{P}_1] - 2(Q^* - H)\, H \,\mathrm{cov}\Big[\tilde{P}_1,\ \tilde{P}_1 \ \min(0, \tilde{\gamma})\Big] \\ &+ H^2 \,\mathrm{var}\Big[\tilde{P}_1 \ \min(0, \tilde{\gamma})\Big]\,. \end{aligned} \tag{24}$$

The first-order condition for program (24) is given by

$$\begin{aligned} &(Q^* - H^{mv})\,\mathrm{var}[\tilde{P}_1]\Big\{1 + \mathrm{E}\Big[\min(0, \tilde{\gamma})\Big]\Big\} \\ &- H^{mv}\Big\{\mathrm{var}\Big[\tilde{P}_1 \ \min(0, \tilde{\gamma})\Big] + \mathrm{var}[\tilde{P}_1]\,\mathrm{E}\Big[\min(0, \tilde{\gamma})\Big]\Big\} \ = \ 0\,, \end{aligned} \tag{25}$$

where we have used the fact that $\mathrm{cov}[\tilde{P}_1,\ \tilde{P}_1 \ \min(0, \tilde{\gamma})] = \mathrm{E}[\min(0, \tilde{\gamma})]\mathrm{var}[\tilde{P}_1]$, and H^{mv} is the minimum-variance futures position. Since $1 + \gamma \geq 0$, the first term on the left-hand side of equation (25) is positive [negative] if $H^{mv} < [>]\ Q^*$. The expression inside the curly brackets of the second term of equation (25) is indeterminate since $\mathrm{var}[\tilde{P}_1 \ \min(0, \tilde{\gamma})] > 0$ but $\mathrm{var}[\tilde{P}_1]\,\mathrm{E}[\min(0, \tilde{\gamma})] < 0$. Thus, without knowing the joint probability density function of $\tilde{P}_1$ and $\tilde{\gamma}$ it is unclear whether H^{mv} is above or below Q^*.[26] This is illustrated in Column 8 in Table II where the minimum-variance futures position monotonically increases in δ. In particular, it is an under-hedge for values of δ ranging from 20 to 22, but turns into an over-hedge for higher values of δ.

Now, the optimal futures position, H^*, as given in Column 7 of Table II can be interpreted by comparing it to the minimum-variance position, H^{mv}. H^* is consistently smaller than H^{mv}; their difference increases in the variability of $\tilde{P}_1$. Suppose the firm starts from the minimum-variance futures position, H^{mv}. Since the firm's preferences exhibit prudence, it would like to reduce the variability of

[25] Due to the independence of $\tilde{P}_1$ and $\tilde{\gamma}$, we have $\mathrm{cov}(\tilde{P}_1, \tilde{P}_x) = \mathrm{var}(\tilde{P}_1)$.

[26] For the case of two deliverable grades whose prices are jointly lognormally distributed with equal variance, Kamara and Siegel (1987) derive a related result where the minimum-variance hedge ratio may be above or below one.

its profit at $t = 1$ and to protect itself from very low realizations of profit (precautionary motive). The only way to satisfy the latter aim is to sell less futures than H^{mv} for the following reason: The lowest realizations of profit occur when the delivery option expires worthless, that is when $P_x > P_1$ due to $\gamma > 0$ and grade 1 is the cheapest to deliver. In these states, the firm's profit only depends on the realization of $\tilde{P}_1$. Generating additional profit for states with high P_1 requires selling less futures contracts. Starting from the minimum-variance position and taking the precautionary motive into account, it is optimal for the prudent firm to sell less futures contracts than H^{mv}.[27] As shown in Column 7, the optimal futures position, H^*, is an under-hedge for values of δ between 20 and 26. For larger values of δ, an over-hedge is optimal.

Columns 9 to 17 report the optimal decisions of the firm for higher values of CRRA. As CRRA increases, the optimal output decreases. This can be attributed to the fact that there is always some non-hedgeable delivery risk left which, taken in isolation, tends to decrease production. Under higher risk aversion, this effect is given a higher weight.

For a given joint density function, the minimum-variance hedge ratio, H^{mv} / Q^*, is always the same. Hence, for a given spread parameter, δ, the fraction, $(H^{mv} - Q^*) / Q^*$, is the same for all levels of CRRA. However, as CRRA increases, the optimal futures position, H^*, decreases both in relation to the exposure, Q^*, as well as in relation to the minimum-variance position, H^{mv}. For CRRA = 3 and CRRA = 4, is it always optimal to opt for an under-hedge. This can be attributed to the firm's prudence which increases in parallel with the coefficient of relative risk aversion.

To sum up, a clear-cut statement about the optimal futures position relative to the firm's optimal output cannot be derived because the minimum-variance futures position under Assumption M can be an over-hedge or an under-hedge, depending on the marginal density functions of $\tilde{P}_1$ and $\tilde{\gamma}$. Preferences other than quadratic utility may lead to higher or smaller futures positions such that the difference, $H^* - Q^*$, cannot be signed. In the example where preferences exhibit prudence, the optimal futures position is always below the minimum-variance position. In contrast, the minimum-variance position under Assumption A is always an under-hedge.[28] As is shown in Proposition 2, the optimality of an under-hedge is preserved whatever the firm's preferences are.

[27] Under quadratic utility, the optimal and the minimum-variance futures position coincide when the futures market is unbiased.

[28] This can be easily seen from the condition for the minimum-variance futures position under Assumption A which is given by $(H^{mv} - Q^*)\ \mathrm{var}[\tilde{P}_1] + H^{mv}\ \mathrm{var}[\min(0, \tilde{\varepsilon}_2, \ldots, \tilde{\varepsilon}_n)] = 0$.

5. Conclusions

Delivery options are a feature of nearly all commodity futures contracts. Quality options allow the seller to choose among several grades of the underlying commodity. Due to the quality option, the futures price on the delivery day does not converge to the spot price of the par-delivery grade but to the spot price of the cheapest-to-deliver grade of the commodity. Hence, any futures position carries an additional delivery risk vis-à-vis the commodity price risk. Hedgers seeking to manage their commodity price risk inevitably encounter this non-hedgeable delivery risk.

This paper has examined the optimal production and futures hedging decisions of a risk-averse competitive firm in the presence of delivery risk arising from a quality option. The three main results can be summarized as follows. First, we have shown that the value of the delivery option enhances the firm's incentives to produce if the firm gauges this value higher than the market. The presence of the non-tradable delivery risk, however, renders the firm's optimal output a function of its preferences and of the underlying uncertainty, thereby invalidating the separation theorem. Second, we have shown that the firm's optimal futures position is an under-hedge if the delivery risk is additively related to the commodity price risk. This is attributable to the fact that reducing the commodity price risk via futures trading creates exposure to the delivery risk. The firm limits its futures position to an under-hedge. Finally, if the delivery risk is multiplicatively related to the commodity price risk, the firm's optimal futures position may be an under- or over-hedge. This indeterminacy is due to two opposing effects: (i) futures trading gives rise to an exposure to the delivery risk, and (ii) the delivery risk in the multiplicative case is negatively correlated with the commodity price risk. The first effect tends to decrease the futures position while the second tends to raise it. A priori, it is unclear which effect dominates without specifying the firm's preferences and the underlying uncertainty.

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