European Finance Review **7:** 249–276, 2003.

Analytical Approach to Value Options with State Variables of a Lévy System

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Abstract. In this paper we present an analytical method in pricing European contingent assets, whose state variables follow a multi-dimensional Lévy process. We give an explicit formula for the hypothetical European "two-price" call option price by means of the conditional characteristic transform. The work not only unifies and extends the option pricing literature, which focuses on the use of the characteristic function, but also provides the way to formalize and unify the valuation of the option price, the valuation of the discount bond price, the valuation of the scaled-forward price, and the valuation of the pricing measure in incomplete markets.

Key words: characteristic function, Fourier transform, jump-diffusion process, Lévy process, option pricing

JEL Classification codes: G13

1. Introduction

In an influential paper in the option-pricing literature, Heston (1993) showed that the risk-neutral exercise probabilities appearing in the call option pricing formulae for bonds, currencies and equities can be computed by a Fourier inversion of the conditional characteristic function, which he showed is known in closed-form for his particular affine, stochastic volatility model. Building on this insight, a variety of option-pricing models (see Bakshi et al. (1997), Bates (1996), Bakshi and Madan (2000), among others) have been developed for state vectors having at most a single jump type (in the asset return) and whose behavior between jumps is that of a Gaussian or "square-root" diffusion. Recently, Duffie et al. (2000) generalized this strand option-pricing literature and developed it into the so-called affine jump-diffusion framework.

This paper unifies and extends the above mentioned literature stream. In the spirit of the work of Bakshi and Madan (2000), we explore the use of the so-called "conditional characteristic transform" of the state vector in pricing European option. We show that knowing the conditional characteristic transform will also automatically determine other elementary securities values such as (i) the discount bond price; (ii) the scaled-forward price[1] (the fair price of a commitment to delivery the underlying asset at expiration); (iii) the two probabilities of option finishing in-the-money (or two Arrow–Debreu securities); and (iv) the pricing measure in incomplete markets.

Our framework also generalizes the affine Poisson jump-diffusion economy of Duffie et al. (2000) by relaxing the affine requirement and allowing more general specifications of the jump structure, since the state variables in our model are multi-dimensional Lévy processes.

Over the last 20 years, the benchmark model for security prices of assets in highly liquid financial markets has been geometric Brownian motion. A relatively simple yet powerful generalization is the class of all continuous-time processes where all non-overlapping increments are independent random variables with stationary distributions. This set of processes are the Lévy processes or processes with stationary independent increments. They consist of a combination of a linear drift, a Brownian motion, and an independent jump process. With the state variables of a multi-dimensional Lévy process in use, all models of an affine jump-diffusion framework of Duffie et al. (2000), such as the compound Poisson processes model proposed by Merton (1976), the stochastic volatility model proposed by Heston (1993), models analyzed by Bakshi et al. (1997), Bates (1996) ...etc can be good fits to our general framework.

The first contribution of this paper is a formula pricing the hypothetical "two-price" call option of European type where the price processes of the underlying assets are some positive functions related to monotone transformation of the underlying state variables. In addition, to make our model more flexible, we propose to use the so-called "characteristic factor" in weighing the impact of each state variable on the underlying assets price. By manipulating the characteristic factor, our method then can be applied to multi-factor models of the diffusion component of stochastic volatility, and the general multi-dimensional affine jump-diffusion models, with much richer dynamic inter-relations among the state variables and much richer jump-distributions. The "two-price" options with the underlying asset price on the level of the affine, power, exponential form and polynomials of the state variables are universal enough to simultaneously capture all the options payoffs considered in Bakshi and Madan (2000) and Duffie et al. (2002).[2].

This paper shows that in order to price claims in incomplete markets, the conditional characteristic transform could also be used as a tool to determine the pricing measure. Upon having provided an explicit form of the conditional characteristic transform we set up the sufficient conditions, under which the conditional charac-

[1] The term "scaled-forward", which we use throughout this paper, is attributable to Bakshi and Madan (2000).

[2] In this paper, we allow ourselves to omit the applications of the method in pricing exotic options with complex payoff forms since it has been extensively carried out by Duffie et al. (2000) and Bakshi and Madan (2000).

teristic transform can be used to determine the pricing measure.[3] This also is the main contribution of the paper.

An outline of this paper is as follows. Section 2 presents the integro-differential equation for the contingent claim values. Section 3 devotes to the theory of option valuation with the conditional characteristic transform. It contains some new result; for example an explicit guidance on how to compute this transform in term of the Lévy characteristic triplet. Section 4 shows how the conditional characteristic transforms facilitate in incomplete markets. In Section 5, a pricing formula for a "two-price" option of European style written on assets, whose state variables follow a multi-dimensional Lévy process, is proposed. Concluding remarks are offered in Section 6.

2. Integro-Differential Equation for the Contingent Claim Value

We consider a stochastic intertemporal economy in a finite time horizon $T \in \mathbb{R}_+$, where the uncertainty is represented by a probability space $(\Omega, \mathcal{F}, P)$, which is equipped with a filtration $I\!F = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness. We assume that $\mathcal{F}_0$ is trivial.

In the following, the word "asset" represents a general financial instrument. The market consists of two assets, which could be stocks, bonds, or any other traded financial securities. Their prices $S_0(t)$ and $S(t)$ are $\mathcal{F}_t$-semimartingales and almost surely strictly positive for all $t \in [0, T]$.

The asset S_0 will play a particular role as a "numéraire asset". As a convention we assume that $S_0(0) = 1$. This numéraire asset can in principle be any traded asset. It is chosen for convenience, and in most cases (but not in all) as a so called "risk free" asset, with P-dynamics of the form

$$S_0(t) = \exp\left(\int_0^t R(s)ds\right), \quad S_0(0) = 1$$

i.e.,

$$\begin{aligned} dS_0(t) &= R(t)S_0(t)dt, \\ S_0(0) &= 1, \end{aligned} \tag{1}$$

where R is a given adapted stochastic process. One interpretation is that this describes a bank with the stochastic *short rate of interest rate* R. The asset S_0 in (1) is termed "locally risk free" since its infinitesimal rate of return $R(t)$ is known at time t.

We assume that the asset price S and the numéraire S_0 are influenced and completely determined by the state vector X. We suppose further that the state vector

[3] However, as the conditional characteristic transform is defined on the complex plane, our new measure must also be complex-valued.

X is a $\mathbb{R}^d$-valued Lévy process. Fix a truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$, i.e., a bounded function with compact support that satisfies $h(x) = x$ in a neighborhood of 0. We assume that the $\mathbb{R}^d$-valued state vector X is admitting a version (B, C, ν^X) of its characteristics that is *deterministic* and has the form:

$$B(t) = b_t dt, \quad C(t) = c_t dt, \quad \nu^X(dt, dx) = dt F(dx),$$

where $b \in \mathbb{R}^d$, c is a symmetric, nonnegative $d \times d$ matrix, F is a positive measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge 1)$ and satisfies $F(\{0\}) = 0$. By Jacod and Shiryaev (1987), Corollary II.4.19 such a representation always exists. In the sequel, we call (b, c, F) from the previous representation the *characteristic triplet* of the Lévy process X.

Let us consider a generic European style contingent claim with expiration date T, strike price K and claim payoff of the form $H(k(\xi' \cdot X(T)); K)$ (throughout this paper, the superscript $'$ stands for the transpose). We assume that the mapping $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is measurable and represents the underlying asset price, i.e., $S(t) = k(\xi' \cdot X(t))$. A $\mathbb{R}^d$-valued vector ξ is called the *characteristic factor*. From an economic viewpoint, the characteristic factor weighs the impact of each state variable and indicates the way how each state variable influences on the underlying asset prices. The aim of providing this factor to the framework is to extent the cope and improve the results of the Bakshi and Madan work, since it allows us to work with multi-dimensional state variables. In particular, by manipulating the characteristic factor one can yield the multi-factor volatility system of asset price, or a general affine jump-diffusion model with much richer dynamic inter-relation among the state variables. Hereafter, we call the variable $\xi' \cdot X$ the *transformed state variable*.

Throughout the paper, we make the following natural standing assumption:

STANDING ASSUMPTION 2.1. There exists an equivalent probability $Q \sim P$ on the probability space $(\Omega, \mathcal{F}, P)$, under which all asset price processes discounted by the numéraire, such that they are Q-integrable, are assumed to be Q-martingales.

In a financial literature, the measure Q is called an *equivalent martingale measure* or a *risk-neutral measure*.

This assumption is related to the no–arbitrage condition with the change of numéraire, for a discussion of this point and a general version of the fundamental theorem of asset pricing, the interested reader may consult Delbaen and Schachermayer (1994, 1998).

