



Capital Market Equilibrium with Differential Taxation^{*}

SULEYMAN BASAK

*London Business School and CEPR, Institute of Finance and Accounting, Regent's Park, London
NW1 4SA, United Kingdom
E-mail: sbasak@london.edu*

MICHAEL GALLMEYER

*Graduate School of Industrial Administration and Center for Computational Finance, Carnegie
Mellon University, Tech and Frew Streets, Pittsburgh, PA 15213-3890, United States
E-mail: mgallmeyer@cmu.edu*

Abstract. This paper studies the effect of investor-specific differential dividend taxation on the dynamics of equilibrium security prices and allocations. In order to deal with the inherent Pareto inefficiency of such an equilibrium as well as the preclusion of tax arbitrage, we construct a continuous-time equilibrium via a representative investor with state-dependent utility. Investors differ in their pricing of risk, inducing investor-specific consumption-based CAPMs, with differential taxation appearing as an additional factor. The interest rate, stock price, and consumption dynamics are also impacted. Under logarithmic preferences, risk is transferred from the higher-taxed to the lower-taxed investor, and the interest rate decreases to counteract extra precautionary savings against this suboptimally shared risk. Numerical analysis reveals further tax rate, time-to-horizon, and dividend risk effects on equilibrium quantities. For most wealth allocations, the stock return volatility is increased above the no-tax benchmark.

Key words: differential taxation, equilibrium asset pricing, tax arbitrage, stock volatility

JEL classification codes: G10, G12, D51, D58, H20

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1. Introduction

A pervasive feature of financial markets is the presence of market participants who face different tax rates on the payoffs from the same set of securities. In U.S. security markets alone, a broad range of differentially-taxed investors coexist, ranging from tax-exempt pension and endowment funds to highly-taxed individual investors. This market structure of tax-induced differences in opportunity sets has potentially important implications for equilibrium price and allocation dynamics.

Examining the effect of differential taxation on security prices is not a novel endeavor – a large body of theoretical work exists which has studied this topic. One focus of this work has been to examine joint restrictions on taxation systems and security prices to guarantee each investor faces a strictly positive set of after-tax state prices (no tax arbitrage) (Schaefer, 1982; Dammon and Green, 1987; Ross, 1987; Jones and Milne, 1992). The other main focus has been to examine equilibrium prices and quantities in static mean-variance environments (Brennan, 1970; Long, 1977; Elton and Gruber, 1978; Litzenberger and Ramaswamy, 1980; Singer, 1979). Little is known about the effects of differential taxation on asset prices in a dynamic context. Such an analysis comes up against two main difficulties. First, in any equilibrium, differential taxation results in investors facing different after-tax opportunity sets, leading to Pareto inefficient allocations, and thus complicating the determination of equilibrium. Second, at the outset it is unclear whether equilibrium can in general be attained without additional frictions since Dammon and Green (1987) have demonstrated in a single-period finite-state setting that tax-arbitrage cannot be eliminated under taxation sufficiently heterogeneous across investors.

Our objective is to provide a tractable dynamic asset pricing model under differential taxation. Our primary contribution is to investigate the dynamic behavior of security prices and consumption allocations, requiring an intertemporal model. In particular we go beyond the mean-variance models to analyze endogenous interest rates, stock price-to-dividend ratios, and the dynamic return and volatility of the stock price. To our knowledge, our model is the first to explicitly solve such an equilibrium. For tractability and clarity of our results we adopt a continuous-time environment, which has the added advantage of an abundance of well-understood benchmark models for comparison. We will demonstrate that this continuous-time formulation circumvents the Dammon and Green (1987) tax-arbitrage problem, allowing for no tax-arbitrage under arbitrarily heterogeneous taxation. While ideally we would like to incorporate all the nuances of a taxation system such as that in the U.S. (progressive marginal tax rates on income, dividends, and interest; realized capital gains taxation; etc.), to retain tractability, we impose the differential taxation only on the dividend paid by the stock.

In our economy, two investors finance risky consumption streams in a complete market by trading in a locally riskless bond and a risky stock whose exogenous dividend stream is taxed. The taxation for each investor is assumed to be linear in

the dividends received, and the tax rate is assumed constant. The differentially-taxed investors must agree on the value of the stock, implying that they also agree on the stock's pre-tax instantaneous expected return and volatility. From this agreement in stock valuation, the difference across investors in their after-tax stock expected rates of return is quantified by their difference in tax rates weighted by the dividend-to-price ratio.

In contrast to the problems in static finite-state differential taxation settings, where tax-arbitrage may persist for all prices (Dammon and Green, 1987), we may always construct a no-arbitrage price system in our infinite-state setting under standard conditions (see Remark 2.3). Specifically, the assumptions of no arbitrage and market completeness allow for the construction of a unique system of after-tax Arrow-Debreu prices for each investor. Heterogeneity in taxation leads to a divergence between each investor's after-tax state prices and hence market prices of risk. The lower-taxed investor faces a higher market price of risk than the higher-taxed, with the disagreement in market prices of risk shown to be given by the differential taxation, weighted by the dividend-to-price ratio and normalized by the stock return volatility. Once the investor-specific after-tax state prices have been specified, each investor's consumption-portfolio problem is solved using standard martingale techniques (Cox and Huang, 1989; Karatzas et al., 1987).

All quantities are restricted to constitute a rational expectations equilibrium. We assume a "closed" economy in which all taxes paid by the investors are exogenously returned to the economy as "endowment" streams to each investor.¹ The equilibrium is characterized via the construction of a representative investor with stochastic weights for the two investors in the economy (i.e., a representative investor having state-dependent preferences) (Cuoco and He, 1994).² The stochastic weighting captures the differential-taxation induced stochastic shifts in the distribution of wealth between the two investors and is explicitly characterized. In this work, the stochastic weighting and the stock price jointly solve two stochastic differential equations which are an example of a backward-forward stochastic differential equation (Ma et al., 1994; El Karoui et al., 1997).

The equilibrium market prices of risk exhibit additional terms due to differential taxation which act to reduce the higher-taxed investor's market price of risk while increasing that of the lower-taxed investor. Accordingly, the differential taxation appears as an additional driving factor in the consumption-based CAPM. This extends the differential taxation CAPM of Brennan (1970) to a dynamic, non-mean-variance framework. Since we are studying an economy with intertemporal

¹ Such a redistribution of taxes collected is also used by Werner (1994) in the context of studying the distortionary effects of capital income taxation on consumption and investment decisions in a dynamic two-country model with exogenous production technologies and a world interest rate.

² This work joins a growing body of work that explores general equilibrium models with frictions using a state-dependent representative investor. Other works include Basak (2000) (heterogeneous beliefs), Basak and Cuoco (1998) (restricted market participation), and Detemple and Serrat (2002) (liquidity constraints).

consumption, we can also capture the effect of differential taxation on the equilibrium interest rate. The equilibrium interest rate exhibits additional components that are attributed to changes in investors' precautionary saving motives induced by the differential taxation. The equilibrium consumption dynamics are also characterized. For the special case of homogeneous logarithmic preferences, differential taxation causes risk to be transferred from the higher-taxed to the lower-taxed investor. This sub-optimal risk-sharing unambiguously decreases the equilibrium interest rate as compared to the no-tax benchmark. We note that all equilibrium effects discussed so far arise from differential taxation, not from the taxation itself.

Due to the complicated relationship between the stock price and the ratio of marginal utilities (the weighting process), it appears difficult to achieve a fully analytical analysis of the equilibrium. To provide additional characterization, we numerically solve for the stock price under the assumptions of a geometric Brownian motion dividend process, logarithmic preferences with one taxed and one untaxed investor, and all collected taxes reallocated to the untaxed investor. In this setting, the stochastic weighting equals the ratio of each investor's total wealth. The equilibrium stock price is calculated by solving a quasi-linear partial differential equation using a method of lines algorithm.

The numerical analysis reveals further results and intuition. The qualitative features discussed henceforth were found to be robust to all parameter choices over reasonable ranges. The amount of risk transferred (as measured by the absolute difference between consumption volatilities) from the taxed to the untaxed investor is found to be increasing in the tax rate, decreasing in the time-to-horizon, and decreasing in the dividend risk. The interest rate decrease compared to the benchmark exhibits the same dependencies. The greatest deviation occurs when wealth is equally distributed across the investors. Some of these trends appear somewhat surprising, such as why should the sub-optimal risk-sharing due to taxation have a larger impact when there is less risk to share? We provide intuition for these results.

Even though both investors have preferences exhibiting constant relative risk aversion, in equilibrium they exhibit wealth and time-to-horizon effects in their portfolio holdings. The taxed investor's portfolio strategy is shown to exhibit some familiar features in the presence of differential taxation; he takes on more risk when he is wealthier, and takes on less risk as he approaches his horizon. The untaxed investor exhibits the opposite tendencies.

Finally, we also investigate the behavior of the stock price and stock return volatility. As in the benchmark, for all exogenous parameter values chosen, the stock return covaries positively with the dividend process and its volatility is increasing in the dividend risk. For almost all wealth allocations, the stock return volatility is increased above the benchmark value. The intuition is that a positive dividend shock preferentially benefits the lower-taxed investor since he holds more stock, thus increasing his weight in the economy, leading to a higher aggregate valuation of the stock relative to its dividends. Accordingly, the stock price-to-dividend ratio covaries positively with the aggregate dividend (unlike the

benchmark economy) implying a stock return more volatile than the dividends. Hence the benchmark model without differential taxation underestimates the volatility of the stock price. The stock price itself exhibits non-monotonicity in the taxation by falling below its benchmark value for low enough taxation and rising above the benchmark for higher taxation.

