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Pricing and Hedging American Options Using Approximations by Kim Integral Equations ⋆

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Abstract. We present an approximation method for pricing and hedging American options written on a dividend-paying asset. This method is based on Kim (1990) equations. We demonstrate that a simple approximation of the Kim integral equations by quadrature formulas leads to an efficient and accurate numerical procedure. This approximation is accompanied by the Newton–Raphson iteration procedure in order to compute the optimal exercise boundary at each time point. The proposed sequence of approximations converges monotonically, convergence is fast and accuracy is high, even for long maturity options. We compare numerically our results with other competing approaches by different authors.

Key words: American options, early exercise boundary, hedging parameters, integral equations, numerical procedures, optimal exercise.

JEL classification codes: G12, G13, C63.

1. Introduction

The difficulty in pricing of American options stems from the possibility of early exercise. So far a closed form formula for pricing of American options has not been found. For this reason many efforts have been concentrated on numerical approximation methods. Numerical methods such as the finite difference method of Brennan and Schwartz (1977) and the binomial tree model of Cox et al. (1979) are among the earliest numerical methods and are still widely used. The binomial tree method is quite accurate if the number of time steps is large enough. In practice it is supposed that the "true" value of American options can be computed by binomial tree method with $N = 10000$ time steps. However, this method is very time consuming and so finance mathematicians have been looking for some faster methods. The most recent papers like Carr and Faguet (1994), Broadie and

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Detemple (1996), Huang et al. (1996), Ju (1998), Bunch and Johnson (2000) and Sullivan (2000), provide different types of methods and a good overview of the literature (see also Kwok, 1998).

In this paper we derive a numerical approximation method for Kim (1990) equations using quadrature formulas and the Newton–Raphson iteration procedure. On the one hand Kim (1990) and Ju (1998) pointed out that solving the integral equations of Kim should be "a simple task that involves straightforward numerical integration". On the other hand, Kim (1990) and Huang et al. (1996), Broadie and Detemple (1996), were not very optimistic "since the numerical method requires solving ... many integral equations ... and it does not necessarily help us to improve computational efficiency ..." In our paper we shall show that a simple numerical method provides an excellent approximation for solutions to Kim equations. Although the idea of our approximation seems to be natural we did not find any references to this type of approximation.

The remainder of this article is organized as follows. In Section 2 we describe the Kim equations. In Section 3 we show that the early exercise boundary $B(t)$ can be computed from an integral equation using the trapezoidal rule and the Newton–Raphson iteration. Once the boundary $B(t)$ is obtained we deduce the value of American put option using the Simpson's rule. In a similar manner the valuation of American call option can be considered. In Section 4 we shall consider some hedging parameters of American options. The most essential part of our paper is Section 5, where we will compare our numerical results with other methods like the binomial models, the analytical method of Geske and Johnson (1984), the quadratic method of Barone-Adesi and Whaley (1987), the lower and upper bound approximations of Broadie and Detemple (1996), the recursive method of Huang et al. (1996) and the multipiece exponential functions method of Ju (1998). The results of this comparison will be demonstrated in Tables I–IV (see Appendix) and in Figures 2–9.

It turns out that our sequence of approximations converges monotonically, convergence is fast and accuracy is high, even for long maturity options. In addition, the computer experiments suggest that our procedure improves upon standard approaches in the field.

2. Kim Equations for the Analytic Valuation of American Put Options

Let $S = S(t)$ denote the underlying security price (at time t) and K be the exercise price of an American put option expiring at time T. We assume also that the security price follows a lognormal diffusion process with constant volatility parameter σ, and that interest rate r is constant. We consider American put written on securities that pay continuous proportional dividends at a constant rate of δ. Assume that the early exercise boundary $B(t)$ for this American put is a well-defined, unique and continuous function $t \mapsto B(t)$, where $B(t) \geq 0$ and $0 \leq t \leq T$. Then by Kim

(1990), the function $t \mapsto B(t)$ is non-decreasing on $[0, T]$. Moreover (Kim, 1990, p. 560),

$$B(T) = K \min(1, \frac{r}{\delta}). \tag{1}$$

Under the usual assumptions, i.e., that markets are perfect and trading occurs continuously (see, e.g., Hull, 2000), the value of American puts, denoted $P(S, K, t)$, is governed for $S \geq B(t)$ by the Black-Scholes differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - \delta) S \frac{\partial P}{\partial S} - rP = 0$$

with terminal and boundary conditions

$$P(S, K, T) = \max(0, K - S), \tag{2}$$

$$\lim_{S \to \infty} P(S, K, t) = 0,$$

$$\lim_{S \downarrow B(t)} P(S, K, t) = K - B(t), \tag{3}$$

$$\lim_{S \downarrow B(t)} \frac{\partial P(S, K, t)}{\partial S} = -1.$$

For the value of an American put $P_0 = P(S, K, 0)$ Kim (1990) (see also Jacka, 1991 and Carr et al., 1992) obtained the following equation

$$P_0 = p(S, K, T) \; + \; \int_0^T \Big[rKe^{-rt} N(-d_2(S, B(t), t)) - \; \delta S e^{-\delta t} N(-d_1(S, B(t), t)) \Big] dt. \tag{4}$$

Here $p(S, K, T)$ is the corresponding European put, where by Black and Scholes (1973)

$$p(S, K, T) = Ke^{-rT} N(-d_2(S, K, T)) - Se^{-\delta T} N(-d_1(S, K, T)). \tag{5}$$

Further, let $N(\cdot)$ be the standard cumulative normal distribution function,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy,$$

and let us denote

$$d_1(x, y, \tau) = \frac{\log(x/y) + (r - \delta + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \tag{6}$$

$$d_2(x, y, \tau) = \frac{\log(x/y) + (r - \delta - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}. \tag{7}$$

Equation (4) states that the American put is equivalent to the corresponding European put plus an early exercise premium in integral form, which is the value of the right to exchange the European put for the American one.

Also by Kim (1990), the early exercise boundary $B(t)$ solves the following integral equation

$$\begin{aligned} K - B(t) = p(B(t), K, T - t) + \int_t^T \Big[& rKe^{-r(v-t)} N(-d_2(B(t), B(v), v - t)) \\ & -\delta B(t) e^{-\delta(v-t)} N(-d_1(B(t), B(v), v - t))\Big] dv. \end{aligned} \tag{8}$$

Let us remark that the right-hand side of equation (8) is obtained from the right-hand side of equation (4). In equation (8) we integrate on the time interval $[t, T]$ and the asset price S in formula (4) is replaced by the free boundary $B(t)$. For getting the left-hand side of equation (8) from the left-hand side of formula (4) the boundary condition (3) is used.

3. Approximate Solutions of Kim Equations by Quadrature Formulas

Now we present our method for pricing American options by solving the Kim equations (4) and (8). This method consists of the following three steps. The first two steps are needed to find the numerical values of the early exercise boundary $B(t)$ from equation (8). When the values of $B(t)$ are obtained, the third step, numerical integration of equation (4), yields the value of a given American option. The idea of using quadrature formulas at this stage is not new (see, e.g., Sullivan, 2000), but apparently nobody tried to solve equation (8) in order to compute the early exercise boundary $B(t)$ at each time point of the interval $[0, T]$. The crucial step of our method is applying the Newton–Raphson iteration procedure in solving equation (8).