By the so-called risk-neutral pricing rule with the change of numéraire (see i.e., Gema et al. (1995)), the price $V(t, T; K)$ for the claim $H(k(\xi' \cdot X(T)); K)$ at time t is determined as follows:

$$V(t, T; K) = S_0(t)\mathbb{E}^Q[H(k(\xi' \cdot X(T)); K)S_0(T)^{-1}|\mathcal{F}_t]. \tag{2}$$

REMARK 2.1. In order for (2) to be meaningful. The discounted payoff $H(k(\xi' \cdot X(T)); K)S_0(T)^{-1}$ has to be Q-integrable. This is true if $H(k(\xi' \cdot X); K)$ is bounded by some affine function $S \mapsto S_0(a + bS)$. Since by assumption, the discounted asset price S/S_0 is a Q-martingale, and hence integrable. In the next section we will provide additional conditions in a general case.

Recall that if the state vector X is in the classical Brownian setting then from the strong Markov property, the value $V(t, T; K)$ admits a Feynman–Kac type representation $f(t, X(t))$, and f is the solution to the Cauchy problem associated with a linear PDE (see, e.g., Karatzas and Shreve (1988)).

Guided by the Feynman–Kac formula, we suppose that $V(t, T; K) = f(t, X(t))$ where $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, is a function of class $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ that it is twice continuously differentiable in the $\mathbb{R}^d$- valued variable $X(t)$ and once continuously differentiable in the variable t. The following proposition shows that the contingent claim value is a solution to an integro-differential equation.

PROPOSITION 2.1. Assume that $g(t,\ X(t)) \triangleq S_0(t)^{-1} f(t,\ X(t)) < \infty$ Q-a.s., then $f(t, X(t))$ satisfies the following integro-differentiable equation:

$$\begin{aligned} 0 \ =\ & D_t f(t,\ X(t_-)) + Df(t,\ X(t_-))' \cdot b + \frac{1}{2}\sum_{i,j=1}^{d} D_{ij} f(t,\ X(t_-))c^{ij} \\ & + \int_{\mathbb{R}^d} (f(t,\ X(t_-) + x) - f(t,\ X(t_-)) - Df(t,\ X(t_-))' \cdot h(x))F_t(dx) \\ & - R(t) f(t, X(t_-)), \quad Q - \text{a.s.}, \quad \text{for any } t \in [0, T], \end{aligned} \tag{3}$$

subject to the boundary condition:

$$f(T, X(T)) = H(k(\xi' \cdot X(T)); K),$$

where $D_t f$ is the derivative of f with respect to t, $Df = (D_1 f, \ldots, D_d f)$ and $(D_{ij} f)_{i,j=1,\ldots,d}$ denote the first and second derivative of f with respect to state variables X_i, $i = 1, \ldots, d$, respectively.

Proof. Recall that $X = X(0) + X^c + h * (\mu^X - \nu^X) + (x - h(x)) * \mu^X + B$ denotes the decomposition of Lévy process X (cf. Jacod and Shiryaev (1987), Theorem II.2.34), with X^c is the continuous local martingale part. By Ito's formula, we have

$$\begin{aligned} dg(t,\ X(t)) \ =\ & f(t, X(t_-))d(\frac{1}{S_0(t)}) + \frac{1}{S_0(t_-)} df(t, X(t)) = \frac{1}{S_0(t_-)} \\ & \times \Bigg\{ D_t f(t, X(t_-))dt + Df(t, X(t_-))' \cdot dM \\ & + Df(t, X(t_-))' \cdot bdt \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^d}\Big(f(t,X(t_-)+x)-f(t,X(t_-))\\
&-Df(t,X(t_-))'\cdot h(x)\Big)\mu^X(dt,dx)\\
&-R(t)f(t,X(t_-))+\frac{1}{2}\sum_{i,j=1}^{d}Df_{ij}(t,X(t_-))c^{ij}dt\Bigg\}\\
=\;&\frac{1}{S_0(t_-)}\times\Bigg\{Df(t,X(t_-))'\cdot dM+\int_{\mathbb{R}^d}\Big(f(t,X(t_-)+x)\\
&-f(t,X(t_-))-Df(t,X(t_-))'\cdot h(x)\Big)(\mu^X-\nu^X)(dx,dt)\Bigg\}+\\
&+\frac{1}{S_0(t_-)}\times\Bigg\{D_t f(t,X(t_-))dt+Df(t,X(t_-))'\cdot bdt\\
&+\int_{dR^d}\Big(f(t,X(t_-)+x)\\
&-f(t,X(t_-))-Df(t,X(t_-))'\cdot h(x)\Big)F_t(dx)dt\\
&+\frac{1}{2}\sum_{i,j=1}^{d}Df_{ij}(t,X(t_-))c^{ij}dt-R(t)f(t,X(t_-))dt\Bigg\},
\end{aligned}
\tag{4}
$$

here M is the martingale part of X and defined by:

$$
\begin{aligned}
M &= X-\sum_{s\leq t}[\triangle X(s)-h(\triangle X(s))]-X(0)-B\\
&= X^c+h*(\mu^X-\nu^X)
\end{aligned}
$$

Let us consider the right-hand side of (4): the first term is a Q-local martingale, but since Q is a risk–neutral measure and g is Q-integrable, then it is already a Q-martingale. Being a martingale, g is a Q-special semimartingale, then the last term – as a predictable process of finite variation – must vanish, and the result follows. □

As a result of the last Proposition, the value V of a considered contingent claim is the solution to the integro-differential Equation (3). From (2) it follows that Equation (3) is the analogue of the Feynman–Kac formula (see e.g., Karatzas and Shreve (1988)). The difference is that the Brownian motion is replaced by a general Lévy process, and the direction taken in the Feynman–Kac approach is the opposite of the one taken in proposition (2.1): Feynman–Kac starts with the solution of some parabolic partial differential equation. If the solution satisfies some set of regularity conditions, then it can be represented as a conditional expectation.

3. Analytical Approaches to Value European Options

From formula (2) it follows that the contingent claim valuation leads to the calculation of its conditional expected final value under the risk-neutral probability, and when the risk-neutral probability density function (PDF) of the underlying asset price is known in closed-form, claim prices can be obtained by a single integration of their payoff against this probability density function. Unfortunately, for most realistic option pricing and derivative-security valuation, the PDFs are either unknown or cannot be expressed in terms of special functions of mathematics. Therefore in order to determine the conditional expectation and hence the claim price, researchers would have directly solved contingent claim valuation equation such as in (3) (e.g., see the seminal work by Cox and Ross (1976), Cox et al. (1985) on this topic). In this paper we will investigate different approaches in computing these values: the analytical approaches.

Since the pioneering work of Heston (1993), much of the literature on option valuation has successfully applied analytical analysis to determine option prices, see Bakshi et al. (1997), Bates (1996), Bakshi and Madan (2000), for example. The key component of these works is the Fourier inversion of the conditional characteristic function, which is known in closed-form.

The conditional characteristic function is a complex-valued function which is in a one-to-one correspondence with the probability density function. The conditional characteristic function is the Fourier transform of the state-price density function (the product of the risk-neutral density and the discounting factor), therefore it inherits the smoothness and hence valuation tractability. For most valuation problems of practical interest, the valuation equations for conditional characteristic functions are remarkably simple, even though the counterpart ones for state-price density, or option prices, are not. Actually, one can use the probability theory to arrive at the characteristic functions for a comprehensive class of stochastic processes. In one extreme, a wide class of processes are recognized and mathematically represented through their characteristic functions. For example, the class of infinitely divisible processes of independent increments, or pure-jump processes. In these cases, the characteristic functions arise naturally from the Lévy Khintchine representations for such processes. For more about the role of the conditional characteristic function in finance we refer the reader to Carr and Madan (1999) and the references therein.

In this section, we generalize the above mentioned analytical techniques and switch from using the conditional characteristic function to using the so-called "conditional characteristic transform" of a transformed state variable, which, in fact, is a *generalized* conditional characteristic function.

Let us consider now a hypothetical European "two-price" call option with payoff

$$H(S_1(T), S_2(T); K_1, K_2) = (S_1(T) - K_1)1_{\{S_2(T)\geq K_2\}},$$

at date T, for given strike prices K_1 and K_2. The underlying asset prices S_1, S_2 are assumed to be positive functions with respect to the transformed state variables, and have the forms:

$$S_1 = k_1(\xi_1' \cdot X), \quad S_2 = k_2(\xi_2' \cdot X),$$

where $\xi_1, \xi_2 \in \mathbb{R}^d$ are the characteristic factors. We assume further that $k_2 : \mathbb{R} \to \mathbb{R}_+$ is a monotone function.

Define the exercise region of the option as $\chi \triangleq \{S_2(T) \geq K_2\}$. From formula (2), it follows that the t-time price $C(t, T; K_1, K_2)$ of a considered call option is defined as

$$\begin{aligned} C(t, T; K_1, K_2) \triangleq\ & V(t, T; K_1, K_2) = S_0(t)\mathbb{E}^Q[S_0(T)^{-1}(S_1(T) \\ & -K_1)1_{\{S_2(T)\geq K_2\}}|\mathcal{F}_t]. \end{aligned} \tag{5}$$

Let us define a conditional characteristic transform of a transformed state variable $(\xi' \cdot X)$ by the mapping f_X^Q: $[0, T] \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$

$$f_X^Q(t, \xi, u) \triangleq S_0(t)\mathbb{E}^Q[e^{u(\xi'\cdot X(T))}S_0(T)^{-1}|\mathcal{F}_t] \tag{6}$$

Hereafter, any $u = \phi_1 + i\phi_2 \in \mathbb{C}$ with $\phi_1, \phi_2 \in \mathbb{R}$, will be abbreviated to (ϕ_1, ϕ_2).