In addition to the differential taxation literature already discussed, further work complementary to our own includes the large literature on capital gains taxation. Constantinides (1983) explores the effect of capital gains taxes on an investor's behavior and on equilibrium asset prices in a dynamic continuous-time economy with identical investors. Constantinides (1984), Dammon et al. (1989), Dammon and Spatt (1996), Dybvig and Koo (1996), and Dammon et al. (2001) explore investor behavior in the presence of capital gains taxation in a variety of settings.

The remainder of the paper is organized as follows. The economic setup is described in Section 2. Section 3 presents the notion of equilibrium and characterizes equilibrium quantities. Section 4 presents additional results for the special case in which each investor has logarithmic preferences. Section 5 concludes and the appendix provides all proofs.

2. General Formulation

We present a continuous-time variation on the Lucas (1978) pure-exchange economy populated by two investors, indexed by $n \in \{H \text{ (high taxed)}, L \text{ (low taxed)}\}$, facing differential (linear) dividend taxation.³ The economy has a finite horizon $[0, T]$ in which there is a single consumption good. The uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$, on which is defined a one-dimensional standard Brownian motion W . The common information of both investors is given by the augmented filtration $\mathbf{F} = \{\mathcal{F}_t : t \in [0, T]\}$ generated by W under the probability measure $\mathbf{P}$. $\mathbf{P}$ represents the common belief set of both investors. All the stochastic processes appearing in the analysis are assumed to be progressively measurable with respect to $\mathbf{F}$, all the stated equalities involving random variables hold $\mathbf{P}$ -a.s., and all the stochastic differential equations are assumed to have solutions without explicitly stating the required regularity conditions.

2.1. INVESTOR-SPECIFIC INVESTMENT OPPORTUNITIES

The investment opportunities are represented by a locally riskless tax-free bond (a money market) and a risky stock. The bond is in zero net supply and earns the locally riskless interest rate r . The stock is in constant net supply of one share and represents a claim to an exogenously specified strictly positive dividend process, δ , satisfying

$$d\delta(t) = \delta(t) [\mu_\delta(t)dt + \sigma_\delta(t)dW(t)], \quad \delta(0) = \delta_0, \quad (2.1)$$

³ The extension of this model to multiple investors is straightforward.

where μ_δ and $\sigma_\delta > 0$ are arbitrary stochastic processes. The dividend process is assumed to pay a dividend at rate δ over $[0, T]$ and a liquidating lump sum dividend at time T given by the value of the process δ at time T : $\delta(T)$. The presence of the lump sum dividend is for technical reasons (see footnote 5), but can be interpreted as a liquidating value of the stock to the investors at time T . However, ignoring the technicality, all our results and intuition follow through in the absence of the time T dividend.

Investor n is taxed on the dividend received δ at a constant rate τ_n where $0 \leq \tau_L < \tau_H < 1$ with no taxation on the time T lump sum dividend. The taxation is linear implying that short positions are taxed symmetrically. If investor n shorts one share of stock, he pays the after-tax dividend $\delta(t) (1 - \tau_n) dt$ over each instant in addition to net capital gains. Under current U.S. tax code, such a tax “rebate” ($\tau_n \delta$) exists since investors can deduct dividends paid on short positions as an investment interest expense. The collected taxes are reallocated back into the economy and will be discussed later.

Due to differential dividend taxation, each investor views the after-tax equilibrium dynamics of the stock differently. It will be shown in equilibrium that the bond price B and the investor-specific stock price S_n are given by

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1, \quad (2.2)$$

$$dS_n(t) + (1 - \tau_n)\delta(t)dt = S_n(t) [\mu_n(t)dt + \sigma_n(t)dW(t)], \quad (2.3)$$

$S_n(T) = \delta(T)$, $n \in \{H, L\}$, where $S_n(T)$ is the cum-dividend stock price. The quantities μ_n and σ_n represent respectively the after-tax (instantaneous) expected return and volatility of the stock to each investor to be determined in equilibrium. Note that by appealing to no arbitrage considerations, the liquidating value of the stock at time T is just the level of the liquidating dividend.

REMARK 2.1. Our choice of a differential taxation scheme is, admittedly, quite simplistic. Specifically, we assume that each investor faces no realized capital gains taxation, no bond interest taxation, no consumption taxation, and a constant rate of dividend taxation, unlike the current U.S. tax code where the tax rate is based on an investor’s taxable income. Incorporating realized capital gains taxation in our dynamic differential taxation setting along the lines of the seminal work of Constantinides (1983) would be especially appealing. However, even in a partial equilibrium setting, the associated consumption-portfolio problem of an investor is very complex due to the need to track the price basis of all securities held when calculating realized capital gains. This is apparent from the original Constantinides (1983) work as well as two more recent works which also examine portfolio selection under realized capital gains taxation by Dybvig and Koo (1996) (capital gains with short sale constraints) and Dammon et al. (2001) (capital gains using an average price basis with short sale constraints).

Extending our current model to include nonlinear dividend taxation appears tractable, as does taxing the interest received from holding the money market (see Remark 3.1). These are potential subjects for future work.

REMARK 2.2. Our analysis and characterization until the end of Section 3.2 are equally valid for stochastic taxation in the form of two progressively measurable tax rate processes, τ_H and τ_L . We assume they are constant for clarity.

The investors in the economy face no additional restrictions on their trading strategies in the stock and the bond other than requiring typical price-taking behavior (a non-anticipative and non-doubling portfolio strategy). Hence, a necessary condition for no-arbitrage opportunities is that the investors agree on the value of the stock ($S_H = S_L \equiv S$). From (2.3), this leads to the following restrictions on the after-tax stock price dynamics:

$$\begin{aligned}\mu_L(t) - \mu_H(t) &= (\tau_H - \tau_L) \frac{\delta(t)}{S(t)}, \\ \sigma_H(t) &= \sigma_L(t) \equiv \sigma(t).\end{aligned}$$

Differential dividend taxation implies each investor faces a different instantaneous after-tax gross expected rate of return on the stock. The investors' divergence ($\mu_L - \mu_H$) is driven by the differential taxation ($\tau_H - \tau_L$) weighted by the dividend-to-price ratio (δ/S). The dividend-to-price ratio captures the relative importance of today's dividend (and hence today's taxation) in the overall stock valuation. Since investor H receives lower dividends due to higher taxes, he must receive a smaller instantaneous after-tax expected return on the stock to preclude arbitrage. Independently of the differential taxation, investors must agree on the stock return volatility (σ); however, the heterogeneity in taxation plays a role in determining this volatility in equilibrium.

In order to aid in the characterization of the equilibrium, we construct a set of after-tax Arrow-Debreu prices for each investor to be determined in equilibrium. An *admissible* price system is defined as a price system $\mathcal{P} \equiv (r, \mu_H, \mu_L, \sigma)$ in which: we can construct a well-defined after-tax state price density process for each investor, each investor values the stock identically, and markets are complete. Accordingly, we define the investor-specific after-tax state price density process ξ_n (a system of Arrow-Debreu securities for each investor) as the process with dynamics

$$d\xi_n(t) = -\xi_n(t) [r(t)dt + \theta_n(t)dW(t)], \quad n \in \{H, L\}, \quad (2.4)$$

where θ_n is each investor's after-tax market price of risk process, $\theta_n(t) \equiv \sigma(t)^{-1} [\mu_n(t) - r(t)]$.⁴ $\xi_n(t, \omega)$ is interpreted as investor n 's after-tax Arrow-

⁴ Sufficient conditions for a well-defined state price density process for each investor are for $\theta_n(t)$ to satisfy the Novikov condition, $E \left[\exp \left\{ \frac{1}{2} \int_0^T \|\theta_n(t)\|^2 dt \right\} \right] < \infty$, and the interest rate to be bounded from below. Market completeness is satisfied if the stock volatility does not vanish.

Debreu price per unit probability P of one unit of consumption good in state $\omega \in \Omega$ at time t . Here, due to differential taxation, investors face differing state prices or investment opportunity sets. From coincidence of each investor's stock valuation, the investors' after-tax market prices of risk are linked via

$$\theta_L(t) - \theta_H(t) = (\tau_H - \tau_L) \frac{\delta(t)}{S(t)\sigma(t)}. \quad (2.5)$$

Consequently, investors disagree on the after-tax market price of risk, with their divergence driven by the differential taxation, weighted by the dividend-to-price ratio and normalized by the stock return volatility.⁵ Assuming $\sigma > 0$, the higher taxed investor's after-tax market price of risk must be lower than the other investor's, due to his lower after-tax gross return on the risky asset. Holding the risky asset is less favorable to him, which must be captured by his lower price on the associated risk.

The existence of an admissible price system leads to the well-known "no-arbitrage" asset pricing equation for the stock.

LEMMA 2.1. Given an admissible price system $\mathcal{P}$, the price of the stock is given by

$$S(t) = \frac{1}{\xi_n(t)} E \left[\int_t^T \xi_n(s)(1 - \tau_n)\delta(s)ds + \xi_n(T)\delta(T) | \mathcal{F}_t \right] \quad (2.6)$$

for $n \in \{H, L\}$.

Equation (2.6) yields the familiar asset pricing equation adjusted for differential dividend taxation; the stock price equals the expected present value of future after-tax dividends discounted by the investor-specific state prices. Equation (2.5) automatically guarantees equality of the right hand side of (2.6) across both investors.