For the numerical approximation we divide the interval of integration $[0, T]$ into M subintervals $[t_{i-1}, t_i]$ with lengths $\Delta t = t_i - t_{i-1}$ for $i = 1, \dots, M$. Denote $B_i = B(t_i)$ for $i = 0, \dots, M$. Since $t_M = T$, we get by the boundary condition (1)

$$B_M = K \min(1, \frac{r}{\delta}). \tag{9}$$

For what follows, let us denote (see the right-hand sides of (4) and (5), respectively)

$$f(x, y, \tau) = rKe^{-r\tau} N(-d_2(x, y, \tau)) - \delta x e^{-\delta\tau} N(-d_1(x, y, \tau)), \tag{10}$$

$$p(x, y, \tau) = ye^{-r\tau} N(-d_2(x, y, \tau)) - xe^{-\delta\tau} N(-d_1(x, y, \tau)). \tag{11}$$

Now the Kim's equations (4) and (8) take the forms

$$P_0 = p(S, K, T) + \int_0^T f(S, B(t), t)dt, \tag{12}$$

$$K - B(t) = p(B(t), K, T - t) + \int_t^T f(B(t), B(v), v - t)dv, \tag{13}$$

respectively.

The value B_M is known due to (9). Putting $t = t_{M-1}$ in (13) we have the equation

$$K - B_{M-1} = p(B_{M-1}, K, \Delta t) + \int_{t_{M-1}}^{t_M} f(B_{M-1}, B(v), v - t)dv,$$

from which B_{M-1} is quite difficult to find. Fortunately, this equation is simpler, if the trapezoidal rule is used (see, e.g., Press, 1992). Hence,

$$\begin{aligned} K - B_{M-1} &= p(B_{M-1}, K, \Delta t) \\ &\quad + \frac{\Delta t}{2}\Big[f(B_{M-1}, B_M, \Delta t) + f(B_{M-1}, B_{M-1}, 0 \cdot \Delta t)\Big], \end{aligned} \tag{14}$$

which gives us the value B_{M-1}. Similarly, for the values B_i, $i = M - 2, M - 3, \ldots, 0$, the trapezoidal rule gives

$$\begin{aligned} K - B_i &= p(B_i, K, (M - i)\Delta t) \\ &\quad + \frac{\Delta t}{2}\Big[f(B_i, B_M, (M - i)\Delta t) + f(B_i, B_i, 0 \cdot \Delta t)\Big] \\ &\quad + \Delta t \sum_{j=i+1}^{M-1} f(B_i, B_j, (j - i)\Delta t). \end{aligned} \tag{15}$$

The value $f(x, x, 0)$ in formulas (14) and (15) is calculated with the aid of the limit process

$$f(x, x, 0) = \lim_{\tau \to 0+} f(x, x, \tau) = \frac{1}{2}(rK - \delta x). \tag{16}$$

Numerical values of $B_i, i = M-1, \ldots, 0$ are found from (15) (or from (14), which coincides with (15) in the case $i = M - 1$) using the Newton–Raphson iteration (see, e.g., Press, 1992). Keeping in mind equation (15) we denote

$$\begin{aligned} F(B_i) &= B_i - K + p(B_i, K, (M - i)\Delta t) + \frac{\Delta t}{2}\Big[f(B_i, B_M, (M - i)\Delta t) \\ &\quad + f(B_i, B_i, 0 \cdot \Delta t)\Big] + \Delta t \sum_{j=i+1}^{M-1} f(B_i, B_j, (j - i)\Delta t). \end{aligned} \tag{17}$$

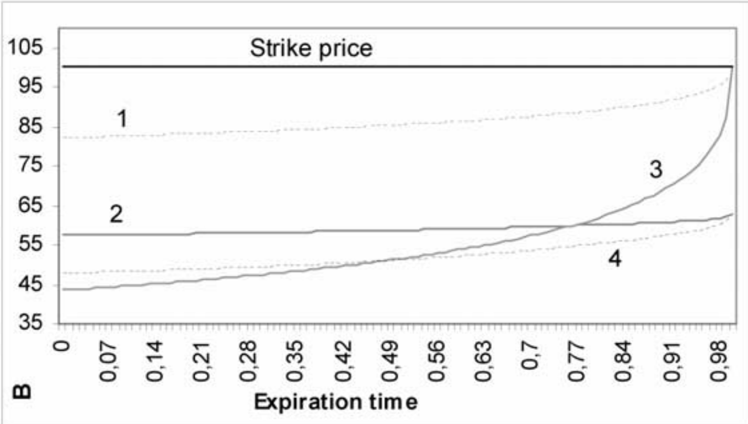

Figure 1. Early exercise boundary $B(t)$ (K=100) of different puts
Put 1: $T = 0.5, \sigma = 0.2, r = 0.08, \delta = 0.05$;
Put 2: $T = 0.5, \sigma = 0.2, r = 0.05, \delta = 0.08$;
Put 3: $T = 0.5, \sigma = 0.7, r = 0.08, \delta = 0.05$;
Put 4: $T = 5, \sigma = 0.2, r = 0.05, \delta = 0.08$.

Now by the Newton–Raphson iteration the values B_i $(i = M-1, M-2, \dots, 0)$ have approximations $B_i^{(k)}$ of order k, where for $k = 0, 1, \dots$

$$B_i^{(k+1)} = B_i^{(k)} - \frac{F(B_i^{(k)})}{F'(B_i^{(k)})}. \tag{18}$$

For the first approximation of B_i we use the value $B_i^{(0)} = B_{i+1}$ $(i = M-1, M-2, \dots, 0)$. For example, $B_{M-1}^{(0)} = B_M$, which is defined by (9). Now we are able to get B_{M-1} from the formula (18). When the value B_{M-1} is obtained we take $B_{M-2}^{(0)} = B_{M-1}$ for getting B_{M-2} by formula (18). Similarly we get all the other values $B_{M-3}, \dots, B_0$. Convergence of the iteration process (18) is very fast; three to four iterations give the accuracy of size 10^{-10}.

Figure 1 shows the graphs of some early exercise boundaries $B(t)$ derived from values $\{B_0, \dots, B_M\}$. Options 1, 2, 3 and 4 all have the same strike price $K = 100$. Options 1 and 3 have different volatilities, $\sigma = 0.2$ and $\sigma = 0.7$, and the same interest and dividend rate $r = 0.08$ and $\delta = 0.05$, respectively. The only difference between options 2 and 4 is the different expiration time $T = 0.5$ and $T = 5$. The interest rate and continuous dividend rate for options 2 and 4 are equal, namely $r = 0.05$ and $\delta = 0.08$, respectively. The time interval $[0, T]$ is re-scaled to the unit interval [0, 1]. The graphs show that the boundary $B(t)$, and so also the price of corresponding option, are more sensitive to the volatility than to the expiration time. Options 1 and 2 differ in interest and dividend rates only. However, it is also seen how different early exercise boundaries of options 1 and 2 are. Because the early exercise boundary represents the asset price below which American puts are exercised optimally, it is very important for practitioners to know the shape of that boundary. Therefore, computing the values $\{B_0, \dots, B_M\}$ for an American put gives practitioners hints when it will be exercised optimally.

The derivative F' in (18) has by (17), (10) and (11) the following form

$$F'(B_i) = \quad 1 + \frac{\partial p(x, y, \tau)}{\partial x}\Bigg|_{(B_i, K, (M-i)\Delta t)} + \Delta t \sum_{j=i+1}^{M-1} \frac{\partial f(x, y, \tau)}{\partial x}\Bigg|_{(B_i, B_j, (j-i)\Delta t)}$$
$$+ \frac{\Delta t}{2}\left[\frac{\partial f(x, y, \tau)}{\partial x}\Bigg|_{(B_i, B_M, (M-i)\Delta t)} + \frac{d}{dx}\big(f(x, x, 0)\big)\Bigg|_{(B_i, B_i, 0)}\right].$$

By (10), (11) and (16) we have

$$\frac{\partial f(x, y, \tau)}{\partial x} = e^{-\delta\tau}\Big[\frac{n(d_1(x, y, \tau))}{\sigma\sqrt{\tau}}(\delta - \frac{rK}{y}) - \delta N(-d_1(x, y, \tau))\Big], \tag{19}$$

$$\frac{d}{dx}\big(f(x, x, 0)\big) = -\frac{\delta}{2}, \qquad \frac{\partial p(x, y, \tau)}{\partial x} = -e^{-\delta\tau} N(-d_1(x, y, \tau)). \tag{20}$$