Notice that the conditional characteristic transform is in fact the price of a hypothetical claim that pays $\exp(u(\xi' \cdot X(T)))$ at date T with some parameter of the contract fixed by u. Ignoring extreme counterexamples, it is well acknowledged from Fourier theory that the conditional characteristic transform as a function of u is infinitely differentiable and has finite algebraic moments of all orders, hence it satisfies Equation (3), subject to a boundary condition $f_X^Q(T, \xi, u) = \exp(u(\xi' \cdot X(T)))$. The solution[4] is provided below:

$$f_X^Q(t, \xi, u) = \exp\left(A(\tau, u) + G(\tau, u)' \cdot X(t)\right) \tag{7}$$

with $\tau = T - t$ and the mappings A: $[0, T] \times \mathbb{C} \to \mathbb{C}$, G: $[0, T] \times \mathbb{C} \to \mathbb{C}^d$ satisfy the complex-valued ordinary differential equation (ODE)

$$\begin{aligned} 0\ =\ & -A_\tau - G_\tau X(t) + G' \cdot b + \frac{1}{2}G' \cdot c \cdot G - R(t) \\ & + \int_{\mathbb{R}^d} \left(\exp(G' \cdot x) - 1 - G' \cdot h(x) \right) F_t(dx), \end{aligned} \tag{8}$$

subject to the conditions

$$A(0, u) = 0, \tag{9}$$

[4] The solution is sketched in Appendix.

and

$$G(0,u) = u\xi. \tag{10}$$

Traditionally, the ordinary differential equation (ODE) (8) can be unbundled into two ODEs separately for A and G, and in some cases these ODEs have explicit solutions. These include independent square-root diffusion models for the short-rate process (see Chan et al. (1992)), the stochastic volatility models of asset prices (see Bates (1996), Bakshi et al. (1997)), or the affine jump-diffusions models (see Duffie et al. (2000)). In other cases, one can solve the ODEs for A and G numerically, even for high-dimensional applications.

REMARK 3.1. Proposition 2.1 implies that formula (6) is well-defined if for any $t \in [0,T]$, $g(t, X(t)) = S_0(t)^{-1} \exp\big(A(\tau,u) + G(\tau,u)' \cdot X(t)\big) < \infty$ Q-a.s., that is:

$$\mathbb{E}^Q[e^{u(\xi' \cdot X(T))} S_0(T)^{-1} | \mathcal{F}_t] < \infty \quad Q - \text{a.s. for any } t \in [0,T]. \tag{11}$$

To ensure that (11) holds true, we impose the following conditions:

$$|\gamma(t) - Dg(t, X(t_-))h(x)| * \nu_t^X \; < \; \infty, \tag{12}$$

$$\mathbb{E}^Q\bigg(\Big(\int_0^T \eta(t) \cdot \eta(t) dt\Big)^{\frac{1}{2}}\bigg) \; < \; \infty, \tag{13}$$

$$\mathbb{E}^Q\big(|g(T, X(T))|\big) \; < \; \infty, \tag{14}$$

Q-a.s. for any $t \in [0,T]$, where

$$\begin{aligned} \gamma(t) &= g(t, X(t_-) + x) - g(t, X(t_-)) \\ \eta(t) &= g(t, X(t_-)) G'(\tau,u) \cdot \sigma(t, X(t)), \quad with \quad c = \sigma' \cdot \sigma. \end{aligned}$$

Following the line of Duffie et al. (2000), we formulate the next definition.

DEFINITION 3.1. Given $u \in \mathbb{C}$ and a truncation function h, the state vector X is *well-behaved* at u if its conditional characteristic transform and characteristic triplet satisfy (12), (13) and (14).

We are now ready to show how powerful is the conditional characteristic transform as a tool employed in solving valuation problems in finance. Recall that the state vector in our analysis is a multi-dimensional Lévy process, which is general enough to encompass all continuous-time, discrete-time or mixed with jumps processes, used in previous works. The principle approach then will be applicable regardless of the source of uncertainty: whether in discrete-time, continuous-time, pure-jump, or mixture environments.

The next proposition generalizes Theorem 1 and other results from the work of Bakshi and Madan (2000).

PROPOSITION 2.1. Let the state vector X be a $\mathbb{R}^d$-valued Lévy process with the characteristic triplet (b, c, F), assume that X is well-behaved at $u = (0, \phi)$ for any $\phi \in \mathbb{R}$ and

$$\int_{\mathbb{R}} |f_X^Q(0, \xi, u)| d\phi < \infty.$$

Define the transform $M_S^Q(t, \xi, u)$: $[0, T] \times \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$

$$M_S^Q(t, \xi, u) \triangleq S_0(t)\mathbb{E}^Q[S_0(T)^{-1} e^{u(\xi' \cdot X(T))} S(T) | \mathcal{F}_t], \tag{15}$$

and assume that it is knowledgeable, then:

1. The price (5) of a "two-price" claim can be written as

   $$C(t, T; K_1, K_2) = F(t, T)\Pi_1(t, T; K_2) - K_1 p(t, T)\Pi_2(t, T; K_2) \tag{16}$$

   where $p(t, T)$ is the time-t of a discount bond price with the expiration time T; $F(t, T)$ is the scaled-forward price, which represents the time-t price of a commitment to delivery at time T the quantity $S_1(T)$; and $\Pi_1(t, T; K_2)$, $\Pi_2(t, T; K_2)$ are the risk–neutral probability of finishing in-the-money (the Arrow–Debreu securities).
2. The discount bond price, the scaled-forward price, and the risk-neutral probabilities can be recovered from the conditional characteristic transform $f_X^Q(t, \xi_j, u)$ and the transform $M_{S_1}^Q(t, \xi_j, u)$, $j = 1,\ 2$, as follows:

   $$p(t, T) = f_X^Q(t, \xi_1, (0, 0)) = f_X^Q(t, \xi_2, (0, 0)) \tag{17}$$
   $$F(t, T) = M_{S_1}^Q(t, \xi_1, (0, 0)) = M_{S_1}^Q(t, \xi_2, (0, 0)) \tag{18}$$
   $$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\frac{e^{-i\phi k_2^{(-1)}(K_2)} M_{S_1}^Q(t, \xi_2, (0, \phi))}{i\phi M_{S_1}^Q(t, \xi_2, (0, 0))}\right\} d\phi \tag{19}$$
   $$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\frac{e^{-i\phi k_2^{(-1)}(K_2)} f_X^Q(t, \xi_2, (0, \phi))}{i\phi f_X^Q(t, \xi_2, (0, 0))}\right\} d\phi, \tag{20}$$

   where $k_2^{(-1)}$ is the inverse function of k_2.
3. If $k_1 \in C^\infty(\mathbb{R})$, i.e., k_1 is infinitely differentiable with respect to the transformed state variable, and $\det|\xi_2| \neq 0$, i.e., ξ_2 is invertible, then

   $$M_{S_1}^Q(t, \xi_2, (0, \phi)) \simeq f_X^Q(t, \xi_2, (0, \phi)) k_1(x_1^*) + \exp(i\phi x_1^*) \sum_{n=1}^{\infty} \frac{\langle A_n, B_n \rangle}{i^n \times n!} \tag{21}$$

here $A_n \triangleq \frac{\partial^n k_1}{\partial X_1^{*n}}\Big\|_{x_1^*}$; $B_n \triangleq \xi_1^n([\xi_2]^{-1})^n \frac{\partial^n \overline{f}_X^Q}{\partial\phi^n}(t, \xi_2, (0, \phi))$; $[\xi_2]^{-1}$ is the $\mathbb{R}^d$-valued inverse vector of ξ_2, and

$$\overline{f}_X^Q(t, \xi_2, (0, \phi)) = e^{-i\phi x_1^*} f_X^Q(t, \xi_2, (0, \phi)),$$

for some constant $x_1^* \in \mathbb{R}^d$.

If in addition $\xi_1 = \xi_2 = \xi$, then (21) holds true with $B_n \triangleq \frac{\partial^n \overline{f}_X^Q}{\partial\phi^n}(t, \xi, (0, \phi))$.