REMARK 2.3 (Tax Arbitrage). A major point of concern in our analysis is whether there even exists an admissible price system (i.e. a set of security prices under differential taxation such that no tax arbitrage opportunities arise). This non-existence issue has been examined extensively in finite state space settings.

One source of tax arbitrage highlighted by Schaefer (1982) arises when there exist redundant securities with differential tax treatments. For example, consider a one period discrete-time economy with two riskless bonds – one taxable and one tax-exempt. If the economy is populated with investors from more than one tax bracket, then no set of prices for the two bonds exists precluding tax arbitrage,

⁵ In the absence of the liquidating dividend at time T for the stock, no-arbitrage requires $S(T) = 0$ and hence, the market price of risk disagreement (2.5) and subsequent equilibrium quantities at time T may not be well-defined. The presence of the liquidating dividend at time T implies $S(T) = \delta(T)$ allowing us to avoid this potential problem.

unless trading restrictions are imposed on the investors. Such an approach is taken by Dybvig and Ross (1986) to study the equilibrium effect of differential taxation on a one-period model with redundant securities. To avoid confounding our results due to differential taxation with the equilibrium effects of other frictions used to rule out tax arbitrage, we assume that the asset structure in the economy remains fixed with no redundant securities. This restriction implies that an investor is not allowed to repackage his portfolio for direct resale to others. However, an investor can still hold an arbitrary position in either security subject to the standard doubling strategy portfolio restriction. Without prohibiting such repackaging, the low-taxed investor could repackage the stock and resell it directly to the high-taxed investor to circumvent the high-taxed investor's dividend taxation.

Even in a security market without redundant assets, Dammon and Green (1987) in a finite state space show that for sufficiently different tax rates across investors, there might not exist any set of prices that rules out tax arbitrage (i.e., no security prices exist such that each investor faces a strictly positive set of after-tax Arrow-Debreu prices). For example, consider a one-period analogue of our economy with a stock with price S paying a liquidating dividend δ in two states of the world $\{U, D\}$ with $\delta(U) > \delta(D)$ and a tax-exempt bond with price B which pays the riskless quantity 1. Computing each investor's after-tax set of Arrow-Debreu prices leads to the following restriction on the price system (S, B) to preclude tax arbitrage: $(1 - \tau_L)\delta(D) < S/B < (1 - \tau_H)\delta(U)$. For sufficiently high differential taxation $(\tau_H - \tau_L)$, these two inequalities may not be satisfied for any price system. This leads to additional conditions on differential taxation and exogenous dividends to guarantee the existence of no tax arbitrage state prices.

A benefit of our infinite-dimensional state space formulation is that it is always possible to construct a set of no-arbitrage prices for any taxation across investors. Intuitively, this is because the "local" return on the stock to either investor is dominated by the underlying Brownian motion uncertainty, while the differential taxation provides a relatively infinitesimal contribution. Mathematically, following He (1990), taking the limit (as $h \rightarrow \infty$) of a constant price coefficient discrete-time economy with h trading intervals over $[0, T]$, the condition for an investor to face a positive set of state prices at t is

$$\frac{\sqrt{h}(S_t\mu - \delta(1 - \tau))}{h + r} - \frac{h\sigma S_t}{h + r} < \frac{\sqrt{h}rS_t}{h + r} < \frac{\sqrt{h}(S_t\mu - \delta(1 - \tau))}{h + r} + \frac{h\sigma S_t}{h + r},$$

which holds trivially as $h \rightarrow \infty$. An example in our setting of such an admissible price system for a given taxation τ_n is $r = 1$, $\mu_n = \mu_\delta - (1 - \tau_n)$, and $\sigma = \sigma_\delta$. Under mild regularity conditions on the dividend process δ , an equivalent martingale measure exists for each investor ruling out tax arbitrage.

2.2. INVESTORS' ENDOWMENTS, PREFERENCES, AND OPTIMIZATION

Each investor is endowed with an initial e_n shares of the stock and with a tax redistribution process ϵ_n from the government in units of the consumption good. Taking prices and the tax redistribution process as given, investor n chooses a consumption process c_n , a portfolio process α_n (in number of stock shares held), and a final wealth $X_n(T)$. At each instant, investor n then pays taxes of value $\tau_n \alpha_n(t) \delta(t) dt$. Given an endowment, an *admissible policy* $(c_n, \alpha_n, X_n(T))$ is defined as one for which the associated financial wealth process, $X_n(t)$, satisfies for all $t \in [0, T]$ the dynamic budget constraint

$$\begin{aligned} dX_n(t) = & X_n(t) r(t) dt + (\epsilon_n(t) - c_n(t)) dt \\ & + \alpha_n(t) S(t) [(\mu_n(t) - r(t)) dt + \sigma(t) dW(t)], \end{aligned} \quad (2.7)$$

obeys the no-bankruptcy condition $X_n(T) \geq 0$, and satisfies standard regularity conditions. The tax redistribution process appears as an exogenous income stream to each investor. Since in equilibrium, each investor faces complete markets, the tax refund process can be spanned by traded securities. Each investor is assumed to derive time-additive, state-independent utility $u_n(\cdot)$ from intertemporal consumption in $[0, T]$ and from final wealth, $X_n(T)$. The function $u_n(\cdot)$ is assumed to be three times continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{c \rightarrow 0} u'_n(c) = \infty$ and $\lim_{c \rightarrow \infty} u'_n(c) = 0$. An investor's dynamic consumption-portfolio problem is to maximize $E \left[\int_0^T u_n(c_n(t)) dt + u_n(X_n(T)) \right]$ over all admissible $(c_n, \alpha_n, X_n(T))$ pairs. The utility of final wealth $X_n(T)$ captures the utility obtained from the liquidating dividend.

It is well-known (Cox and Huang, 1989; Karatzas et al., 1987) that each investor's dynamic consumption-portfolio problem can be converted into the following static variational problem given the investor's Arrow-Debreu prices, ξ_n :

$$\max_{(c_n, X_n(T))} E \left[\int_0^T u_n(c_n(t)) dt + u_n(X_n(T)) \right] \quad (2.8)$$

subject to

$$\begin{aligned} E \left[\int_0^T \xi_n(t) c_n(t) dt + \xi(T) X_n(T) \right] \\ \leq \xi_n(0) e_n S(0) + E \left[\int_0^T \xi_n(t) \epsilon_n(t) dt \right]. \end{aligned}$$

The necessary and sufficient conditions for optimality of the consumption stream and final wealth of this problem are

$$\hat{c}_n(t) = I_n(y_n \xi_n(t)), \quad t \in [0, T], \quad (2.9)$$

$$\hat{X}_n(T) = I_n(y_n \xi_n(T)), \quad (2.10)$$

for $n \in \{H, L\}$ where I_n is the inverse of u'_n , and y_n is the unique positive number such that investor n 's static budget constraint holds with equality at the optimum, i.e., y_n satisfies

$$\begin{aligned} E \left[\int_0^T \xi_n(t) I_n(y_n \xi_n(t)) dt + \xi_n(T) I_n(y_n \xi_n(T)) \right] \\ = \xi_n(0) e_n S(0) + E \left[\int_0^T \xi_n(t) \epsilon_n(t) dt \right]. \end{aligned} \quad (2.11)$$

Furthermore, investor n 's optimal financial wealth $\hat{X}_n$ is given by

$$\hat{X}_n(t) = \frac{1}{\xi_n(t)} E \left[\int_t^T \xi_n(s) (\hat{c}_n(s) - \epsilon_n(s)) ds + \xi_n(T) \hat{X}_n(T) | \mathcal{F}_t \right]. \quad (2.12)$$

By an application of the martingale representation theorem, his optimal portfolio is

$$\alpha_n(t) = \frac{\theta_n(t)}{\sigma(t)} \frac{\hat{X}_n(t)}{S(t)} + \frac{\psi_n(t)}{\sigma(t) \xi_n(t) S(t)} \quad (2.13)$$

where the stochastic process ψ_n satisfies

$$\begin{aligned} \xi_n(t) \hat{X}_n(t) + \int_0^t \xi_n(s) (\hat{c}_n(s) - \epsilon_n(s)) ds \\ = \xi_n(0) \hat{X}_n(0) + \int_0^t \psi_n(s) dW(s). \end{aligned}$$

3. Equilibrium with Differential Dividend Taxation

In this paper, financial security prices are characterized by appealing to general equilibrium restrictions. Our model differs from a standard Lucas (1978) pure-exchange economy due to differential dividend taxation inducing investor-specific opportunity sets. All taxes collected are redistributed back into the economy by an exogenously-specified government reallocation rule given by the stochastic processes κ_n , where $\kappa_H(t) + \kappa_L(t) = 1$, denoting the fraction of total tax revenues distributed to investor n . Total taxes collected in units of the consumption good are given by $\epsilon(t)$. Investor n 's claim to the tax redistribution is $\epsilon_n(t) \equiv \kappa_n(t)\epsilon(t)$. Equilibrium is defined as follows.