Here the function $n(\cdot)$ is the standard normal density function

$$n(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

Finally, using Simpson's rule (see, e.g., Press, 1992) for the Kim equation (4) we get the value of an American put in the following way: assuming M is an even number we have by (4) and (10)

$$\begin{aligned} P_0 \;=\; & p(S, K, T) + \frac{\Delta t}{3}[f(S, B_0, 0\Delta t) + 4f(S, B_1, \Delta t) \\ +\; & 2f(S, B_2, 2\Delta t) + 4f(S, B_3, 3\Delta t) + \dots \\ +\; & 2f(S, B_{M-2}, (M-2)\Delta t) \\ +\; & 4f(S, B_{M-1}, (M-1)\Delta t) + f(S, B_M, M\Delta t)]. \end{aligned} \tag{21}$$

The value $f(S, B_0, 0)$ in the last formula is not uniquely determined. By (10) we have (see also (16))

$$f(S, B_0, 0) = \begin{cases} rK - \delta S, & S < B_0, \\ \frac{1}{2}(rK - \delta S), & S = B_0, \\ 0, & S > B_0. \end{cases} \tag{22}$$

Fortunately, there is no need to use formula (21) in case $S \le B_0$. Since (see Kim, 1990 and also Figure 1) the optimal early exercise boundary $B(t)$ is defined as the asset price below which it is optimal to exercise American put and the function $t \mapsto B(t)$ is non-decreasing, so $B_0 \leq B(t)$ for all $t \in [0, T]$, and we may conclude (see also Bunch and Johnson, 2000, p. 2336) that

$$P_0 = K - S \qquad (S \leq B_0). \tag{23}$$

Finally, in case $S > B_0$ we get from (21) and (22)

$$\begin{aligned} P_0 \;=\; & p(S, K, T) + \frac{\Delta t}{3}[4f(S, B_1, \Delta t) + 2f(S, B_2, 2\Delta t) \\ +\; & 4f(S, B_3, 3\Delta t) + \ldots + 2f(S, B_{M-2}, (M-2)\Delta t) \\ +\; & 4f(S, B_{M-1}, (M-1)\Delta t) + f(S, B_M, M\Delta t)]. \end{aligned} \tag{24}$$

The applications of the formula (24) are illustrated in Section 6.

4. Approximate Valuation Formula for American Call Options and for Hedge Parameters

Although American calls can be evaluated using the parity result of McDonald and Schroder (1990)

$$C(S, K, r, \delta, \sigma, T) = P(K, S, \delta, r, \sigma, T),$$

we present here Kim (1990) equations also for American calls and their approximate solutions in a similar manner as in Section 3. Let us denote by $G(t)$ the early exercise boundary for an American call option. Then under the assumptions and notations of Section 2 the value of an American call for $S \leq G(t)$, denoted by C_0, is as follows:

$$\begin{aligned} C_0 = c(S, K, T) + \int_0^T \Big[& -rKe^{-rt} N(d_2(S, G(t), t)) \\ & + \delta S e^{-\delta t} N(d_1(S, G(t), t)) \Big] dt. \end{aligned} \tag{25}$$

Here the corresponding European call option is

$$c(S, K, T) = Se^{-\delta T} N(d_1(S, K, T)) - Ke^{-rT} N(d_2(S, K, T)), \tag{26}$$

and the early exercise boundary $G(t)$ solves the integral equation

$$\begin{aligned} G(t) - K \;=\; c(G(t), K, T-t) + \int_t^T \Big[& -rKe^{-r(v-t)} N(d_2(G(t), G(v), v-t)) \\ & + \delta G(t) e^{-\delta(v-t)} N(d_1(G(t), G(v), v-t)) \Big] dv. \end{aligned} \tag{27}$$

For the early exercise boundary $G(t)$ we denote $G_i = G(t_i)$. Thus

$$G_M = G(T) = K \max(1, \frac{r}{\delta}).$$

Denote

$$g(x, y, \tau) = -rKe^{-r\tau}N(d_2(x, y, \tau)) + \delta x e^{-\delta\tau}N(d_1(x, y, \tau)), \tag{28}$$

$$c(x, y, \tau) = -ye^{-r\tau}N(d_2(x, y, \tau)) + xe^{-\delta\tau}N(d_1(x, y, \tau)). \tag{29}$$

From (27) using the trapezoidal rule for $G_{M-1}, \ldots, G_0$ the equation

$$\begin{aligned} G_i - K &= c(G_i, K, (M-i)\Delta t) \\ &+ \frac{\Delta t}{2}\Big[g(G_i, G_M, (M-i)\Delta t) + g(G_i, G_i, 0 \cdot \Delta t)\Big] \\ &+ \Delta t \sum_{j=i+1}^{M-1} g(G_i, G_j, (j-i)\Delta t) \end{aligned}$$

is obtained, where in the case of $i = M - 1$ the last sum turns out to be zero. Moreover,

$$g(x, x, 0) = \lim_{\tau \to 0+} g(x, x, \tau) = \frac{1}{2}(\delta x - rK).$$

The derivatives needed for the Newton–Raphson iteration are as follows:

$$\frac{\partial g(x, y, \tau)}{\partial x} = e^{-\delta\tau}\Big[\frac{n(d_1(x, y, \tau))}{\sigma\sqrt{\tau}}(\delta - \frac{rK}{y}) + \delta N(d_1(x, y, \tau))\Big],$$

$$\frac{d}{dx}g(x, x, 0) = \frac{\delta}{2}, \qquad \frac{\partial c(x, y, \tau)}{\partial x} = e^{-\delta\tau}N(d_1(x, y, \tau)).$$

When the values $G_0, G_1, \ldots, G_M$ are obtained, we have

$$C(S, K, T) = S - K,$$

if $S \geq G_0$. In the case of $S < G_0$ we have

$$\begin{aligned} C(S, K, T) &= c(S, K, T) + \frac{\Delta t}{3}[4g(S, G_1, \Delta t) + 2g(S, G_2, 2\Delta t) \\ &+ 4g(S, G_3, 3\Delta t) + \ldots + 2g(S, G_{M-2}, (M-2)\Delta t) \\ &+ 4g(S, G_{M-1}, (M-1)\Delta t) + g(S, G_M, M\Delta t)]. \end{aligned} \tag{30}$$

Also the valuation formula (24) offers a simple way to calculate the hedge parameters

$$\text{delta:} \quad \Delta = \frac{\partial P}{\partial S}, \qquad \text{gamma:} \quad \Gamma = \frac{\partial^2 P}{\partial S^2}, \qquad \text{vega:} \quad \vartheta = \frac{\partial P}{\partial \sigma}.$$

For example, equation (24) yields

$$\begin{aligned}\Delta = \frac{\partial P}{\partial S} &= \frac{\partial p}{\partial x}\bigg|_{(S,K,T)} + \frac{\Delta t}{3}\Bigg[4\frac{\partial f}{\partial x}\bigg|_{(S,B_1,\Delta t)} \\ &\quad + 2\frac{\partial f}{\partial x}\bigg|_{(S,B_2,2\Delta t)} + \ldots + \frac{\partial f}{\partial x}\bigg|_{(S,B_M,M\Delta t)}\Bigg],\end{aligned} \tag{31}$$

where the derivatives $\partial p/\partial x$ and $\partial f/\partial x$ will be calculated by the formulas (19) and (20), respectively. For gamma the following partial derivatives

$$\frac{\partial^2 f(x,y,\tau)}{\partial x^2} = e^{-\delta\tau}\frac{n(d_1(x,y,\tau))}{x\sigma\sqrt{\tau}}\left[\delta - \frac{d_1(x,y,\tau)}{\sigma\sqrt{\tau}}(\delta - \frac{rK}{y})\right],$$

$$\frac{\partial^2 p(x,y,\tau)}{\partial x^2} = e^{-\delta\tau}\frac{n(d_1(x,y,\tau))}{x\sigma\sqrt{\tau}}$$

have to be found. Using straightforward formulas

$$d_1 - d_2 = \sigma\sqrt{\tau}, \qquad \frac{n(d_2(x,y,\tau))}{n(d_1(x,y,\tau))} = \frac{x}{y}e^{(r-\delta)\tau},$$

$$d_1 = -\sigma\frac{\partial d_2}{\partial \sigma}, \qquad d_2 = -\sigma\frac{\partial d_1}{\partial \sigma},$$

we get, for finding vega, the equations

$$\frac{\partial f}{\partial \sigma} = \sigma\tau x^2\frac{\partial^2 f}{\partial x^2}, \tag{32}$$

$$\frac{\partial p}{\partial \sigma} = \sigma\tau x^2\frac{\partial^2 p}{\partial x^2}. \tag{33}$$

5. Numerical Applications and Comparisons with Known Results

In this section we present some numerical examples in order to demonstrate the convergence and accuracy of our approximate solutions of Kim equations.