Proof. 1. From (5) we have:

$$\begin{aligned} C(t, T; K_1, K_2) &= S_0(t)\mathbb{E}^Q[S_0(T)^{-1}(S_1(T) - K_1)1_\chi|\mathcal{F}_t] \\ &= S_0(t)\mathbb{E}^Q[S_1(T)S_0(T)^{-1}1_\chi|\mathcal{F}_t] - K_1 S_0(t)\mathbb{E}^Q[S_0(T)^{-1}1_\chi|\mathcal{F}_t] \end{aligned}$$

Recall that $S_0(0) = 1$, since $S_1(T)S_0(T)^{-1}$ is a strictly positive Q-martingale then we have $\mathbb{E}^Q[S_1(T)S_0(T)^{-1}] = S(0)$. Hence we may define a new probability measure Q^{S_1} by putting: $\frac{dQ^{S_1}}{dQ} \triangleq \frac{S_1(T)S_0(T)^{-1}}{S(0)}$. Bayer's formula implies that

$$\begin{aligned} S_0(t)\mathbb{E}^Q[S_1(T)S_0(T)^{-1}1_\chi|\mathcal{F}_t] &= S_0(t)\mathbb{E}^Q[S_1(T)S_0(T)^{-1}|\mathcal{F}_t]\mathbb{E}^{Q^{S_1}}[1_\chi|\mathcal{F}_t] \\ &= F(t, T)\Pi_1(t, T; K_2) \end{aligned}$$

where the scaled-forward price $F(t, T)$ is defined as

$$F(t, T) \triangleq S_0(t)\mathbb{E}^Q[S_1(T)S_0(T)^{-1}|\mathcal{F}_t] \tag{22}$$

and

$$\Pi_1(t, T; K_2) = \mathbb{E}^{Q^{S_1}}[1_\chi|\mathcal{F}_t], \tag{23}$$

is the conditional probability that the option finishes in-the-money under the martingale measure Q^{S_1}. Using similar arguments as for the second integral we have

$$K_1 S_0(t)\mathbb{E}^Q[S_0(T)^{-1}1_\chi|\mathcal{F}_t] = K_1 p(t, T)\Pi_2(t, T; K_2)$$

where

$$p(t, T) \triangleq S_0(t)\mathbb{E}^Q[S_0(T)^{-1}|\mathcal{F}_t] \tag{24}$$

$$\Pi_2(t, T; K_2) \triangleq \mathbb{E}^{Q^T}[1_\chi|\mathcal{F}_t], \tag{25}$$

with the probability measure Q^T determined from the relation $\frac{dQ^T}{dQ} = \frac{S_0(T)^{-1}}{\mathbb{E}^Q[S_0(T)^{-1}]}$.

In summary, we have shown that the price at time t of a considered call option is given by

$$C(t, T; K_1, K_2) = F(t, T)\Pi_1(t, T; K_2) - K_1 p(t, T)\Pi_2(t, T; K_2)$$

2. We now derive formulae (17), (18, (19), (20) and (21) from (6) and (16). By the definitions of $p(t,T)$ and $F(t,T)$ in (22) and (24), clearly we have (17) and (18). We are now ready to prove (19) and (20).

Since $\Pi_j(t,T;K_2)$ with $j = 1, 2$ are the risk-neutral probabilities, hence it is enough to be able to compute the characteristic functions of $\widetilde{\Pi}_j(t,T;\phi)$ of $\Pi_j(t,T;K_2)$, for then well-known Fourier inversion methods can be used to compute terms of the form $\Pi_j(t,T;K_2)$:

$$\Pi_j(t,T;K_2) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\frac{e^{-i\phi k_2^{(-1)}(K_2)}\widetilde{\Pi}_j(t,T;\phi)}{i\phi}d\phi.$$

The characteristic functions correspond to $\Pi_j(t,T;K_2)$, are given as follows (see (9.1.1) Lukcs (1960)):

$$\widetilde{\Pi}_1(t,T;\phi) = \mathbb{E}^{Q^{S_1}}[e^{i\phi X_2^*}|\mathcal{F}_t], \quad \text{and} \quad \widetilde{\Pi}_2(t,T;\phi) = \mathbb{E}^{Q^T}[e^{i\phi X_2^*}|\mathcal{F}_t],$$

By Bayer's formula we get the desired results: (19) and (20).

3. Since k_1 is smooth and infinitely differentiable, taking Taylor series of $k_1(X_1^*)$ around x_1^* and reformulating the transform $M_{S_1}^Q(t,\xi_2,(0,\phi))$ we should get (21). This completes the proof to proposition 3.1. □

Proposition 3.1 shows that the "two-price" call option of European type can be valued without any necessary specification on the asset price or interest rate, but only with the contracts that mandates the knowledge of the transform $M_{S_1}^Q(t,\xi_2,(0,\phi))$. Clearly, this transform can not be explicitly demonstrated for an arbitrary contract. However, as the next corollary shows, when the claim is contingent on the level of a transformed state variable, on its powers, on its exponential, and/or on polynomials, the transform $M_{S_1}^Q(t,\xi_2,(0,\phi))$ can be obtained from the corresponding conditional characteristic transform. As a result, in these cases the valuation equation for the conditional characteristic transform (7) is sufficient to recover the call option price and other underlying primitive claim prices such as the discount bond or the forward contract.

COROLLARY 3.1. Assume that $\det|\xi_2| \neq 0$, $k_2\colon \mathbb{R} \to \mathbb{R}_+$ is a monotone function, and $k_1\colon \mathbb{R} \to \mathbb{R}_+$ is in the form:

$$k_1(x) = \sum_{n=1}^{N} a_n x^n + \sum_{m=1}^{M} b_m e^{c_m x} + \sum_{p=1}^{P} d_p e^{g_p x} x^p, \tag{26}$$

for a_n, b_m, c_m, d_p, g_p are arbitrary constants with $n = 1, \ldots, N$; $m = 1, \ldots, M$; and $p = 1, \ldots, P$, then

1.

$$p(t,T) = f_X^Q(t,\xi_j,(0,0)) \quad j = 1,2 \tag{27}$$

$$\widetilde{\Pi}_2(t,T;\phi) = \frac{f_X^Q(t,\xi_2,(0,\phi))}{f_X^Q(t,\xi_j,(0,0))}, \tag{28}$$

$$F(t,T) = \left\{ \sum_{n=1}^{N} \frac{a_n[\xi_2]^{-1} \cdot \xi_1}{i^n} \times \frac{\partial^n f_X^Q(t,\xi_2,(0,\phi))}{\partial \phi^n} \right\|_{\phi=0} \tag{29}$$

$$+ \sum_{m=1}^{M} b_m f_X^Q(t,\xi_2,(c_m \xi_1 [\xi_2]^{-1}, 0))$$

$$\left. + \sum_{p=1}^{P} \frac{d_p[\xi_2]^{-1} \cdot \xi_1}{i^p} \times \frac{\partial^p f_X^Q(t,\xi_2,(g_p \xi_1 [\xi_2]^{-1}, \phi))}{\partial \phi^p} \right\|_{\phi=0} \Bigg\} \tag{30}$$

$$\widetilde{\Pi}_1(t,T;\phi) = \frac{1}{iF(t,T)} \times \left\{ \sum_{n=1}^{N} \frac{a_n[\xi_2]^{-1} \cdot \xi_1}{i^n} \times \frac{\partial^n f_X^Q(t,\xi_2,(0,\phi))}{\partial \phi^n} \right.$$

$$+ \sum_{m=1}^{M} b_m f_X^Q(t,\xi_2,(c_m \xi_1 [\xi_2]^{-1}, \phi))$$

$$\left. + \sum_{p=1}^{P} \frac{d_p[\xi_2]^{-1} \cdot \xi_1}{i^p} \times \frac{\partial^p f_X^Q(t,\xi_2,(g_p \xi_1 [\xi_2]^{-1}, \phi))}{\partial \phi^p} \right\}. \tag{31}$$

2. In addition, if $\xi_1 = \xi_2 = \xi$ then

$$p(t,T) = f_X^Q(t,\xi,(0,0)), \tag{32}$$

$$\widetilde{\Pi}_2(t,T;\phi) = \frac{f_X^Q(t,\xi,(0,\phi))}{f_X^Q(t,\xi,(0,0))}, \tag{33}$$

$$F(t,T) = \left\{ \sum_{n=1}^{N} \frac{a_n}{i^n} \times \frac{\partial^n f_X^Q(t,\xi,(0,\phi))}{\partial \phi^n} \right\|_{\phi=0} + \sum_{m=1}^{M} b_m f_X^Q(t,\xi,(c_m,0))$$

$$\left. + \sum_{p=1}^{P} \frac{d_p}{i^p} \times \frac{\partial^p f_X^Q(t,\xi,(g_p,\phi))}{\partial \phi^p} \right\|_{\phi=0} \Bigg\} \tag{34}$$

$$= p(t,T) \times \left\{ \sum_{n=1}^{N} \frac{a_n}{i^n} \times \frac{\partial^n \widetilde{\Pi}_2(t,T;\phi)}{\partial \phi^n} \right\|_{\phi=0}$$

$$+ \sum_{m=1}^{M} b_m \widetilde{\Pi}_2(t,T;-ic_m)$$

$$\left. + \sum_{p=1}^{P} \frac{d_p}{i^p} \times \frac{\partial^p \widetilde{\Pi}_2(t,T;\phi)}{\partial \phi^p} \right\|_{\phi=-ig_p} \Bigg\} \tag{35}$$

$$\widetilde{\Pi}_1(t,T;\phi) = \frac{1}{iF(t,T)} \times \left\{ \sum_{n=1}^{N} \frac{a_n}{i^n} \times \frac{\partial^n f_X^Q(t,\xi,(0,\phi))}{\partial \phi^n} \right.$$

$$
\begin{aligned}
& + \sum_{m=1}^{M} b_m f_X^Q(t, \xi, (c_m, \phi)) \\
& \left. \sum_{p=1}^{P} \frac{d_p}{i^p} \times \frac{\partial^p f_X^Q(t, \xi, (g_p, \phi))}{\partial \phi^p} \right\} \\
= \; & \frac{p(t,T)}{iF(t,T)} \times \left\{ \sum_{n=1}^{N} \frac{a_n}{i^n} \times \frac{\partial^n \widetilde{\Pi}_2(t, T; \phi)}{\partial \phi^n} \right. \\
& + \sum_{m=1}^{M} b_m \widetilde{\Pi}_2(t, T; \phi - i c_m) \\
& \left. + \sum_{p=1}^{P} \frac{d_p}{i^p} \times \frac{\partial^p \widetilde{\Pi}_2(t, T; \phi - i g_p)}{\partial \phi^p} \right\}
\end{aligned} \tag{36}
$$