DEFINITION 3.1. Given preferences, endowments, and tax reallocation rules $(u_H(\cdot), u_L(\cdot), e_H, e_L, \kappa_H, \kappa_L)$, an equilibrium is a collection of allocations $((\hat{c}_H, \hat{\alpha}_H, \hat{X}_H(T)), (\hat{c}_L, \hat{\alpha}_L, \hat{X}_L(T)))$ and an admissible price system $\mathcal{P}$ such that $(\hat{c}_n, \hat{\alpha}_n, \hat{X}_n(T))$ is an optimal solution to investor n 's optimization problem, all markets clear for all $t \in [0, T]$:

$$\hat{c}_H(t) + \hat{c}_L(t) = \delta(t), \quad \hat{\alpha}_H(t) + \hat{\alpha}_L(t) = 1, \quad \hat{X}_H(t) + \hat{X}_L(t) = S(t), \quad (3.1)$$

$$\hat{X}_H(T) + \hat{X}_L(T) = \delta(T), \quad (3.2)$$

and tax revenues are fully redistributed back to the investors in the economy:

$$\epsilon_H(t) + \epsilon_L(t) = \tau_H \alpha_H(t) \delta(t) + \tau_L \alpha_L(t) \delta(t). \quad (3.3)$$

The assumption of the tax collected equaling the tax redistributed leads to the simplicity of the consumption market clearing. Namely, the aggregate consumption is equated to the exogenously specified aggregate dividend (as in a standard pure-exchange economy). In this way, we focus on the distributional effects of differential taxation and not on government inefficiencies causing goods to leave the economy. A number of related works have instead implicitly assumed that all taxes collected are subject to free disposal implying no clearing in the good market (Schaefer, 1982; Dammon, 1987; Dammon and Green, 1987).

3.1. DETERMINATION OF EQUILIBRIUM WITH DIFFERENTIAL TAXATION

When investors face differential dividend taxation and hence different after-tax state prices, the equilibrium consumption allocation need not be Pareto efficient. Risk sharing across investors is sub-optimal. Hence, our equilibrium is not characterized by the usual representative investor constructed as a constant-weighted linear combination of each investor's utility. However, using a weaker notion of aggregation introduced by Cuoco and He (1994), we can still construct a representative investor, but with a stochastic weighting (i.e., state dependent utility). With this construction, the market clearing conditions are easily verified and the search for an equilibrium is reduced to the determination of the appropriate weighting process.

Specifically, the representative investor's utility function over intertemporal consumption and terminal wealth is defined by

$$\begin{aligned} U(\delta; \lambda) &\equiv \max_{c_H, c_L} u_H(c_H) + \lambda u_L(c_L) \quad \text{s.t. } c_H + c_L = \delta, \\ &\equiv \max_{X_H, X_L} u_H(X_H) + \lambda u_L(X_L) \quad \text{s.t. } X_H + X_L = \delta, \end{aligned}$$

where λ is a strictly positive process. Identifying

$$\lambda(t) = \frac{u'_H(\hat{c}_H(t))}{u'_L(\hat{c}_L(t))} = \frac{y_H \xi_H(t)}{y_L \xi_L(t)} \quad \text{and} \quad \lambda(T) = \frac{u'_H(\hat{X}_H(T))}{u'_L(\hat{X}_L(T))} = \frac{y_H \xi_H(T)}{y_L \xi_L(T)},$$

we obtain the following equilibrium conditions in terms of the representative investor's utility function, the stochastic weighting process λ , and the stock price. For compactness, we suppress the dependence of the representative investor's utility

on δ and λ by denoting $U(t) \equiv U(\delta(t), \lambda(t))$. The first (second) derivative of the representative investor's utility with respect to the dividend process δ is denoted $U'(t)$ ($U''(t)$).

PROPOSITION 3.1. If equilibrium exists, the equilibrium investor-specific after-tax state price density processes are given by

$$\xi_H(t) = \frac{U'(t)}{U'(0)}, \quad \xi_L(t) = \frac{\lambda(0)}{\lambda(t)} \frac{U'(t)}{U'(0)}, \quad (3.4)$$

$$\xi_H(T) = \frac{U'(T)}{U'(0)}, \quad \xi_L(T) = \frac{\lambda(0)}{\lambda(T)} \frac{U'(T)}{U'(0)}, \quad (3.5)$$

where $\lambda(0)$ solves either investor's static budget constraint, i.e.,

$$E \left[\int_0^T U'(t) I_H(U'(t)) dt + U'(T) I_H(U'(T)) \right] = \quad (3.6)$$

$$E \left[\int_0^T U'(t) (e_H(1 - \tau_H) \delta(t) + \epsilon_H(t)) dt + U'(T) e_H \delta(T) \right],$$

and λ and S satisfy

$$d\lambda(t) = \lambda(t) \frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)} \times \quad (3.7)$$

$$\left[\left(A(t) \delta(t) \sigma_\delta(t) + \frac{A(t)}{A_H(t)} \frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)} \right) dt + dW(t) \right],$$

$$S(t) = \frac{1}{U'(t)} E \left[\int_t^T U'(\delta(s); \lambda(s)) (1 - \tau_H) \delta(s) ds \right. \quad (3.8)$$

$$\left. + U'(\delta(T); \lambda(T)) \delta(T) | \mathcal{F}_t \right],$$

with absolute risk aversions defined as

$$A(t) \equiv -\frac{U''(t)}{U'(t)} = \frac{1}{1/A_H(t) + 1/A_L(t)}, \quad A_n(t) \equiv -\frac{u_n''(\hat{c}_n(t))}{u_n'(\hat{c}_n(t))}.$$

The equilibrium consumption and final wealth allocations are given by

$$\begin{aligned} \hat{c}_H(t) &= I_H(U'(t)), \\ \hat{c}_L(t) &= I_L(U'(t)/\lambda(t)), \end{aligned} \quad (3.9)$$

$$\begin{aligned}\hat{X}_H(T) &= I_H(U'(\delta(T); \lambda(T)), \\ \hat{X}_L(T) &= I_L(U'(\delta(T); \lambda(T))/\lambda(T)).\end{aligned}\tag{3.10}$$

Conversely, if there exists ξ_H , ξ_L , λ , and S satisfying (3.4)–(3.8), then the associated optimal consumption-wealth-portfolio policies clear all markets.

Given differential taxation leads investors to face differing state prices, the representative investor's utility in this economy is state-dependent. The stochastic weighting process λ acts as a proxy for stochastic shifts in the distribution of wealth between the two investors induced by differential taxation. Inspection of (3.7) reveals that the stochastic weighting is positively correlated with the stock price and (assuming $\sigma > 0$) the dividends. The weighting, then, acts to put more weight on the lower-taxed investor as the stock dividend and price increases. Differential taxation causes the stock to become relatively less favorable to the more highly taxed investor, causing him to hold less in the stock whereby a high stock price preferentially benefits the lower-taxed investor. Furthermore, the multiplicative stochastic term in the dynamics of λ is increasing in the differential taxation ($\tau_H - \tau_L$) and in the dividend-to-price ratio (capturing how important current dividends are in the stock valuation). When investors have no dividend taxation ($\tau_H = \tau_L = 0$) or have equal dividend taxation ($\tau_H = \tau_L$), investors face the same set of state prices in equilibrium, and the weighting process collapses to a constant as in a standard pure-exchange complete markets economy.

The converse in Proposition 3.1 gives a convenient method for constructing the equilibrium. By jointly solving the two stochastic differential equations (3.7) and (3.8) for λ and S , we can recover the equilibrium price system and allocations. This equilibrium construction has an extra level of complexity over that of Basak and Cuoco (1998), where the stochastic weighting process is shown to implicitly satisfy a single stochastic differential equation in which it is the only unknown. Basak (2000) has a yet simpler characterization with the stochastic weighting dynamics explicitly determined in terms of exogenous quantities. The joint solution of (3.7)–(3.8) is, then, a harder problem than in other work which has been able to explicitly characterize λ . This system of equations is an example of a forward-backward stochastic differential equation (FBSDE) (Ma and Yong, 1999; El Karoui et al., 1997; Ma et al., 1994).⁶

In order to study the characteristics of an equilibrium with differential taxation, it will be useful to solve the FBSDE given by (3.6)–(3.8). Specifically, we can solve

⁶ We note that our forward-backward system does not fall into the class of systems whose existence has been formally shown (Ma and Yong, 1999). Existence in our setting is still an open issue. In fact, very few continuous-time equilibrium models with frictions have shown existence (Basak and Cuoco (1998) is a notable exception). Although it is far from a proof of existence, we have conducted an extensive numerical analysis of our model for a wide-range of parameters with no evidence of non-existence. Section 4 includes a subset of this analysis.

for an adapted solution to this FBSDE using the “Four Step Scheme” of Ma et al. (1994) requiring us to assume that (δ, λ) are jointly Markovian. Heuristically explained, we first recognize that the forward stochastic differential equation (3.7) for the weighting process λ is also linked to the stock’s backward stochastic differential equation (3.8) through the stock volatility σ . Taking this linkage into account, we can solve (3.8) “backwards” as a function of $(\lambda(t), \delta(t), t)$ by a parabolic quasi-linear partial differential equation. Using this solution as well as the restriction on the stock’s instantaneous volatility σ , we can find λ by solving (3.7) “forward.” The final step of the “Four Step Scheme” is to then substitute our solution of the weighting process λ back into the stock price S . For the purpose of our equilibrium characterization, the last two steps need not be implemented. Our main focus will be to solve (3.8) in terms of δ and λ .

Appealing to this motivation behind the “Four Step Scheme” under a Markovian structure and $\kappa_H(t) = 0$, Proposition 3.2 characterizes the equilibrium price system via a system of non-linear partial differential equations. The results of this proposition will be applied in the next section to provide additional equilibrium characterization.