Suppose that the "true" value of an American option (put and call) is based on the binomial tree model with $N = 10000$ time steps. Let $\overline{V}_i$ be the "true" value of an American option for a sample i of some fixed parameters $(K, T, \sigma, r, \delta)$. Using another valuation method of the same option for the same sample i we get another value, say V_i. If we consider N different samples, then the widely used error estimation of that valuation method is the root of the mean squared errors (abbreviated as RMSE)

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(V_i - \overline{V_i})^2}.$$

For clarification purposes, let us introduce the following abbreviations for different valuation methods of American options.

"TRUE" – the binomial tree model (see, e.g., Hull, 2000) for "true" values, $N = 10000$ time steps;

B – the binomial tree model of Cox and Rubinstein;

BBS – the binomial tree model, where the binomial formula is replaced by the Black and Scholes formula at the last time step (Broadie and Detemple, 1996);

BBSR – the binomial tree with Black and Scholes using the Richardson extrapolation (Broadie and Detemple, 1996);

BW – the quadratic method of Barone-Adesi and Whaley (1987) (see also, e.g., Hull, 2000);

GJ4 – the analytical method of Geske and Johnson (1984) using four point Richardson extrapolation;

LUBA – the lower and upper bound approximations of Broadie and Detemple (1996);

RAN6 – the six-point randomization method of Carr (1998);

HSY6 – the six-point recursive method of Huang et al. (1996);

EXP3 – the multipiece exponential functions method of Ju (1998) using the three-point Richardson extrapolation;

QFK – the quadrature formulas approximations of Kim (1990) equations, the method which is implemented in this paper.

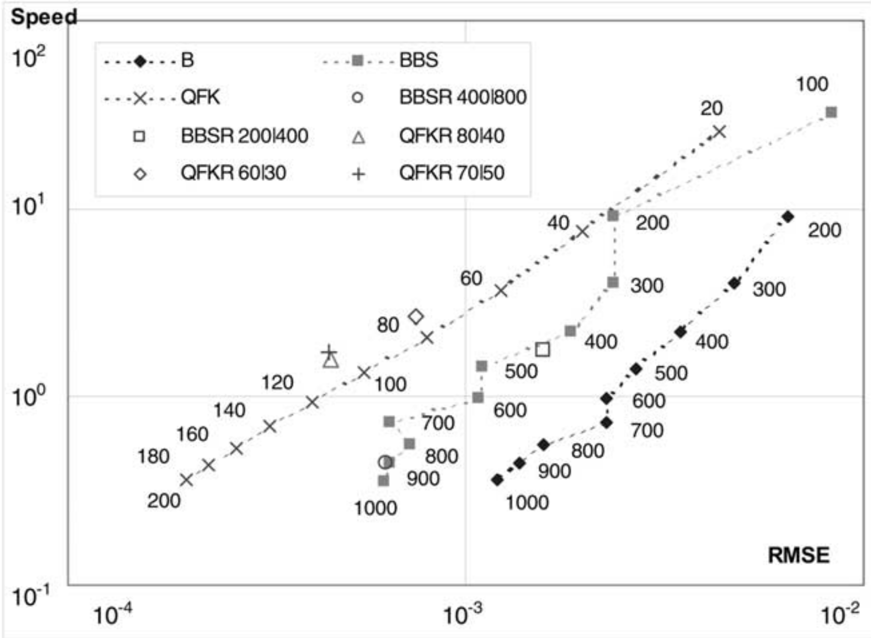

Figure 2. Speed-accuracy trade-off using 20 option values in Tables I–III.

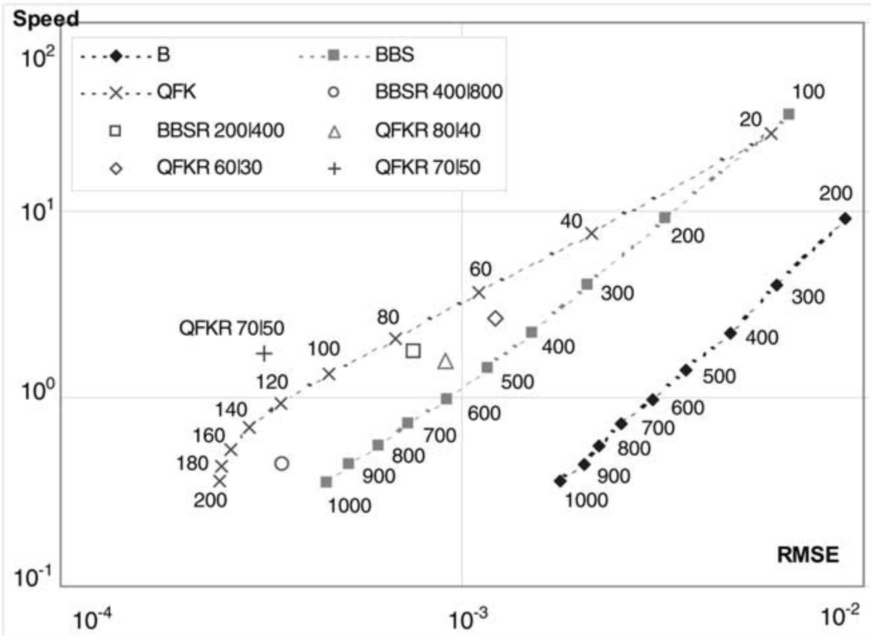

Figure 3. Speed-accuracy trade-off using 1050 option values: $S = 85, 90, \ldots, 115$ (7 values); $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.1, 0.2, \ldots, 0.6$ (6 values) and $T = 1/12, 1/4, 1/2, 1, 3$.

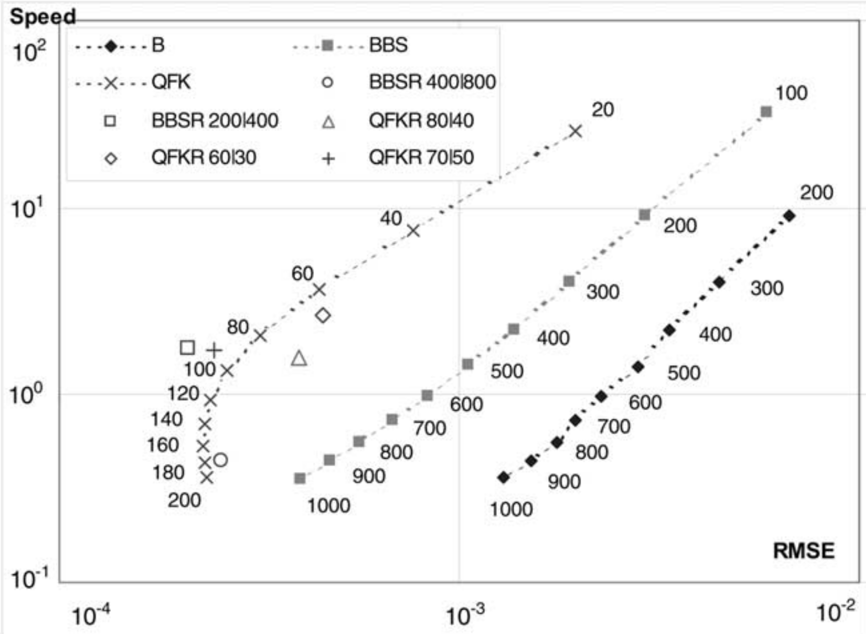

Figure 4. Speed-accuracy trade-off for short maturity options ($T < 1.0$) using 1050 option values: $S = 85, 90, \ldots, 115$ (7 values); $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.1, 0.2, \ldots, 0.6$ (6 values) and $T = 1/12, 1/6, 1/4, 1/2, 3/4$.