Proof. Observer that $M_{S_1}^Q(t, \xi_2, (0, \phi))$ is obtained by translating and differentiating the transform $f_X^Q(t, \xi_2, (0, \phi))$ with respect to ϕ. Hence we are able to calculate explicitly the formulae of the characteristic function $\widetilde{\Pi}_1(t, T;\ \phi)$ and $\widetilde{\Pi}_2(t, T;\ \phi)$. □

REMARK 3.2.

1. The conditional characteristic transform at $(0, \phi)$, $f_X^Q(t, \xi, (0, \phi))$, is in fact the conditional characteristic function of a state vector X. Proposition 3.1 shows that if the conditional characteristic transform f_X^Q is available in closed-form and k_1 is continuously infinitely differentiable, then the option price (15), the scaled-forward price (17) and two risk-neutral probabilities (18)–(19) can be numerically approximated with the use of (21).
2. The general case, in which the characteristic factors ξ_1, ξ_2 or parameters appeared in (26) are deterministic, time-dependent and bounded functions, can be similarly treated.

Corollary 3.1 shows that knowing the conditional characteristic transform $f_X^Q(t, \xi, u)$ in closed-form enables us to value derivative-security with payoff at a future time T of the form $(k_1(\xi_1' \cdot X(T)) - K_1)1_{k_2(\xi_2' \cdot X(T)) \geq K_2}$, with $k_1(x)$ given as in (26), and $k_2(x)$ is monotone. This payoff form is general enough to encompass many options of different types available in the market such as plain-vanilla options on currencies and equities, quanto options (such as an option on a common stock or bond struck in a different currency), options on zero-coupon bonds, caps, floors, chooser options, "slope-of-the-yield-curve" options and certain Asian options ... etc. We refer the reader to Duffie et al. (2000) for examples.

Corollary 3.1 also shows that in some case it suffices to manipulate the conditional characteristic transform in order to simultaneously and jointly recover (i) the discount bond price (ii) the scaled-forward price; and (iii) the two neutral-risk probabilities of finishing in-the-money (the Arrow–Debreu securities prices or the delta claims).

Our framework significantly extends the result of Bakshi and Madan (2000) and Duffie et al. (2000). In fact, the last Corollary covers all models considered by the last mentioned authors. In particular, the Poisson jump-diffusion model proposed by Duffie et al. (2000) can be regarded as an example of our framework, where the three components of the characteristic triplet of the state vector X are affine in X.

With the use of state variables of a Lévy system, our framework generalizes and outperforms the affine Poisson jump-diffusion framework. While the limitation of the latter arises due to the exclusive use of compound Poisson processes to model jumps, these processes generate only a finite number of jumps within a finite time interval, our model considers more general jump structures, which permit an infinite number of jumps to occur within any finite time interval.

In addition, the presence of the characteristic factors along with the state vector provides us with much more flexibility in dealing with the multi-dimensional framework, and allows us to work with much richer dynamic inter-relations among the state variables.

Moreover, the superiority of working under this framework, as Bakshi and Madan pointed out, is that, while the two characteristic functions corresponding to two risk-neutral probabilities of finishing in-the-money in Heston (1993), Bates (1996), Bakshi et al. (1997) are, for instance, obtained mostly by solving two separately valuation equations (see equations (12) and (22) in Heston), we only need to solve a single valuation integro-differential Equation (8) in order to obtain a conditional characteristic transform.

4. Change of Measure or Option Pricing in Incomplete Markets

Although the general theory of Lévy processes is usually formulated in terms of semimartingales. In this section, we restrict our attention to special semimartingales since these are a little easier to understand from an intuitive point of view. This allows us to replace the truncation function $h\colon \mathbb{R}^d \to \mathbb{R}^d$ appearing in the valuation equation (8) with the identity mapping $h\colon \mathbb{R}^d \to \mathbb{R}^d$, $x \to x$. In the case of Lévy processes, our restriction means that we consider only those with existing first moment, (cf. Jacod and Shiryaev (1987), Proposition II.2.29a) and we exclude, e.g., α-stable Lévy process with $\alpha \leq 1$ (cf. Kallsen (1998)).

For future reference, we denote by $\wp^d$ the predictable σ-field on $[0, T] \times \mathbb{R}^d$.

Now let us consider one of the most frequently encountered market dynamics in the financial literature, where the state of a system is governed by the SDE:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t) + \int_E w(t, X(t_-), x)\widetilde{p}(dt, dx) \tag{37}$$

$$X(0) = X_0, \quad X_0 \in \mathbb{R}^d \tag{38}$$

where X, a $\mathbb{R}^d$-valued semimartingale; W, a $\mathbb{R}^d$-valued standard Wiener process; b, a $\wp^d$-measurable mapping $[0, T] \times \mathbb{R}^d \to \mathbb{R}^d$; σ, a $\wp^{d\times d}$-measurable mapping $[0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$; w, a $(\wp^d \otimes \mathcal{E})$-measurable mapping $[0, T] \times \mathbb{R}^d \times E \to \mathbb{R}^d$; $\widetilde{p}$, the compensated martingale $\widetilde{p} \triangleq p - q$, where p is a $\wp \otimes \mathcal{E} - \sigma$-finite integer-valued random measure (adapted to the filtration) with compensator $q(dt, dx) \triangleq dtQ(dx)$.

Here $(E, \mathcal{E})$ is a Lusin space (in applications, usually, $E = \mathbb{R}^d$ or $E = I\!N$ or a finite set). Equation (37) implies that we have $\triangle X(t) = \int w(t, X(t_-), x)p(t \times dx)$ for any $t \in \mathbb{R}_+$. Since p is an integer-valued random measure, we have $\triangle X(\omega, t) \in \Gamma \setminus \{0\}$ if and only if $p(\omega, t, x) = 1$ for some x with $w(t, X(t), x) \in \Gamma \setminus \{0\}$. This yields

$$\mu^X([0, t] \times \Gamma) = \int_{[0,t]\times E} 1_{\Gamma\setminus\{0\}}(w(s, X(s_-), x))p(ds, dx) \tag{39}$$

$$\nu^X([0, t] \times \Gamma) = \int_{[0,t]\times E} 1_{\Gamma\setminus\{0\}}(w(s, X(s_-), x))q(ds, dx), \tag{40}$$

for any $t \in \mathbb{R}_+$, $\Gamma \in \mathcal{B}^d$. The coefficients must be such that all integrals are well-defined. Furthermore, we shall make the following assumptions:

STANDING ASSUMPTION 4.1.

1. (The linear growth condition.) For every $T > 0$ there exists a measurable mapping M: $[0, T] \to \mathbb{R}_+$ such that

$$|b(t, x)|^2 + |\sigma(t, x)|^2 + \int_E |w(t, x, y)|^2 Q(dy) \le M(t)(1 + |x|^2) \tag{41}$$

for all $(t, x) \in ([0, T], \mathbb{R}^d)$.

2. (The local Lipschitz condition.) For every $T > 0$ and $R > 0$ there exists a positive constant $M(t; R)$ such that

$$\begin{aligned}|b(t, x) \; - \; b(t, x')|^2 + |\sigma(t, x) - \sigma(t, x')|^2 \\ + \int_E |w(t, x, y) - w(t, x', y)|^2 Q(dy) \le M(t; R)|x - x'|^2,\end{aligned} \tag{42}$$

for all $t \in [0, T]$, $|x| \le R$, $|x'| \le R$.

3. $$(|x|^2 \wedge |x|) * \nu^X < \infty. \tag{43}$$

Assumptions on b, σ, w guarantee the existence and the uniqueness of a strong Markov solution with an initial condition to (37) (see Gihman and Skorohod (1972), or Kallsen (1998)), and the condition (43) assures that X is a special semimartingale.