PROPOSITION 3.2. Suppose there exists an equilibrium for the economy with (δ, λ) jointly Markovian and no tax reallocation to investor H .⁷ Define the functions F and P as follows:

$$F(\delta(t), \lambda(t), t) = \frac{1}{U'(t)} E \left[\int_t^T U'(s) I_H(U'(s)) ds + U'(T) I_H(U'(T)) | \delta(t), \lambda(t) \right], \quad (3.11)$$

$$P(\delta(t), \lambda(t), t) = \frac{1}{U'(t)} E \left[\int_t^T U'(s) (1 - \tau_H) \delta(s) ds + U'(T) \delta(T) | \delta(t), \lambda(t) \right]. \quad (3.12)$$

If F and P are twice continuously differentiable with respect to δ and λ and continuously differentiable with respect to t , then F and P solve the following quasi-linear system of partial differential equations

$$\left(\mathcal{L} + \frac{\partial}{\partial t} \right) U'(\delta; \lambda) F(\delta, \lambda, t) + U'(\delta; \lambda) I_H(U'(\delta, \lambda)) = 0, \quad (3.13)$$

⁷ The assumption that investor H receives no tax reallocation is only required when recovering each investor’s optimal trading strategy in Section 4; we make the assumption here purely for consistency. This proposition extends directly to the case where investor H receives part of reallocation. The only change is in the equation for the function F .

$$\left(\mathcal{L} + \frac{\partial}{\partial t}\right) U'(\delta; \lambda) P(\delta, \lambda, t) + U'(\delta; \lambda) (1 - \tau_H) \delta = 0, \quad (3.14)$$

where $d\lambda(t) = \lambda(t)\mu_\lambda(\delta(t), \lambda(t), t)dt + \lambda(t)\sigma_\lambda(\delta(t), \lambda(t), t)dW(t)$ with

$$\mu_\lambda(\delta, \lambda, t) = \sigma_\lambda(\delta, \lambda, t) \quad (3.15)$$

$$\times \left(\frac{U''(\delta; \lambda)}{U'(\delta; \lambda)} \delta \sigma_\delta(\delta, t) - \frac{U'_\lambda(\delta; \lambda)}{U_c(\delta; \lambda)} \lambda \sigma_\lambda(\delta, \lambda, t) + \sigma_\lambda(\delta, \lambda, t) \right),$$

$$\sigma_\lambda(\delta, \lambda, t) = \frac{(\tau_H - \tau_L) \delta}{P(\delta, \lambda, t) \sigma(\delta, \lambda, t)}, \quad (3.16)$$

$$\sigma(\delta, \lambda, t) = \frac{1}{P(\delta, \lambda, t)} \quad (3.17)$$

$$\times \left(P_\delta(\delta, \lambda, t) \delta \sigma_\delta(\delta, t) + P_\lambda(\delta, \lambda, t) \lambda \frac{(\tau_H - \tau_L) \delta}{P(\delta, \lambda, t) \sigma(\delta, \lambda, t)} \right),$$

subject to the boundary conditions $P(\delta, \lambda, T) = \delta(T)$ and $F(\delta, \lambda, T) = I_H(U'(\delta(T); \lambda(T)))$, where $\mathcal{L}$ denotes the differential generator of (δ, λ) . Furthermore, $F(\delta(t), \lambda(t), t)$ and $P(\delta(t), \lambda(t), t)$ define the optimal wealth process for investor H and the stock price respectively. The initial value of the weighting process, $\lambda(0)$, is given by $F(\delta(0), \lambda(0), 0) = e_H P(\delta(0), \lambda(0), 0)$.

Proposition 3.2 provides a characterization of the equilibrium suitable for numerical analysis. Note that the system of partial differential equations is a decoupled system in that the stock price differential equation (3.14) can be solved independently of investor H 's wealth differential equation (3.13). Within this economy arriving at analytical closed-form solutions of equilibrium quantities appears difficult. Equation (3.17) displays a quadratic restriction on the stock volatility implied by the BFSDE (3.7) and (3.8). In general, solving (3.17) for σ will not be unique, implying the potential for multiple equilibria.

3.2. CHARACTERIZATION OF EQUILIBRIUM

Although not explicitly solved in terms of exogenous quantities, the behavior of the security price dynamics and investors' consumption dynamics can still be examined under differential dividend taxation. Throughout our discussion, the term *benchmark* refers to an economy with no dividend taxation ($\tau_H = \tau_L = 0$). Proposition 3.3 summarizes the main results concerning the market price of risk for each investor.

PROPOSITION 3.3. Assuming an equilibrium exists, the investors' after-tax market prices of risk are given by

$$\theta_H(t) = A(t) \delta(t) \sigma_\delta(t) - \frac{A(t)}{A_L(t)} \frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)}, \quad (3.18)$$

$$\theta_L(t) = A(t)\delta(t)\sigma_\delta(t) + \frac{A(t)}{A_H(t)} \frac{(\tau_H - \tau_L)\delta(t)}{S(t)\sigma(t)}. \quad (3.19)$$

Furthermore, if $\sigma(t) > 0$, $\theta_L(t) > \theta_H(t)$.

The first term in each after-tax market price of risk is the benchmark term, the aggregate risk in the economy ($\delta\sigma_\delta$) normalized by the representative investor's risk tolerance ($1/A$). The second term is proportional to the differential in taxation normalized by the other investor's risk tolerance, and output (δ) per unit of overall stock volatility ($S\sigma$). Note that in an economy with equal dividend taxation ($\tau_H = \tau_L$), the investor-specific market prices of risk collapse to the common benchmark value.

We can derive the after-tax risk premium for the stock as faced by the two investors as

$$\begin{aligned} \mu_H(t) - r(t) &= A(t) \operatorname{cov} \left(\frac{dS(t)}{S(t)}, d\delta(t) \right) - \frac{A(t)}{A_L(t)} \frac{(\tau_H - \tau_L)\delta(t)}{S(t)}, \\ \mu_L(t) - r(t) &= A(t) \operatorname{cov} \left(\frac{dS(t)}{S(t)}, d\delta(t) \right) + \frac{A(t)}{A_H(t)} \frac{(\tau_H - \tau_L)\delta(t)}{S(t)}. \end{aligned}$$

As in the standard consumption-based CAPM (Breedon, 1979; Duffie and Zame, 1989), the stock's risk premium is positively related to the covariance of its return with the change in aggregate consumption. Under differential taxation, an additional term appears, proportional to the differential in taxation and the dividend-to-price ratio. For a given covariance and risk aversions, the lower-taxed investor demands a higher risk premium, while the higher-taxed investor demands a lower risk premium. For a large enough differential in taxation, it appears possible for the higher taxed investor to demand a negative risk premium. We also note that even if investor L is not taxed ($\tau_L = 0$), his consumption-based CAPM does not collapse to the benchmark.

Another implication of Proposition 3.3 is on the pre-tax equilibrium risk premium in the presence of differential taxation. As the proof of Proposition 3.1 in Appendix I demonstrates, we may define a pre-tax market price of risk by

$$\theta(t) \equiv \theta_n(t) + \frac{\tau_n \delta(t)}{S(t)\sigma(t)},$$

which increases each investor's expected stock gross return appropriately, thereby removing the effect of differential taxation on investors' market prices of risk (and hence state prices). We may interpret θ as the market price of risk for a non-taxed investor outside the model. Proposition 3.3 then implies that the equilibrium pre-tax market price of risk is given by

$$\theta(t) = A(t)\delta(t)\sigma_\delta(t) + \frac{A(t)\delta(t)}{S(t)\sigma(t)} \left(\frac{\tau_L}{A_L(t)} + \frac{\tau_H}{A_H(t)} \right)$$

yielding the pre-tax equilibrium risk premium

$$\mu(t) - r(t) = A(t) \operatorname{cov} \left(\frac{dS(t)}{S(t)}, d\delta(t) \right) + \frac{A(t) \delta(t)}{S(t)} \left(\frac{\tau_L}{A_L(t)} + \frac{\tau_H}{A_H(t)} \right).$$

Hence, the standard consumption-based CAPM expression is adjusted by adding the risk-tolerance weighted tax rate of each investor scaled by the dividend-to-price ratio. This extends the static differential-taxation CAPM of Brennan (1970) and others to a dynamic non-mean-variance context.⁸ Note that estimating the stock's risk premium just by using the standard consumption-based CAPM will lead to an underestimate of the true risk premium.

Proposition 3.4 reports the equilibrium interest rate.