QFKR – the quadrature formulas approximations of Kim (1990) equations using the Richardson extrapolation.

The numbers, associated with an abbreviation will indicate numbers of time-steps or some other control parameters. For example, B10000 denotes the binomial tree method with $N = 10000$ time steps. Tables I, II and III (see Appendix) contain the results of put option prices derived from the eighteen alternative methods including the method QFK presented in this paper. For all options we fix $K = 100, T = 3, \sigma = 0.2$ and $r = 0.08$. All columns consist of twenty options forming four groups A, B, C, D, where in each group the current asset price S has values 80, 90, 100, 110 and 120. The groups A, B, C, D are determined by different dividend rates $\delta = 0.12, 0.08, 0.04$ and 0, respectively. In Table I the entries of columns (1) and (5-9) are taken from Ju (1998), for other methods the computer programs are implemented by ourselves. At the top of each column the abbreviation (see the list of abbreviations) of the pricing method under consideration is shown. At the bottom of each column the root of the mean squared error RMSE is presented.

In Table I we can see that the most accurate analytical methods are LUBA, EXP3 and QFK70. Numerical methods B800 and BBS800 (800 stands for the

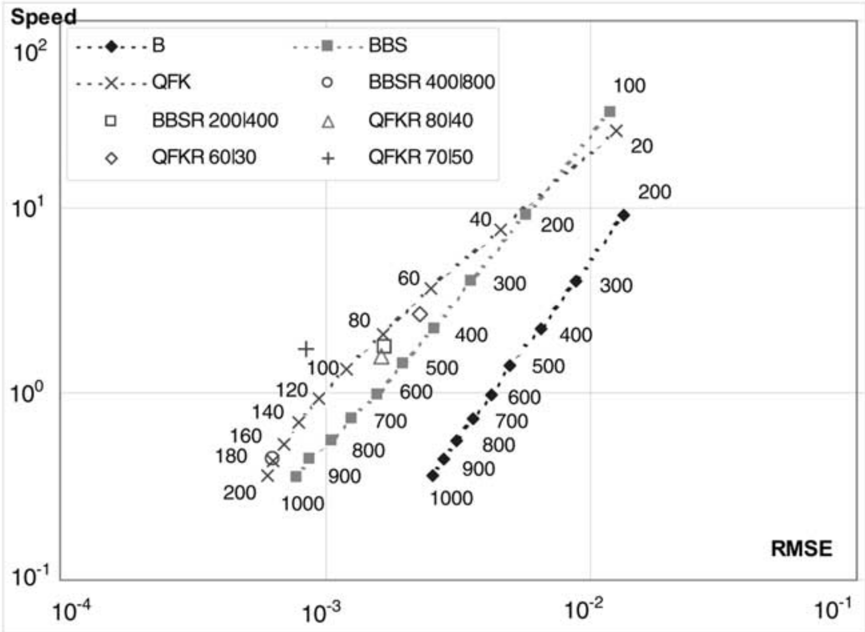

Figure 5. Speed-accuracy trade-off for long maturity options ($T \geq 1.0$) using 1050 option values: $S = 85, 90, \ldots, 115$ (7 values); $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.1, 0.2, \ldots, 0.6$ (6 values) and $T = 1, 1.5, 2, 3, 5$.

number of time steps) are a little more accurate than the analytical ones, but at the same time more time consuming. Our method QFK70 needs, for the accuracy on the level of numerical methods (columns (2), (3) and (10)), about 70 time-step iterations, but it is much faster than the numerical methods. The QFK method is also easier to implement than some analytical methods like LUBA or EXP3. Another advantage of our method is, as we can see in Table III, that the convergence of RMSE is monotone, and for high accuracy we do not need a large number of time-steps.

Table II contains results concerning the binomial Black-Scholes BBS and QFK methods using Richardson extrapolation. For example, QFKR80/40 means that the price is approximated by $2P_2 - P_1$, where P_1 and P_2 are computed using the QFK method with $M = 40$ and $M = 80$, respectively.

Table III shows the dynamic of accuracy of the method QFK. The put options in Table III are the same as in Table I, but in the present case each column consists of prices evaluated by QFK with different time-steps numbers $M = 10, 20, \ldots, 150$. The accuracy of QFK with $M = 10$ time steps is comparable to HSY6 in Table I, but in the case $M = 20$ time-steps, QFK is about twice as accurate as HSY6 is. The time-steps number $M = 100$ gives more accurate results than almost any other in Table I.

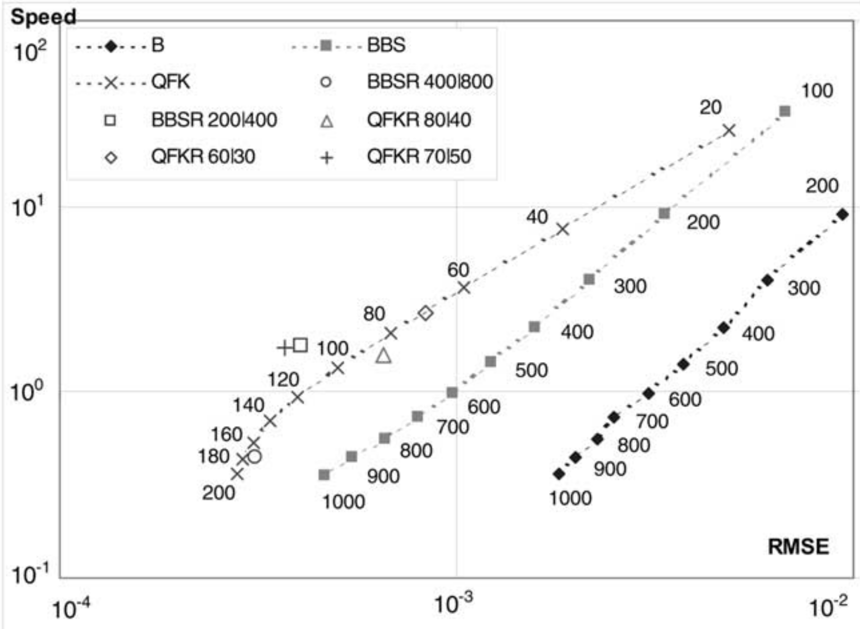

Figure 6. Speed-accuracy trade-off for at-the-money options ($0.9 < S/K < 1.1$) using 1050 option values: $S = 91, 94, \ldots, 109$ (7 values); $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.1, 0.2, \ldots, 0.6$ (6 values) and $T = 1/12$, 1/4, 1/2, 1, 3.

Table IV presents the accuracies of different methods for calculating the hedging ratio Δ of American puts. Good results are obtained by RAN6, EXP3 and B800. Our method QFK by formula (31) attains the same level of accuracy using $M \geq 50$ step numbers.