Since we are dealing mainly with the transformed state variable $X^* = \xi' \cdot X$, let us slightly extend the notion of a characteristic factor.

DEFINITION 4.1. Let X be a P-special semimartingale in $\mathbb{R}^d$, $\xi \in \mathbb{R}^d$ is called an admissible characteristic factor if $X^* = \xi' \cdot X$ is also a P-special semimartingale.

Notice that if ξ is bounded in $\mathbb{R}^d$ then clearly it is an admissible characteristic factor. We denote by $\mathcal{A}^d$ the set of all $\mathbb{R}^d$-valued admissible characteristic factors. Note that X^* may not be special even if X is special (and vice versa).

In complete markets, there is a unique measure which is equivalent to the canonical measure (the "real world" measure) and which makes the integrable discounted price process a martingale. The unique fair price of a contingent claim is then the expectation of the discounted payoff at maturity under this martingale measure.

With the dynamics of the state vector given as in (37)–(38), we suppose that there should be many equivalent martingale measures under which the discounted price process is a martingale. Since there does not exist a unique equivalent martingale measure, the value of a European contingent claim corresponds to the expected value of the discounted terminal payoff with respect to a not necessarily unique equivalent martingale measure.

One of the most important problem one faces in incomplete markets is which martingale measure to choose as a pricing measure. Upon the pricing measure Q has been chosen, the price of any contingent claim can be defined as in (2) and calculated by application the framework described in the previous section. Many different approaches in selecting an appropriate martingale measure among all available equivalent martingale measures have been proposed in recent years but there is as yet no definitive way of choosing the "right" pricing measure (in a relative sense).

In the previous section, we know that using the conditional characteristic transform we can value not only the options, but also the discount bonds, the forward contracts and the Arrow–Debreu securities. In fact many exotic options with complex payoff forms can also be valued under the aforementioned manner (see Bakshi and Madan (2000), Duffie et al. (2000) for more details). From a practical viewpoint, an often posed question is: If the conditional characteristic transform is available in closed-form, is one able to determine the pricing measure?. In the next theorem, we propose another *candidate* to be a pricing measure, which is formulated with the use of a conditional characteristic transform.

In the sequel, we identify a probability $Q \sim P$ with its density process $Z = (Z_t)_{0 \le t \le T}$, $Z_t = \mathbb{E}[dQ/dP|\mathcal{F}_t]$, $Z_0 = 1$.

THEOREM 4.1. Assume that the well-behaved state vector X is governed by Equation (37), $\xi \in \mathcal{A}^d$, $k \in C^2(\mathbb{R})$ and $S = k(X^*)$ is a P-special semimartingale.

1. The probability measure $Q \sim P$, whose density process Z determined by the following relation

$$Z_t \triangleq \frac{f_X^P(t, \xi, u)}{f_X^P(0, \xi, u) S_0(t)}$$

$$= \mathcal{E}\Big(\int_0^t G' \cdot \sigma dW(s) + (e^{(G' \cdot x)} - 1) * (\mu^X - \nu_t^X)\Big), \tag{44}$$

(here $\mathcal{E}$ is the Doléans-Dade exponential) belongs to the set of equivalent martingale measures if a $(\mathcal{P} \otimes \mathcal{B})$-measurable mapping $Y \colon \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ and a predictable process $\beta \colon \Omega \times \mathbb{R}_+ \to \mathbb{C}$ defined by

$$\beta(\omega, t) = G'(\tau, u) \cdot \sigma(t, X(t)) \tag{45}$$

$$Y\big(\omega, t, (k(\xi' \cdot X(t_-) + \xi' \cdot x) - k(\xi' \cdot X(t_-))\big) = \exp(G'(\tau, u) \cdot x), \tag{46}$$

P-a.s. for any $t \in [0, T]$, satisfy the following conditions:

(a) $$|s(Y(t, s) - 1)| * \nu_t^S < \infty \tag{47}$$

(b) $$\int_0^t |\beta(t) c^S| ds < \infty \tag{48}$$

c) $$\int_0^t \beta(t)^2 c^S ds < \infty \tag{49}$$

(d) $$(b^S - R(t)S(t)) + \sigma^S \beta(t) + S \int_{\mathbb{R}^d} s|Y(t, s) - 1| F^S(dt) = 0. \tag{50}$$

P-a.s. for any $t \in [0, T]$, where (b^S, c^S, F_t^S) is the characteristic triplet of a special semimartingale S and determined by

$$\begin{aligned} b^S(t) &= Dk(X^*(t_-))\xi' \cdot b + \frac{1}{2} D^2 k(X^*(t_-))\xi' \cdot c \cdot \xi \\ &\quad + \int_{\mathbb{R}^d} \big(k(X^*(t_-) + \xi' \cdot x) - k(X^*(t_-)) \\ &\quad Dk(X^*(t_-))\xi' \cdot x\big) F_t(dx) \end{aligned} \tag{51}$$

$$c^S = (Dk(X^*(t_-)))^2 \xi' \cdot c \cdot \xi \tag{52}$$

$$F_t^S(\Gamma) = \int_{\mathbb{R}^d} 1_{\Gamma \setminus \{0\}} \big(k(X^*(t_-) + \xi' \cdot x) - k(X^*(t_-))\big) F_t(dx) \tag{53}$$

2. Under the measure Q, we have:

$$f_X^Q(t, \xi, u) = f_{\widetilde{X}}^P(t, \xi, u),$$

where $\widetilde{X}$ is a $\mathbb{R}^d$-valued Lévy process with characteristic triplet $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ where:

$$\widetilde{b} = b + G'(\tau, u) \cdot \sigma(t, X(t)) \tag{54}$$

$$\widetilde{c} = c \tag{55}$$

$$\widetilde{F}_t(\Gamma) = \int 1_{\Gamma \setminus \{0\}}(s) Y(s, t) F_t(ds). \tag{56}$$

Proof. First, notice that since the conditional characteristic transform is defined as the expected value of the complex exponential of the transformed state variable, the measure $Q \sim P$ defined in (44) must also be complex-valued.

1. Regardless of whether X is a special semimartingale or only a semimartingale, we have:

$$\mu^S([0,t]\times\Gamma) = \int_{[0,t]\times\mathbb{R}^d} 1_{\Gamma\setminus\{0\}}(k(X^*(s_-)+\xi'\cdot x) \\ -k(X^*(s_-)))\mu^X(ds,dx) \tag{57}$$

and hence for its compensator ν^S

$$\nu^S([0,t]\times\Gamma) = \int_{[0,t]\times\mathbb{R}^d} 1_{\Gamma\setminus\{0\}}(k(X^*(s_-)+\xi'\cdot x) \\ -k(X^*(s_-)))\nu^X(ds,dx), \tag{58}$$

for any $t \in [0,T]$ and any $\Gamma \in \mathcal{B}^d$.

Since X is well-behaved, $\xi \in \mathcal{A}^d$, $k \in C^2(\mathbb{R})$ and $S = k(\xi'\cdot X)$ is a special semimartingale, an application of Ito's lemma leads to that S is a Lévy process with characteristic triplet (b^S, c^S, F^S) determined by (51), (52) and (53).

With slight extension of Girsanov's theorem and the Doléans-Dade Exponential formula to a complex space, (see, e.g., Jacod and Shiryaev (1987), Chapter I.4f), after some standard stochastic calculations, (8), (37) and (38) imply that

$$Z_t = \mathcal{E}\left(\int_0^t G'\cdot\sigma dW(s) + (\exp(G'\cdot x)-1)*(\mu^X-\nu^X)\right).$$

From (57) and (46), we have

$$\begin{aligned}\triangle(Y(t,s)-1)*(\mu^S-\nu^S)_t &= \int(Y(t,s)-1)\mu^S(\{t\}\times ds)\\ &= \int_{\mathbb{R}^d}(Y(t,k(X^*+\xi'\cdot x)-k(X^*))-1)\mu_t^X(dx)\\ &= \int_{\mathbb{R}^d}(e^{G'\cdot x}-1)\mu_t^X(dx)\\ &= \triangle(e^{G'\cdot x}-1)*(\mu^X-\nu^X)_t,\end{aligned}$$

for any $t \in [0,T]$. It follows then from Jacod and Shiryaev, Chapter I.4.19 that

$$(Y-1)*(\mu^S-\nu^S) = (e^{G'\cdot x}-1)*(\mu^X-\nu^X)$$

and hence

$$Z(Y-1)*(\mu^S-\nu^S) = Z(e^{G'\cdot x}-1)*(\mu^X-\nu^X)$$

up to indistinguishability (cf. Jacod and Shiryaev, chapter II.I.30). However from (45) we have

$$Z_t\beta(t)dW(t) = Z_tG'(\tau,u)\cdot\sigma(t)dW(t)),$$

then an application of Ito's formula for complex-valued processes, along with (44) imply

$$\begin{aligned}
\mathcal{E}\bigg(& \textstyle\int_0^t \; G'(T-s,u)\cdot\sigma dW(s) + (\exp(G'(T-s,u)\cdot x)-1) * (\mu^X - \nu^X)\bigg) \\
&= \mathcal{E}\bigg(\int_0^t \beta(t) dW(s) + (Y(t,s)-1) * (\mu^S - \nu^S)\bigg).
\end{aligned}$$

Under the conditions (47), (48), (49), and (50), by Girsanov's theorem for complex-valued version (cf. Jacod and Shiryaev (1987), chapter III.3.24 or Kallsen (1998), chapter II.2.6) or by Ito's lemma for complex-valued processes (see e.g., Protter (1990), Theorem II.35), from (44), (37) and (1) it is not difficult to verify that ZSS_0^{-1} is a P-martingale, i.e., Z is an equivalent martingale measure.