PROPOSITION 3.4. Assuming equilibrium exists, the interest rate is given by

$$\begin{aligned} r(t) = & A(t)\delta(t)\mu_\delta(t) - \frac{1}{2}A(t)B(t)\delta(t)^2\sigma_\delta(t)^2 \\ & - \frac{1}{2} \frac{A(t)^3}{A_H(t)^2 A_L(t)^2} (B_H(t) + B_L(t)) \frac{(\tau_H - \tau_L)^2 \delta(t)^2}{S(t)^2 \sigma(t)^2} \\ & + \frac{A(t)^3}{A_H(t) A_L(t)} \left(\frac{B_H(t)}{A_H(t)} - \frac{B_L(t)}{A_L(t)} \right) \frac{(\tau_H - \tau_L) \delta(t)^2 \sigma_\delta(t)}{S(t) \sigma(t)}, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} B(t) &\equiv -\frac{U'''(t)}{U''(t)} = \left(\frac{A(t)}{A_H(t)} \right)^2 B_H(t) + \left(\frac{A(t)}{A_L(t)} \right)^2 B_L(t), \\ B_n(t) &\equiv -\frac{u_n'''(\hat{c}_n(t))}{u_n''(\hat{c}_n(t))}. \end{aligned}$$

The interest rate in the no-tax benchmark economy is given by the first two terms in (3.20), revealing a positive relation to the growth rate of consumption and a negative relation (for investors with decreasing absolute risk aversion (DARA)) to the aggregate risk in the economy (σ_δ^2). The latter relation arises from investors'

⁸ Alternatively, rewriting each investor's after-tax risk premium and manipulating yields

$$\frac{A(t)}{A_H(t)} \mu_H(t) + \frac{A(t)}{A_L(t)} \mu_L(t) - r(t) = A(t) \operatorname{cov} \left(\frac{dS(t)}{S(t)}, d\delta(t) \right).$$

We obtain the standard consumption-based CAPM expression, with the risk premium replaced by a risk-tolerance-weighted average of each investor's after-tax risk premium. This interpretation has been pointed out by Basak (2000) in dynamic equilibrium in the context of investors with heterogeneous beliefs about an extraneous process.

precautionary savings motive (with intensity captured by the prudence coefficient B) when facing uncertain future consumption.⁹ Relative to the benchmark economy, the interest rate under differential taxation is driven by two additional terms. The third term acts to reduce the interest rate (for DARA investors) and compensates for additional precautionary savings induced by suboptimal risk sharing under heterogeneous taxation. The last term increases the interest rate if $\sigma > 0$ and the higher-taxed investor is also more prudent (e.g., $B_H/A_H > B_L/A_L$); risk is then being transferred to the less prudent investor reducing the overall precautionary savings motive in the economy.

Proposition 3.5 reports the equilibrium consumption dynamics for each investor.

PROPOSITION 3.5. The equilibrium consumption dynamics for investor n are given by

$$dc_n(t) = c_n(t) [\mu_{c_n}(t)dt + \sigma_{c_n}(t)dW(t)], \quad (3.21)$$

where

$$c_n(t)\sigma_{c_n}(t) = \frac{A(t)}{A_n(t)}\delta(t)\sigma_\delta(t) - \frac{A(t)}{A_n(t)A_m(t)} \frac{(\tau_n - \tau_m)\delta(t)}{S(t)\sigma(t)}, \quad (n, m) = (H, L), (L, H), \quad (3.22)$$

$$\begin{aligned} c_n(t)\mu_{c_n}(t) = & \frac{A(t)}{A_n(t)}\delta(t)\mu_\delta(t) \\ & + \frac{1}{2} \frac{A(t)^2}{A_n(t)} \left(\frac{B_n(t)}{A_n(t)} - \frac{B(t)}{A(t)} \right) \delta(t)^2 \sigma_\delta(t)^2 \\ & + \frac{1}{2} \left(\frac{A(t)}{A_n(t)A_m(t)} \right)^3 [A_n(t)B_n(t) - A_m(t)B_m(t)] \frac{(\tau_n - \tau_m)^2 \delta(t)^2}{S(t)^2 \sigma(t)^2} \\ & - \frac{A(t)^3}{A_n(t)^2 A_m(t)^3} (B_n(t) + B_m(t)) \frac{(\tau_n - \tau_m)\delta(t)^2 \sigma_\delta(t)^2}{S(t)\sigma(t)}. \end{aligned} \quad (3.23)$$

As in the benchmark economy, each investor's consumption volatility is positively related to the aggregate risk with a sensitivity given by that investor's fraction of the aggregate risk tolerance (A/A_n). Under differential taxation, an additional term arises showing that (if $\sigma > 0$) risk is transferred from the higher-taxed investor to the lower-taxed investor. Equation (3.22) sharpens our previous assertion

⁹ The notion of precautionary savings refers to the additional demand for saving induced due to uncertainty in future consumption (Leland, 1968). The relationship between precautionary savings and the prudence coefficient was originally pointed out by Kimball (1990).

that differential taxation induces suboptimal risk sharing, via the higher-taxed investor divesting shares in the taxed stock. One interesting implication is that, if taxed higher, the more risk tolerant investor may not necessarily absorb more of the aggregate risk.

Investor n 's mean consumption growth is given by (3.23). The aggregate consumption growth in the economy is shared between investors in proportion to their risk tolerances, but is then adjusted by three additional terms. The second term, as in the benchmark, captures the investor's precautionary savings against dividend risk, driven by how prudent he is relative to the whole economy. The third term analogously captures investor n 's precautionary savings against additional risk introduced from the differential taxation induced suboptimal risk sharing. The fourth term is a precautionary savings adjustment due to the tax-induced transfer of risk across investors. For given prudences (and $\sigma > 0$), the investor H has risk transferred away from him so his precautionary saving is reduced, contributing to a reduction in μ_{c_n} .

REMARK 3.1 (Interest Taxation on the Money Market). One possible extension of our analysis is to also incorporate differential taxation on the interest paid by the money market. This extension leads to an additional layer of complexity due to the interest rate being endogenous, as we highlight here.

Assuming the bond interest taxation is the same rate as the dividend taxation leads to the following bond price dynamics:

$$dB_n(t) = B_n(t) \frac{r_n(t)}{(1 - \tau_n)} dt, \quad n \in \{H, L\},$$

where r_n is defined as the after-tax riskless interest rate for investor n . For both investors to agree on the price of the money market, they must agree on its pre-tax interest rate given by

$$r(t) \equiv \frac{r_H(t)}{(1 - \tau_H)} = \frac{r_L(t)}{(1 - \tau_L)}.$$

The after-tax market prices of risk are then related as follows:

$$\theta_L(t) - \theta_H(t) = \frac{(\tau_H - \tau_L)}{\sigma(t)} \left(\frac{\delta(t)}{S(t)} - r(t) \right).$$

Following our earlier analysis, we obtain expressions for the market prices of risk, the (weighted, after-tax) interest rate, and the stochastic weighting as:

$$\begin{aligned} \theta_H(t) &= A(t)\delta(t)\sigma_\delta(t) - \frac{A(t)}{A_L(t)} \frac{(\tau_H - \tau_L)}{\sigma(t)} \left(\frac{\delta(t)}{S(t)} - r(t) \right), \\ \theta_L(t) &= A(t)\delta(t)\sigma_\delta(t) + \frac{A(t)}{A_H(t)} \frac{(\tau_H - \tau_L)}{\sigma(t)} \left(\frac{\delta(t)}{S(t)} - r(t) \right), \end{aligned}$$

$$\begin{aligned}
& A(t) \left(\frac{(1-\tau_H)}{A_H(t)} + \frac{(1-\tau_L)}{A_L(t)} \right) r(t) \\
&= A(t) \delta(t) \mu_\delta(t) - \frac{1}{2} A(t) B(t) \delta(t)^2 \sigma_\delta(t)^2 \\
&\quad - \frac{1}{2} \frac{A(t)^3}{A_H(t)^2 A_L(t)^2} (B_H(t) + B_L(t)) \frac{(\tau_H - \tau_L)^2}{\sigma(t)^2} \left(\frac{\delta(t)}{S(t)} - r(t) \right)^2 \\
&\quad + \frac{A(t)^3}{A_H(t) A_L(t)} \left(\frac{B_H(t)}{A_H(t)} - \frac{B_L(t)}{A_L(t)} \right) \frac{(\tau_H - \tau_L)^2 \delta(t) \sigma_\delta(t)}{\sigma(t)} \left(\frac{\delta(t)}{S(t)} - r(t) \right),
\end{aligned}$$

and

$$\begin{aligned}
d\lambda(t) = & \lambda(t) \frac{(\tau_H - \tau_L)}{\sigma(t)} \left(\frac{\delta(t)}{S(t)} - r(t) \right) \left[\left\{ A(t) \delta(t) \sigma_\delta(t) \right. \right. \\
& \left. \left. + \frac{\sigma(t) S(t) r(t)}{\delta(t) - r(t) S(t)} + \frac{A(t)}{A_H(t)} \frac{(\tau_H - \tau_L)}{\sigma(t)} \left(\frac{\delta(t)}{S(t)} - r(t) \right) \right\} dt + dW(t) \right].
\end{aligned}$$

The first main difference in these expressions is that the dividend-to-price ratio, δ/S , is replaced everywhere by the relative attractiveness of the stock's yield to the bond's interest rate, $\delta/S - r$. This term captures the relative effective tax burden of investing in the stock compared to the bond. For low equilibrium interest rates as compared to the dividend-to-price ratio of the risky asset, taxation on the money market reduces the impact of the differential taxation on investors' risk premia and interest rates. Since the bond is now also taxed, the higher taxed investor has less incentive to divest his position in the risky asset. Risk sharing moves closer to optimal and precautionary saving decreases. If in equilibrium, the interest rate dominates the dividend-to-price ratio of the risky asset, the bond is being taxed more heavily than the risky asset, so the more heavily taxed investor now prefers the risky asset. We would anticipate risk to now be transferred in reverse from the lower taxed investor to the higher taxed. Precautionary saving begins to increase again.

As a result, the interest rate expression is complicated by being quadratic in r . Additionally, the left hand side has the interest rate replaced by a weighted average of the individual investors' after-tax interest rates. However, this expression would still provide the interest rate as a function of the contemporaneous stock price, volatility, and stochastic weighting. Substitution into the stochastic weighting expression again leaves us with the problem of jointly solving for λ and S . This numerical analysis is left for future research.