In conclusion, it could be stated that in solving analytical equations of Kim numerically we have obtained fast algorithms with accuracy as high as desired. In contrast, the analytical methods like EXP3 and LUBA are accurate, but there is no easy way to obtain higher accuracy. The numerical methods like B800 and BBS2500 can give high accuracy, but these algorithms are not fast. Our method's success is probably due to the good numerical approximation of early exercise boundary values $B_0, B_1, \ldots, B_M$.

For all numerical algorithms another important factor is the computation speed. Tables I through IV in Appendix do not give summary information about computational speed for a wide variety of options. More thorough and systematic results concerning the speed-accuracy trade-off of various option pricing methods are given in Figures 2 through 9. Figure 2 illustrates the speed-accuracy trade-off for options in Tables I–III. Figures 3 through 9 represent average results from 1050 options determined by parameters $81 \leq S \leq 120$, $K = 100$, $r = 0.08$, $0 \leq \delta \leq 0.16$, $0.05 \leq \sigma \leq 0.6$, $1/12 \leq T \leq 5$, respectively. The measure of accuracy that we

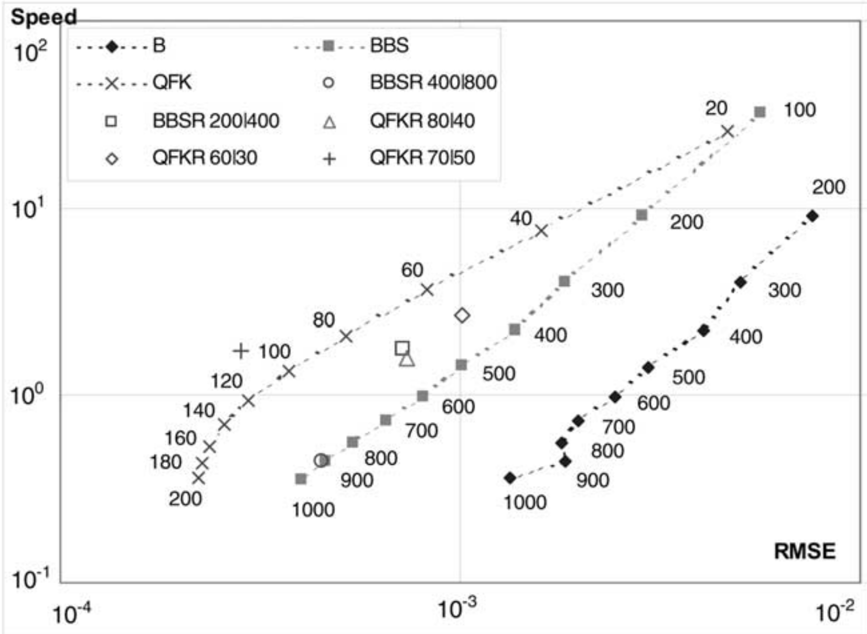

Figure 7. Speed-accuracy trade-off for in- and out-of-the-money options ($S/K \leq 0.9$ or $S/K \geq 1.1$) using 1050 option values: $S = 81, 84, 87, 90, 110, 115, 120$; $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.1, 0.2, \ldots, 0.6$ (6 values) and $T = 1/12, 1/4, 1/2, 1, 3$.

have used is the root mean squared error RMSE, where the "true" value is estimated by the BBS2500. Computation speed is measured in option prices calculated per second. In measuring the computation speed the exact hardware is inconsequential, since only relative speed matters. The results are plotted on a log-log scale. In Figure 2 we present the graphs of RMSE versus computation speed for put options in Tables I–III. We have computed the prices using the binomial methods B, the binomial Black-Scholes methods BBS and the QFK methods. The numbers next to the graphs indicate the important parameters, which control the method. For B and BBS the control parameter is the number of time-steps (see, e.g., Hull, 2000); for the method QFK it is $M = T/\Delta t$ (see Section 3). Symbols like $\triangle, \square, \diamond, \ldots$ indicate the results computed by Richardson extrapolation. Figure 2 shows that our method QFK is very fast, it has high accuracy and its convergence is monotone.

Broadie and Detemple (1996) proved that among methods BW, LUBA, binomial, etc., two very efficient valuation procedures are the BBS and BBSR. In this paper it was not our purpose to provide computations by all these methods for a very wide range of samples. Thus, we have compared the QFK with the BBS and the BBSR. Since we had no "true" values for a wide range of samples we chose the BBS2500 for the "true" value. In Table II at column (12) we can see that in respect

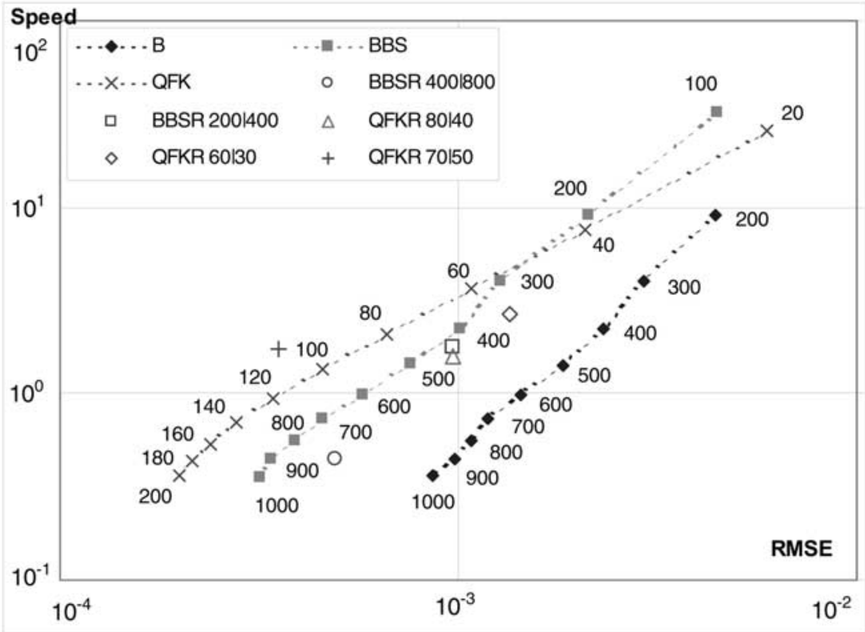

Figure 8. Speed-accuracy trade-off for low volatility options ($\sigma < 0.35$) using 1050 option values: $S = 85, 90, \ldots, 115$ (7 values); $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.05, 0.1, \ldots, 0.3$ (6 values) and $T = 1/12$, 1/4, 1/2, 1, 3.

of the traditional "true" value computed by B10000 the RMSE of the BBS2500 is 0.00025.

Figure 3 presents valuation of 1050 put options. Figures 4 and 5 break the results down by option maturity, Figures 6 and 7 by S/K (the "moneyness" of the option), and Figures 8 and 9 by option volatility.

As already known from Broadie and Detemple (1996) the binomial method B and the binomial method associated with the Black-Scholes formula BBS plot as nearly straight lines in Figures 2 through 9. The same is true for the QFK with the time step number until 120. In this range of time steps the accuracy of the QFK increases very quickly. Next the accuracy of the QFK is slower, although it is higher than that for the methods BBS or B. As for the Richardson extrapolation, computations show that the best one is the QFKR70/50. In some cases (Figures 4, 5, 6 and 9) the BBSR800/400 is very good also, but it seems to be unstable for some specific samples of options (see Figure 7 and 8).

To conclude, we may say that the QFK method seems to dominate the BBS method, which was known as one of the best so far.