2. A slight extension of Girsanov's theorem to complex-valued measure implies that under Q we have:

$$dW^Q(t) = dW(t) - G'\cdot\sigma dt \tag{59}$$

$$F_t^Q(\Gamma) = \int 1_{\Gamma\setminus\{0\}}(s) Y(t,s) F_t ds, \tag{60}$$

where Y defined as in (51). Therefore, any P-special semimartingale X with characteristic triplet (b, c, F) becomes Q-special semimartingale with characteristic triplet $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ defined as in (55), (56) and (57). And this completes the proof. □

REMARK 4.1. Sufficient conditions for $k(X)$ to be a special semimartingale are each of the following (cf. Kallsen (1998, chapter II.2.5)):

1. For any $t \in \mathbb{R}_+$ one has $(|x|^2 \wedge |x|) * \nu_t^{k(X)} < \infty$ P-a.s.
2. There is some $M \in \mathbb{R}_+$ such that for any $X \in \mathbb{R}^d$ one has $\| Dk(X) \| \leq M$.

5. Applications

The purpose of this section is to derive a explicit formulation for a triple-jump diffusion option-pricing model that includes many option-pricing models as special case. The dynamics structure introduced here is not new. However, the aim of this concreted example is to present the framework of pricing option in the manner described in Section 3. For more on application of this method, we refer the reader to Bakshi and Madan (2000) and Duffie et al. (2000).

As such, it is then convenient to follow a standard practice and specify from the outset a stochastic structure under a risk-neutral probability measure. In the sequel we will use the convention that if a parameter is time-dependent this is written explicitly, otherwise all parameters are constant.

Example. Jump-log model with jump-stochastic interest rate and jump-stochastic volatility.

We suppose that the numéraire is governed by (1), and the underlying nondividend-paying stock price $S(t) = S(0)\exp(X(t))$, where the state vector X is governed by the following system:

$$
\begin{aligned}
dX(t) &= \big(R(t) - \frac{1}{2}V(t)\big)dt + \sqrt{V(t)}dW_x(t) \\
&\quad + \int_{-\infty}^{+\infty} x(p_x - q_x)(dx, dt), \quad X(0) = 1 \qquad (61)\\
dV(t) &= (\theta_v - \kappa_v V(t))dt + \sigma_v\sqrt{V(t)}dW_v(t) \\
&\quad + \int_{-\infty}^{+\infty} v(p_v - q_v)(dv, dt), \quad V(0) = 1 \qquad (62)\\
dR(t) &= (\theta_r - \kappa_r R(t))dt + \sigma_r\sqrt{R(t)}dW_r(t) + \int_{-\infty}^{+\infty} r(p_r - q_r)(dr, dt) \qquad (63)\\
R(0) &= 1,
\end{aligned}
$$

where $R(t)$, the time-t instantaneous spot interest rate, which is assumed to be nonnegative; $V(t)$, the variance process, which is assumed to be nonnegative; W_x, W_r, W_v, the standard Brownian motions with

$$
\begin{aligned}
Cov[dW_x(t), dW_v(t)] &= \rho dt \\
Cov[dW_r(t), dW_v(t)] &= 0 \\
Cov[dW_r(t), dW_x(t)] &= 0;
\end{aligned}
$$

p_x, p_v, p_r, the homogeneous Poisson random measures with compensator; $q_x = \lambda_x \otimes H_x$; $q_r = \lambda_r \otimes H_r$; $q_v = \lambda_v \otimes H_v$, respectively (cf. Jacod and Shiryaev, II.1.20). We assume that H_x, H_v, H_r are fixed measures on $(\mathbb{R}, \mathcal{B})$, which satisfy

$$
\begin{aligned}
\int (|x^2| \wedge |x|)H_j(dx) &< \infty, \quad j = x, v, r \\
\int e^x 1_{[1,\infty)}(x)H_x(dx) &< \infty,
\end{aligned}
$$

where the first condition is imposed to ensure that $X(t)$, $V(t)$, $R(t)$ are well-defined Lévy processes, the last one assures that $\exp(X)$ is also a well-defined Lévy process.

θ_j, κ_j with $j = v, r$ are positive.

REMARK 5.1. An additional problem in applying the proposed model (61)–(63) is to keep the variance process $V(t)$ and interest rate process $R(t)$ nonnegative.[5]

In the case when there are no jumps at all in the two mentioned processes. Processes in (62)–(63) are the well-known mean-reverting square root processes

[5] We thank the referee for pointing this out.

used by Cox et al. (1985)) in modeling the term structure, and an examination of the boundary classification criteria shows that if

$$\theta_v > 0, \quad \kappa_v > 0, \quad \frac{\theta_v}{2} \geq \sigma_v^2$$

and

$$\theta_r > 0, \quad \kappa_r > 0, \quad \frac{\theta_r}{2} \geq \sigma_r^2$$

then the upward drifts are sufficiently large to make the origin inaccessible, and this implies that an initially nonnegative value $V(t)$ and $R(t)$ can never subsequently become negative (cf. Cox et al. (1985, p. 391)). However, in our general framework, it is still an open problem to determine the conditions, under which the two considered processes never jump into negative values.

To simplify further the example, we assume that the jumps of the state vector (X, V, R) are uncorrelated.

Without loss of generality we assume that $S(0) = 1$. The variables x, r, v in the formulae (61)–(63) are the square integrable random jump relative sizes of the state vector X, R and V. Alternatively, the integral expressions in (61), (62) and (63) can be written in the following way: for $j \triangleq x, v, r$ then $\mu_j(t) = \int_0^t \int q_j(dx, ds)$ is a Poisson-counting process with parameter λ_j and arrival time $T_i^j = \inf\{t | N_j(t) = i\}$. Furthermore, $\left(L_i^j \triangleq \int_{T_{j-1}}^{T_j} \int xq_j(dx, dt)\right)$ is a sequence of independent identically distributed random variables of the distribution H_j. Then

$$q_j(\omega; dx, dt) = \sum_i 1_{[T_i^j(\omega)]} \delta_{(L_i^j(\omega), T_i^j(\omega))}(dx, dt),$$

where δ_a denotes the Dirac measure in point a and 1 the indicator function. With $J_j = \sum_i L_i^j 1_{(T_{i-1}^j, T_i^j)}$ follows:

$$\int_{-\infty}^{+\infty} x(p_x - q_x)(dx, dt) = \left(J_x(t) d\mu_x(t) - \lambda_x \mathbb{E}[L_i^x] dt\right) \tag{64}$$

$$\int_{-\infty}^{+\infty} v(p_v - q_v)(dv, dt) = \left(J_v(t) d\mu_v(t) - \lambda_v \mathbb{E}[L_i^v] dt\right) \tag{65}$$

$$\int_{-\infty}^{+\infty} r(p_r - q_r)(dr, dt) = \left(J_r(t) d\mu_r(t) - \lambda_r \mathbb{E}[L_i^r] dt\right), \tag{66}$$

where the expected percentage jump size in the state variables $\mathbb{E}[L_i^j]$ with $j = x$, v, r defined by

$$b_j \triangleq \mathbb{E}[L_i^j] = \frac{1}{\lambda_j} \int_{\mathbb{R}} xq_j(dx). \tag{67}$$

The state vector $X = (R, V, X)'$ is a vector of Lévy processes with the characteristic triplet as follows:

$$b = \begin{bmatrix} \theta_r - \kappa_r R \\ \theta_v - \kappa_v V \\ R - \frac{1}{2}V \end{bmatrix} \tag{68}$$

$$c = \begin{bmatrix} \delta_r^2 R & 0 & 0 \\ 0 & \delta_v^2 V & \delta_v V\rho \\ 0 & \delta_v V\rho & V \end{bmatrix} \tag{69}$$

$$F_t(\Gamma) = \begin{bmatrix} \lambda_r \int 1_{\Gamma\setminus\{0\}} H_r(dx) \\ \lambda_v \int 1_{\Gamma\setminus\{0\}} H_v(dx) \\ \lambda_x \int 1_{\Gamma\setminus\{0\}} H_x(dx) \end{bmatrix}, \tag{70}$$

for any $t \in \mathbb{R}_+$, $\Gamma \in \mathcal{B}$.