4. The Case of Logarithmic Preferences

To further explore the impact of differential taxation on the equilibrium quantities, in this section we specialize our economy to both investors having logarithmic

preferences ($u_H(\cdot) = u_L(\cdot) = \log(\cdot)$)¹⁰ and investor H having no claim to the tax reallocation ($\epsilon_H = 0$).¹¹ The representative investor now has utility

$$U(c; \lambda) = \log\left(\frac{c}{1 + \lambda}\right) + \lambda \log\left(\frac{\lambda c}{1 + \lambda}\right),$$

and $U'(c; \lambda) = (1 + \lambda)/c$. In both the no-tax benchmark and our economy, the representative investor's weighting process λ equals the ratio of investors' optimal consumption processes, $\lambda(t) = \hat{c}_L(t)/\hat{c}_H(t)$. Defining an investor's "total wealth" by the sum of his financial wealth and his future tax reallocation, i.e.

$$\tilde{X}_H(t) = \hat{X}_H(t), \tilde{X}_L(t) = \hat{X}_L(t) + \frac{1}{\xi_L(t)} E \left[\int_t^T \xi_L(s) \epsilon_L(s) ds \middle| \mathcal{F}_t \right],$$

λ also equals the ratio of investors' total wealth, $\lambda(t) = \tilde{X}_L(t)/\tilde{X}_H(t)$.

The quantities in the no-tax benchmark economy will be denoted by an overbar " $\bar{\cdot}$ ". The benchmark equilibrium state price density and consumption allocations are $\bar{\xi}(t) = \delta(0)/\delta(t)$, $\bar{c}_H(t) = \delta(t)/(1 + \bar{\lambda}) = e_H \delta(t)$, $\bar{c}_L(t) = \bar{\lambda} \delta(t)/(1 + \bar{\lambda}) = e_L \delta(t)$, $\bar{X}_H(T) = \delta(T)/(1 + \bar{\lambda})$, $\bar{X}_L(T) = \delta(T) \bar{\lambda}/(1 + \bar{\lambda})$, where $\bar{\lambda} = e_L/e_H$. This implies $\bar{S}(t) = (T - t + 1)\delta(t)$, $\bar{r}(t) = \mu_\delta(t) - \sigma_\delta(t)^2$, $\bar{\theta}(t) = \sigma_\delta(t)$, $\bar{\mu}_{c_n}(t) = \mu_\delta(t)$, $\bar{\sigma}_{c_n}(t) = \sigma_\delta(t)$. Furthermore, the benchmark equilibrium portfolio strategies are $\bar{\alpha}_H(t) = e_H$, $\bar{\alpha}_L(t) = e_L$, with no riskless lending or borrowing.

4.1. EQUILIBRIUM IN THE LOG CASE

Specializing Proposition 3.1, we arrive at the following characterization of the price system and optimal consumption allocations.

PROPOSITION 4.1. If equilibrium exists, then the state prices, the consumption allocations, and the final wealth allocations are given by

¹⁰ The analysis of this section readily extends to the case of both agents having identical power preferences. Under such an assumption, the subsequent numerical characterization also becomes dependent on the instantaneous expected growth rate of the dividend process (μ_δ) and the common relative risk aversion of each investor, in addition to differential taxation and the dividend's instantaneous volatility. As will be seen, the simpler logarithmic case is sufficient to derive rich implications. Details about the power case are available from the authors.

¹¹ To allow for the determination of each investor's portfolio strategy in Figures 7 and 8, the tax reallocation assumption is needed. The rest of the analysis is independent of the assumption placed on the tax reallocation.

$$\xi_H(t) = \frac{\delta(0)}{1 + \lambda(0)} \frac{1 + \lambda(t)}{\delta(t)}, \quad \xi_L(t) = \frac{\lambda(0) \delta(0)}{1 + \lambda(0)} \frac{1 + \lambda(t)}{\lambda(t) \delta(t)}, \quad (4.1)$$

$$c_H(t) = \frac{\delta(t)}{1 + \lambda(t)}, \quad c_L(t) = \frac{\lambda(t) \delta(t)}{1 + \lambda(t)}, \quad (4.2)$$

$$X_H(T) = \frac{\delta(T)}{1 + \lambda(T)}, \quad X_L(T) = \frac{\lambda(T) \delta(T)}{1 + \lambda(T)}, \quad (4.3)$$

where $\lambda(0)$ solves $T + 1 = e_H E \left[\int_0^T (1 - \tau_H)(1 + \lambda(t)) dt + 1 + \lambda(T) \right]$ and the weighting process λ and the stock price S jointly solve

$$\begin{aligned} d\lambda(t) &= \lambda(t) \frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)} \\ &\times \left[\left(\sigma_\delta(t) + \frac{(\tau_H - \tau_L) \delta(t)}{(1 + \lambda(t)) S(t) \sigma(t)} \right) dt + dW(t) \right], \end{aligned} \quad (4.4)$$

$$\begin{aligned} S(t) &= \frac{\delta(t)}{1 + \lambda(t)} \\ &\times E \left[\int_t^T (1 + \lambda(s))(1 - \tau_H) ds + 1 + \lambda(T) | \mathcal{F}_t \right]. \end{aligned} \quad (4.5)$$

Due to differential taxation, each investor's optimal consumption process is no longer proportional to the aggregate dividend process. It now also fluctuates due to incomplete risk sharing induced by differential taxation.

Proposition 4.2 reports the investor-specific market prices of risk and the interest rate for the logarithmic investor case.

PROPOSITION 4.2. If equilibrium exists, each investor's market price of risk is given by

$$\theta_H(t) = \sigma_\delta(t) - \frac{\lambda(t)}{1 + \lambda(t)} \frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)}, \quad (4.6)$$

$$\theta_L(t) = \sigma_\delta(t) + \frac{1}{1 + \lambda(t)} \frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)}. \quad (4.7)$$

The interest rate is given by

$$r(t) = \mu_\delta(t) - \sigma_\delta(t)^2 - \frac{\lambda(t)}{(1 + \lambda(t))^2} \left(\frac{(\tau_H - \tau_L) \delta(t)}{S(t) \sigma(t)} \right)^2. \quad (4.8)$$

Consequently, we have (i) $r(t) < \bar{r}(t)$, (ii) $\theta_H(t) < \bar{\theta}(t) < \theta_L(t)$ if $\sigma(t) > 0$.

In the logarithmic case, further comparisons with the benchmark can be made. If $\sigma(t) > 0$, the higher-taxed investor has a market price of risk lower than the benchmark, while the lower-taxed investor has a higher market price of risk. Since $\sigma_{c_n}(t) = \theta_n(t)$ under logarithmic preferences, risk is transferred from the higher-taxed investor to the lower-taxed investor, leading to suboptimal risk sharing. In aggregate, investors attempt more precautionary savings to account for this inefficiency; hence, the equilibrium interest rate falls below its benchmark value to compensate. These effects are driven by the differential in taxation, not by the taxation alone. Also note that if we assume the underlying dividend process is a geometric Brownian motion, the equilibrium market prices of risk and the equilibrium short-rate are now time-varying as opposed to constant in the no-tax benchmark.

4.2. NUMERICAL CHARACTERIZATION

Since it appears intractable to analytically solve for the weighting process and the stock price, we numerically solve the stock price in terms of the dividend and weighting processes to gain additional characterization of the differential taxation equilibrium. We make the additional assumptions that the dividend process follows a geometric Brownian motion (μ_δ and $\sigma_\delta > 0$ are constants) and $\tau_L = 0$. Since the differential in taxation drives most deviations from the benchmark economy, this simplified case should capture much of the effect and intuition of taxation.

PROPOSITION 4.3. For $\tau_L = 0$, if an equilibrium exists, the equilibrium stock price is given by

$$S(t) = (1 - \tau_H)\delta(t) \left(\frac{T-t}{1 + \lambda(t)} + h(\lambda(t), t) \right) \quad (4.9)$$

where $h(\lambda, t)$ is the solution to the following partial differential equation

$$\begin{aligned} \frac{1}{2}\lambda^2\sigma_\lambda^2 [(1 + \lambda)h_{\lambda\lambda} + 2h_\lambda] + \lambda\mu_\lambda [(1 + \lambda)h_\lambda + h] \\ + \lambda + (1 + \lambda)h_t = 0, \end{aligned} \quad (4.10)$$

with $\mu_\lambda(\lambda, t)$ and $1/\sigma_\lambda(\lambda, t)$ given by

$$\mu_\lambda(\lambda, t) = \sigma_\delta\sigma_\lambda + \frac{\sigma_\lambda^2}{1 + \lambda}, \quad (4.11)$$

$$\begin{aligned} 1/\sigma_\lambda(\lambda, t) = \frac{\sigma_\delta(1 - \tau_H) \left(\frac{T-t}{1+\lambda} + h \right)}{2\tau_H} \\ \pm \frac{\sqrt{\sigma_\delta^2(1 - \tau_H)^2 \left(\frac{T-t}{1+\lambda} + h \right)^2 + 4\tau_H(1 - \tau_H)\lambda \left(h_\lambda - \frac{T-t}{(1+\lambda)^2} \right)}}{2\tau_H}, \end{aligned} \quad (4.12)$$

subject to the boundary conditions

$$h(+\infty, t) = \frac{T - t + 1}{1 - \tau_H}, \quad h(0, t) = \frac{1}{1 - \tau_H}, \quad h(\lambda, T) = \frac{1}{1 - \tau_H}. \quad (4.13)$$

Proposition 4.3 reveals that the price-to-dividend ratio is driven only by the exogenous quantities τ_H , σ_δ , the time-to-horizon $T - t$, and by the endogenous stochastic weighting λ . The boundary conditions (4.13) with respect to the weighting process ensure the limits of the equilibrium stock price (4.9) collapse to the appropriate single investor valuations. For example, as $\lambda \rightarrow +\infty$, investor L holds all the wealth in the economy leading to $S(t) = \delta(t)(T - t + 1)$; as $\lambda \rightarrow 0$, $S(t) = (1 - \tau_H)\delta(t)(T - t) + \delta(T)$. The quasi-linear parabolic partial differential equation (4.10) with boundary conditions (4.13) was numerically solved using a method of lines algorithm.¹² From (4.12), two partial differential equations characterize the stock price for each parameterization. In all trials numerically solved, convergence only occurred for those taking the radical term in the denominator of (4.12) as positive. These are the results presented.