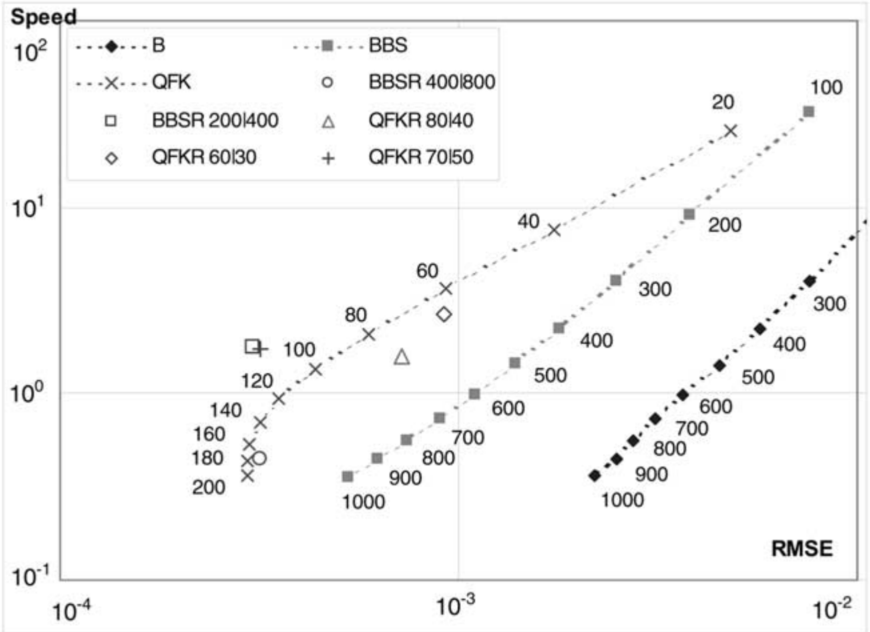

Figure 9. Speed-accuracy trade-off for high volatility options ($\sigma \geq 0.35$) using 1050 option values: $S = 85, 90, \ldots, 115$ (7 values); $K = 100$; $r = 0.08$; $\delta = 0, 0.04, \ldots, 0.16$ (5 values); $\sigma = 0.35, 0.4, \ldots, 0.6$ (6 values) and $T = 1/12$, 1/4, 1/2, 1, 3.

6. Conclusion

We have derived an efficient method for pricing and hedging American options. For this purpose we have solved integral equations of Kim (1990) numerically. The main novelty of our approach is the use of the Newton–Raphson iteration procedure in order to compute the earlier exercise boundary at each time point. Surprisingly we have found that approximation by quadrature formulas combined with Newton–Raphson iteration gives a very fast and accurate algorithm for pricing American options. This algorithm seems to be faster than many numerical and analytical pricing methods known so far. The accuracy and the speed of that algorithm are tested by many computational experiments using American put options values. Similar computations can be done for American call options and hedging parameters as well.

Appendix: Numerical examples

Table I. Prices of American put options ($K = 100$, $T = 3$, $\sigma = 0.2$, $r = 0.08$) valued by different methods

	S	(1) "True"	(2) B800	(3) BBS800	(4) BW	(5) GJ4	(6) LUBA	(7) RAN6	(8) HSY6	(9) EXP3	(10) QFK70
A	80	25.6577	25.6564	25.6573	26.2452	25.6529	25.6568	25.6577	25.7021	25.6570	25.6585
$\delta = 0.12$	90	20.0832	20.0839	20.0831	20.6410	20.1089	20.0834	20.0830	20.0905	20.0817	20.0838
	100	15.4981	15.4945	15.4984	15.9902	15.5122	15.4985	15.4984	15.5020	15.4970	15.4988
	110	11.8032	11.8013	11.8031	12.2212	11.8023	11.8032	11.8032	11.8121	11.8022	11.8034
	120	8.8856	8.8873	8.8851	9.2345	8.8803	8.8855	8.8856	8.8921	8.8850	8.8857
B	80	22.2050	22.2043	22.2049	22.3950	22.2079	22.1985	22.2032	22.1493	22.2084	22.2061
$\delta = 0.08$	90	16.2071	16.2084	16.2075	16.4976	16.1639	16.1986	16.2034	16.2578	16.2106	16.2081
	100	11.7037	11.7013	11.7047	12.0302	11.7053	11.6988	11.6994	11.7237	11.7066	11.7047
	110	8.3671	8.3664	8.3678	8.6871	8.3886	8.3630	8.3621	8.3563	8.3695	8.3677
	120	5.9299	5.9322	5.9303	6.2222	5.9435	5.9261	5.9252	5.9323	5.9323	5.9304
C	80	20.3500	20.3489	20.3490	20.3255	20.5134	20.3335	20.3492	20.3932	20.3511	20.3512
$\delta = 0.04$	90	13.4968	13.4969	13.4963	13.5631	13.5246	13.4982	13.4939	13.4602	13.5000	13.4966
	100	8.9438	8.9423	8.9441	9.1076	8.8414	8.9424	8.9403	8.9891	8.9474	8.9442
	110	5.9119	5.9115	5.9121	6.1225	5.8904	5.9122	5.9077	5.9269	5.9146	5.9123
	120	3.8975	3.8988	3.8977	4.1153	3.9046	3.8980	3.8932	3.8834	3.8997	3.8980
D	80	20.0000	20.0000	20.0000	20.0000	19.7313	20.0000	20.0000	19.9484	20.0000	20.0000
$\delta = 0$	90	11.6974	11.6955	11.6952	11.6342	11.8842	11.6953	11.6960	11.7047	11.6991	11.6943
	100	6.9320	6.9301	6.9310	6.9621	6.9266	6.9346	6.9301	6.9109	6.9346	6.9306
	110	4.1550	4.1539	4.1541	4.2574	4.1033	4.1550	4.1528	4.1897	4.1571	4.1542
	120	2.5102	2.5103	2.5097	2.6402	2.4906	2.5110	2.5082	2.5150	2.5119	2.5100
RSME		**0.0000**	**0.0016**	**0.0007**	**0.2971**	**0.0872**	**0.0048**	**0.0028**	**0.0304**	**0.0023**	**0.0010**

Table II. Prices of American put options ($K = 100$, $T = 3$, $\sigma = 0.2$, $r = 0.08$) valued by different methods

	S	"True"	(11) B2500	(12) BBS2500	(13) BBS800/400	(14) BBS400/200	(15) QFK400	(16) QFKR80/40	(17) QFKR60/30	(18) QFKR70/50
A	80	25.6577	25.6573	25.6576	25.6578	25.6578	25.6578	25.6573	25.6570	25.6580
$\delta = 0.12$	90	20.0832	20.0836	20.0832	20.0832	20.0833	20.0833	20.0829	20.0827	20.0834
	100	15.4981	15.4972	15.4984	15.4984	15.4984	15.4984	15.4982	15.4981	15.4985
	110	11.8032	11.8027	11.8032	11.8032	11.8031	11.8032	11.8031	11.8030	11.8033
	120	8.8856	8.8851	8.8854	8.8855	8.8855	8.8855	8.8854	8.8854	8.8856
B	80	22.2050	22.2047	22.2050	22.2050	22.2049	22.2051	22.2044	22.2043	22.2054
$\delta = 0.08$	90	16.2071	16.2076	16.2072	16.2071	16.2073	16.2071	16.2065	16.2062	16.2074
	100	11.7037	11.7030	11.7041	11.7039	11.7041	11.7039	11.7034	11.7032	11.7042
	110	8.3671	8.3669	8.3673	8.3671	8.3671	8.3671	8.3667	8.3665	8.3673
	120	5.9299	5.9297	5.9300	5.9298	5.9301	5.9299	5.9295	5.9294	5.9301
C	80	20.3500	20.3495	20.3495	20.3487	20.3502	20.3500	20.3491	20.3513	20.3502
$\delta = 0.04$	90	13.4968	13.4969	13.4967	13.4970	13.4963	13.4968	13.4971	13.4975	13.4968
	100	8.9438	8.9434	8.9440	8.9442	8.9438	8.9440	8.9439	8.9439	8.9441
	110	5.9119	5.9118	5.9120	5.9119	5.9123	5.9119	5.9117	5.9116	5.9121
	120	3.8975	3.8974	3.8975	3.8975	3.8977	3.8975	3.8972	3.8971	3.8977
D	80	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000
$\delta = 0$	90	11.6974	11.6969	11.6968	11.6995	11.6908	11.6974	11.6977	11.6959	11.6958
	100	6.9320	6.9314	6.9317	6.9332	6.9296	6.9321	6.9331	6.9338	6.9316
	110	4.1550	4.1547	4.1547	4.1551	4.1550	4.1549	4.1555	4.1557	4.1547
	120	2.5102	2.5100	2.5101	2.5104	2.5102	2.5102	2.5105	2.5105	2.5102
RSME		**0.0000**	**0.00044**	**0.00025**	**0.00063**	**0.00158**	**0.00012**	**0.00046**	**0.00075**	**0.00045**