Suppose we have to value a European call option with the payoff of the form $H(k(\xi' \cdot X(t)); K) = (k(\xi' \cdot X(t)) - K)^+$ where $\xi' = [0, 0, 1]$ and $k(x) = \exp(x)$. By Corollary (3.1), the option price is $C(t, T; K) = F(t, T)\Pi_1(t, T; K) - Kp(t, T)\Pi_2(t, T; K)$ where

$$p(t, T) = f_X(t, \xi, (0, 0)) \tag{71}$$

$$F(t, T) = f_X(t, \xi, (1, 0)) \tag{72}$$

$$\widetilde{\Pi}_1(t, T; x) = \frac{f_X(t, \xi, (1, \phi))}{iF(t, T)} \tag{73}$$

$$\widetilde{\Pi}_2(t, T; x) = \frac{f_X(t, \xi, (0, \phi))}{p(t, T)}. \tag{74}$$

It remains to verify the conditional transform characteristic $f_X(t, \xi, u)$ for $u = (0, 0)$, $u = (1, 0)$, $u = (1, \phi)$ and $u = (0, \phi)$. Upon this function is derived in closed-form, we should have the option-pricing formula in closed-form.

Now let us fix any $u \in \mathbb{C}$. Using (8)–(11) we guess $G(\tau, u)$ in Equation (8) has the form

$$G(\tau, u) = \begin{bmatrix} C(\tau, u) \\ D(\tau, u) \\ u \end{bmatrix},$$

then the conditional characteristic transform satisfies:

$$f_X(t, \xi, u) = \exp(A(\tau, u) + C(\tau, u)R + D(\tau, u)V + uX(t))$$

where

$$\begin{aligned}
0 &= -A_\tau - C_\tau R - D_\tau V + C(\theta_r - \kappa_r R) + D(\theta_v - \kappa_v V) \\
&\quad + u(R - \frac{1}{2}V) + \frac{1}{2}C^2\delta_r^2 R + \frac{1}{2}D^2\delta_v^2 V + uD\delta_v V\rho - \frac{1}{2}u^2 V \\
&\quad + \lambda_x \int_{\mathbb{R}} (e^{ux} - 1 - ux) H_x(dx) + \lambda_r \int_{\mathbb{R}} (e^{iCx} - 1 - iCx) H_r(dx) \\
&\quad + \lambda_v \int_{\mathbb{R}} (e^{iDx} - 1 - iDx) H_v(dx) - R
\end{aligned} \tag{75}$$

$$\begin{aligned}
D(0,u) &= 0 \\
C(0,u) &= 0
\end{aligned}$$

then A, C, and D solve three following ODEs:

$$\begin{aligned}
C_\tau(\tau,u) &= \frac{1}{2}C^2\delta_r^2 + \kappa_r C + (u-1) \\
C(0,u) &= 0
\end{aligned} \tag{76}$$

$$\begin{aligned}
D_\tau(\tau,u) &= \frac{1}{2}D^2\delta_v^2 + (\kappa_v - u\delta_v\rho)D + \frac{1}{2}(u+u^2) \\
D(0,u) &= 0
\end{aligned} \tag{77}$$

$$\begin{aligned}
C(\theta_r - \lambda_r b_r) - \lambda_r \int_{\mathbb{R}} (e^{Cx} - 1) H_r dx - u\lambda_x b_x & \\
+ D(\theta_v - \lambda_v b_v) - \lambda_v \int_{\mathbb{R}} (e^{Dx} - 1) H_v dx &= A_\tau(\tau,u)
\end{aligned} \tag{78}$$

$$A(0,u) = 0 \tag{79}$$

(76) and (77) are Riccati equations and have the solutions:

$$\begin{aligned}
C &= \frac{2(1-u)}{\kappa_r + \triangle_c \frac{1+e^{\triangle_c \tau}}{1-e^{\triangle_c \tau}}} \\
D &= \frac{-(u+u^2)}{\kappa_v - u\delta_v\rho + \triangle_d \frac{1+e^{\triangle_c \tau}}{1-e^{\triangle_c \tau}}}
\end{aligned}$$

where

$$\begin{aligned}
\triangle_c &= \sqrt{\kappa_r^2 - 2\delta_r^2(u-1)} \\
\triangle_d &= \sqrt{(\kappa_v - u\delta_v\rho)^2 - \delta_v^2(u+u^2)}
\end{aligned}$$

hence by (78) we get

$$
\begin{aligned}
A \;=\; & -\frac{(\theta_r-\lambda_r b_r)\tau}{\delta_r^2}(\kappa_r-\triangle_c)-\frac{(\theta_v-\lambda_v b_v)\tau}{\delta_v^2}(\kappa_v-\triangle_d-u\delta_v\rho)\\
& -u\lambda_x b_x\tau-\frac{2(\theta_r-\lambda_r b_r)\tau}{\delta_r^2}\ln\left[1+\frac{1}{2}(\kappa_r-\triangle_c)\frac{1-e^{\triangle_c\tau}}{\triangle_c}\right]\\
& -\frac{2(\theta_v-\lambda_v b_v)\tau}{\delta_v^2}\ln\left[1+\frac{1}{2}(\kappa_v-\triangle_c-u\delta_v\rho)\frac{1-e^{\triangle_d\tau}}{\triangle_d}\right]\\
& -\lambda_x\int_0^\tau\int_{\mathbb{R}}(e^{ux}-1)H_x d(x)\\
& -\lambda_r\int_0^\tau\int_{\mathbb{R}}\left(\exp\left(\frac{2(1-u)}{\kappa_r+\triangle_c\frac{1+e^{\triangle_c\tau}}{1-e^{\triangle_c\tau}}x}\right)-1\right)H_r d(x)\\
& -\lambda_v\int_0^\tau\int_{\mathbb{R}}\left(\exp\left(\frac{-(u+u^2)}{\kappa_v-u\delta_v\rho+\triangle_d\frac{1+e^{\triangle_d\tau}}{1-e^{\triangle_d\tau}}x}\right)-1\right)H_v d(x).
\end{aligned}
\tag{80}
$$

Hence if the integer-valued random measures of the state vector are available in closed-forms, then plugging them into (80) we obtain an explicit formula of A.

Using a different language, a version of the previous example can be found in Duffie et al. (2000). We refer the reader to their paper and Bakshi and Madan (2000) for more applications of this framework in pricing path dependent claims of different types such as the average-rate interest rate options, the correlation options, quanto options, chooser options, discretely-monitored knock-out options, the class of affine-jump-diffusion model, or options written on more than one asset.

6. Concluding Remarks

In this paper, we proposed a general option pricing framework that unifies and extends the option valuation literature. We have used the conditional characteristic transform as an efficient tool to simultaneously and jointly recover (i) the option price, (ii) the discount bond price, (iii) the scaled-forward price and (iv) the pricing measure in incomplete markets. When the conditional characteristic transform is known and tractable, the aforementioned option pricing framework warrants no further simplification in pricing exotic options with payoffs of different forms. We proposed to use the general multi-dimensional Lévy processes in order to capture the richness of the framework. This has the great advantages to encompass many previous works, since Lévy process includes Brownian motion as well as many jump processes.

Moreover, with the presence of the characteristic factor in the framework, our method can be applied directly in pricing options with the underlying assets prices follow, for instance, the multi-factor stochastic volatility or a general multi-dimensional affine jump-diffusion processes. By manipulating the characteristic factor, we can indicate the influence direction and the impact of each state variable on the underlying asset price. In general, it allows us to work with much richer dynamic inter-relation among the state variables and much richer jump distribution. From this paper, there also arises an additional direction for future research: to explore the applications of the complex-valued measure.

Appendix

In this part of the paper, we show the sketch of how to derive the transform characteristic $f_X^Q(t, \xi, u)$ defined in (6).

Recall that for any fixed ξ, from the strong Markov property of X, the conditional characteristic transform admits a Feynman–Kac transform as a function of $X(t)$ and t. Let us denote the conditional characteristic transform by $f(t, X(t))$, then from proposition 2.1 we see that $f(t, X(t))$ solves Equation (3), subject to the boundary condition $f(T, x) = e^{u\xi' \cdot x}$. Guided by the form of the solution of the bond price equation in the term structure model of Cox et al. (1985), we guess a solution of the from

$$f(t, X(t)) = \exp\left(A(\tau, u) + G(\tau, u)' \cdot X(t)\right), \tag{81}$$

where A: $[0, T] \times \mathbb{C} \to \mathbb{C}$, G: $[0, T] \times \mathbb{C} \to \mathbb{C}^d$, $\tau = T - t$, and

$$\begin{aligned} A(0, u) &= 0, \qquad \text{and} \\ G(0, u) &= u\xi. \end{aligned}$$

Since $f(t, X(t))$ defined as in (81) is analytic, taking derivatives and using Itô's formula for a complex version (see e.g., Theorem II.35 in Protter (1990)) then rewrite Equation (3) with the posited solution and rearranging gives the desired result. □

Acknowledgements

I am indebted to Prof. Simon Benninga (the editor) and the referee for their very careful reading of the first and second version of this paper and for their many helpful suggestions.

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