Table III. Prices of American put options ($K = 100$, $T = 3$, $\sigma = 0.2$, $r = 0.08$) valued by QFK with different time steps M

	S	(1) "True"	(2) $M = 10$	(3) $M = 20$	(4) $M = 30$	(5) $M = 40$	(6) $M = 50$	(7) $M = 60$	(8) $M = 70$	(9) $M = 80$	(10) $M = 100$	(11) $M = 150$
A	80	25.6577	25.6703	25.6624	25.6603	25.6595	25.6590	25.6587	25.6585	25.6584	25.6582	25.6580
$\delta = 0.12$	90	20.0832	20.0919	20.0865	20.0851	20.0844	20.0841	20.0839	20.0838	20.0837	20.0835	20.0834
	100	15.4981	15.5042	15.5006	15.4997	15.4992	15.4990	15.4989	15.4988	15.4987	15.4986	15.4985
	110	11.8032	11.8069	11.8047	11.8040	11.8038	11.8036	11.8035	11.8034	11.8034	11.8033	11.8033
	120	8.8856	8.8879	8.8865	8.8860	8.8859	8.8858	8.8857	8.8857	8.8856	8.8856	8.8856
B	80	22.2050	22.2260	22.2110	22.2085	22.2074	22.2068	22.2064	22.2061	22.2059	22.2056	22.2053
$\delta = 0.08$	90	16.2071	16.2212	16.2130	16.2104	16.2093	16.2087	16.2083	16.2081	16.2079	16.2077	16.2074
	100	11.7037	11.7165	11.7088	11.7067	11.7058	11.7053	11.7050	11.7047	11.7046	11.7044	11.7042
	110	8.3671	8.3771	8.3710	8.3693	8.3686	8.3682	8.3679	8.3677	8.3676	8.3675	8.3673
	120	5.9299	5.9375	5.9330	5.9317	5.9311	5.9307	5.9305	5.9304	5.9303	5.9302	5.9300
C	80	20.3500	20.3100	20.3470	20.3520	20.3526	20.3522	20.3517	20.3512	20.3508	20.3503	20.3499
$\delta = 0.04$	90	13.4968	13.4998	13.4948	13.4957	13.4963	13.4965	13.4966	13.4966	13.4967	13.4967	13.4967
	100	8.9438	8.9452	8.9451	8.9447	8.9445	8.9444	8.9443	8.9442	8.9442	8.9442	8.9441
	110	5.9119	5.9181	5.9143	5.9133	5.9128	5.9126	5.9124	5.9123	5.9123	5.9122	5.9120
	120	3.8975	3.9052	3.9003	3.8991	3.8985	3.8983	3.8981	3.8980	3.8979	3.8978	3.8976
D	80	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000
$\delta = 0$	90	11.6974	11.6691	11.6899	11.6914	11.6921	11.6929	11.6937	11.6943	11.6949	11.6957	11.6966
	100	6.9320	6.9092	6.9217	6.9266	6.9286	6.9296	6.9302	6.9306	6.9309	6.9313	6.9317
	110	4.1550	4.1421	4.1505	4.1524	4.1533	4.1537	4.1540	4.1542	4.1544	4.1545	4.1547
	120	2.5102	2.5063	2.5085	2.5092	2.5096	2.5098	2.5099	2.5100	2.5100	2.5101	2.5102
RSME		**0.0000**	**0.0149**	**0.0044**	**0.0026**	**0.0020**	**0.0015**	**0.0012**	**0.0010**	**0.0008**	**0.0006**	**0.0003**

Table IV. Values of the hedging ratio Δ of American puts ($K = 100$, $T = 3$, $\sigma = 0.2$, $r = 0.08$) evaluated by different methods

	S	(1) "True"	(2) B800	(3) GJ4	(4) LUBA	(5) RAN6	(6) HSY6	(7) EXP3	(8) QFK10	(9) QFK30	(10) QFK50
A $\delta = 0.12$	80	−0.6103	−0.6101	−0.6046	−0.6101	−0.6104	−0.6125	−0.6104	−0.6107	−0.6104	−0.6104
	90	−0.5063	−0.5061	−0.5061	−0.5063	−0.5063	−0.5088	−0.5064	−0.5067	−0.5064	−0.5064
	100	−0.4122	−0.4122	−0.4141	−0.4123	−0.4122	−0.4115	−0.4122	−0.4125	−0.4123	−0.4123
	110	−0.3287	−0.3286	−0.3296	−0.3287	−0.3287	−0.3286	−0.3286	−0.3288	−0.3287	−0.3287
	120	−0.2569	−0.2569	−0.2569	−0.2569	−0.2569	−0.2573	−0.2569	−0.2570	−0.2569	−0.2569
B $\delta = 0.08$	80	−0.6878	−0.6877	−0.7013	−0.6887	−0.6881	−0.6788	−0.6877	−0.6897	−0.6878	−0.6878
	90	−0.5189	−0.5187	−0.5161	−0.5186	−0.5190	−0.5150	−0.5190	−0.5190	−0.5190	−0.5189
	100	−0.3871	−0.3871	−0.3831	−0.3869	−0.3872	−0.3929	−0.3872	−0.3874	−0.3872	−0.3871
	110	−0.2847	−0.2847	−0.2845	−0.2846	−0.2847	−0.2847	−0.2847	−0.2849	−0.2847	−0.2847
	120	−0.2064	−0.2065	−0.2077	−0.2064	−0.2063	−0.2048	−0.2064	−0.2066	−0.2065	−0.2064
C $\delta = 0.04$	80	−0.8374	−0.8375	−0.8268	−0.8338	−0.8377	−0.8500	−0.8372	−0.8119	−0.8345	−0.8378
	90	−0.5541	−0.5542	−0.5717	−0.5547	−0.5542	−0.5472	−0.5540	−0.5556	−0.5538	−0.5540
	100	−0.3691	−0.3692	−0.3729	−0.3689	−0.3691	−0.3664	−0.3691	−0.3685	−0.3690	−0.3690
	110	−0.2456	−0.2457	−0.2414	−0.2455	−0.2456	−0.2508	−0.2456	−0.2453	−0.2455	−0.2456
	120	−0.1628	−0.1630	−0.1615	−0.1628	−0.1628	−0.1630	−0.1628	−0.1628	−0.1628	−0.1628
D $\delta = 0$	80	−1.0000	−1.0000	−0.9236	−1.0000	−1.0000	−0.9755	−1.0000	−1.0000	−1.0000	−1.0000
	90	−0.6209	−0.6216	−0.6352	−0.6173	−0.6209	−0.6419	−0.6207	−0.6135	−0.6223	−0.6212
	100	−0.3583	−0.3587	−0.3716	−0.3588	−0.3583	−0.3489	−0.3582	−0.3581	−0.3578	−0.3581
	110	−0.2109	−0.2112	−0.2092	−0.2108	−0.2109	−0.2112	−0.2109	−0.2097	−0.2107	−0.2108
	120	−0.1257	−0.1259	−0.1224	−0.1256	−0.1256	−0.1289	−0.1257	−0.1251	−0.1256	−0.1256
RSME		**0.0000**	**0.0002**	**0.0186**	**0.0012**	**0.0001**	**0.0087**	**0.0001**	**0.0060**	**0.0007**	**0.0001**

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