For a reasonable choice of tax rate ($\tau_H = 0.2$) and aggregate consumption volatility ($\sigma_\delta = 0.04$), Figures 1 and 2 plot the solutions for the stock price to dividend ratio S/δ (relative to the benchmark value, $T - t + 1$) and the stock return volatility σ over the whole space of λ and t , taking T as 100 (in arbitrary units). From the numerical estimation, a typical path of an equilibrium is inlayed on Figures 1 and 2. Since Figure 2 (and all other parameters tried) showed the stock return volatility to be consistently positive, for terseness in the subsequent discussion, we assume σ is always positive.

These figures reveal the price-to-dividend ratio to be lower (higher) than the benchmark when investor H (L) holds the majority of the total wealth in the economy, while the stock price volatility is higher than the benchmark for most of the space. An increased stock volatility over the benchmark persisted for most of the space for several choices of parameters ($\tau_H = 0.05, 0.10, 0.20, 0.30, 0.40$ and $\sigma_\delta = 0.02, 0.04, 0.08, 0.12, 0.16$). Since the instantaneous covariance between the price-to-dividend ratio and the dividend is

$$\text{cov} \left(d \left(\frac{S(t)}{\delta(t)} \right), d\delta(t) \right) = S(t) (\sigma(t) - \sigma_\delta(t)) dt,$$

an increased stock return volatility (σ) over the benchmark (σ_δ) is associated with a stock price-to-dividend ratio which covaries positively with the aggregate dividend.

¹² The method of lines algorithm, commonly used in physics and engineering and recently applied to American option valuation (Carr, 1998; Meyer and van der Hoeck, 1997), approximates a system of non-linear partial differential equations by a system of ordinary differential equations which can then be solved using standard numerical techniques. The major benefit of using the method of lines algorithm is its flexibility in solving a large class of non-linear systems where a specific finite difference algorithm would typically fail. A general reference to the method of lines is Schiesser (1991). Additional details are also available from the authors.

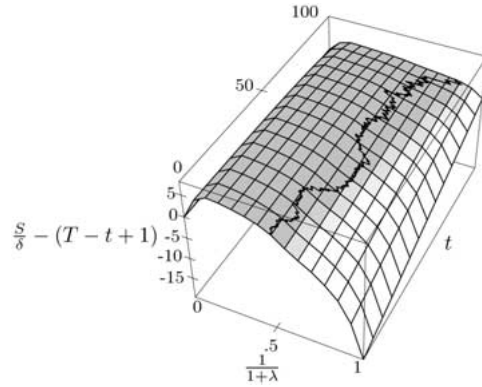


Figure 1. Stock price-to-dividend ratio net of the benchmark. The shaded area represents the region where the stock price-to-dividend ratio is above the benchmark. The solid line represents a typical path of the equilibrium with $e_H = e_L = 0.5$. Parameter choice: $\tau_H = 0.2$, $\sigma_\delta = 0.04$, $T = 100$.

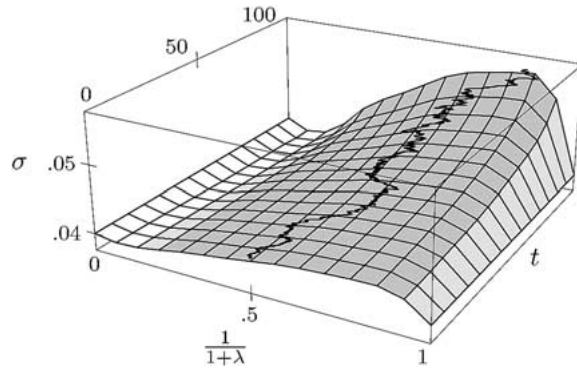


Figure 2. Stock volatility (σ). The shaded area represents the region where the stock volatility is above the benchmark (0.04). The solid line represents a typical path of the equilibrium with $e_H = e_L = 0.5$. Parameter choice: $\tau_H = 0.2$, $\sigma_\delta = 0.04$, $T = 100$.

In a benchmark no-tax economy, the price-to-dividend ratio is time-deterministic yielding zero covariance. In the presence of differential taxation, however, the price-to-dividend ratio must usually covary positively with the dividend, implying a more volatile stock price than dividend. When a positive shock to the dividend occurs, we have argued that the lower-taxed investor benefits more because he favors holding the stock, so his weight in the economy increases, leading the representative investor to more closely resemble the lower-taxed investor. This shift must cause an increase in the stock-to-dividend ratio due to an increased weighting on the lower-taxed investor who values the stock more highly. This leads to an increased stock return volatility.

Once the stock price and volatility have been generated, the remaining quantities of interest may be deduced from Propositions 4.1 and 4.2. As expected from (4.6)–(4.7), the equilibrium market price of risk (and hence consumption volatil-

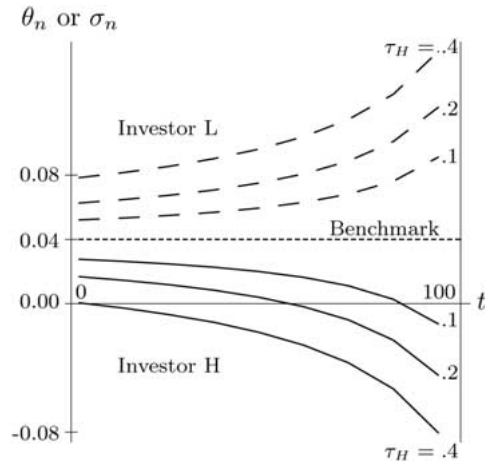


Figure 3. Each investor's market price of risk versus time for a given τ_H . Investor H 's (L 's) market prices of risk for $\tau_H = 0.1, 0.2, 0.4$ are given by the solid (dashed) lines. Parameters fixed: $1/(1 + \lambda) = 0.5$, $\sigma_\delta = 0.04$, $T = 100$.

ity) of the taxed investor is always lower than the benchmark, while the untaxed investor's market price of risk is always higher. The taxed investor's market price of risk may go negative while the untaxed investor's does not. Similarly, the taxed investor's risk premium and portfolio weight ϕ_n (proportion of wealth invested in the stock) are lower than the benchmark and can go negative, while investor L 's are higher than the benchmark and do not appear to ever go negative for all parameter choices examined. This pattern of investment is reversed as compared to earlier mean-variance framework literature with differential taxation (Elton and Gruber, 1978). In a mean-variance setting, the higher-taxed invests more in the risky stock due to a reduction in the exogenous after-tax volatility of the stock. Due to the endogenous stock volatility in our setting, such an effect does not occur. Figures 3, 5, and 7 plot the market prices of risk, risk premia, and portfolio weights versus time for a fixed weighting λ where both investors are consuming equal quantities, with τ_H as a varying parameter. Figures 4, 6, and 8 plot these quantities versus the taxed investor's weightings with σ_δ as a varying parameter. Note also that the pre-tax risk premium of the stock is given in Figures 5 and 6 by investor L 's risk premium since $\tau_L = 0$.

The deviation from the benchmark for all these quantities is seen to be increasing in the tax rate τ_H , and decreasing in the appropriate investor's share of aggregate total wealth, the aggregate consumption risk (dividend risk), and the time-to-horizon. As a result, even though investors exhibit identical CRRA preferences, their market prices of risk, risk premia, and portfolio weights show both wealth and time-to-horizon effects unlike in the benchmark economy. Specifically, a taxed investor puts more weight in the risky asset the wealthier he is, and less in the risky asset as he approaches his horizon (moving to the left in Figure 7). He exhibits behavior, then, similar to that seen in the panel data (Blume and Zeldes,

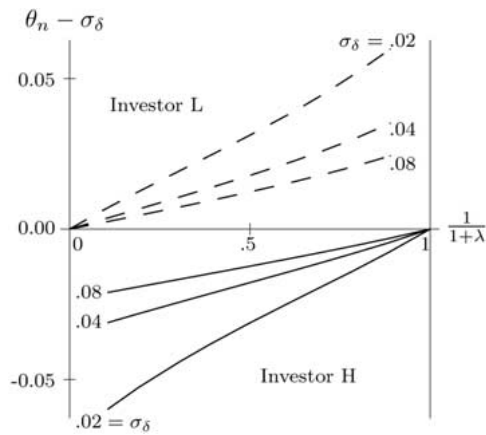


Figure 4. Each investor's market price of risk over the benchmark versus $1/(1+\lambda)$ for a given σ_δ . Investor H 's (L 's) market prices of risk over the benchmark for $\sigma_\delta = 0.02, 0.04, 0.08$ are given by the solid (dashed) lines. Parameters fixed: $t = 50$, $\tau_H = 0.2$, $T = 100$.

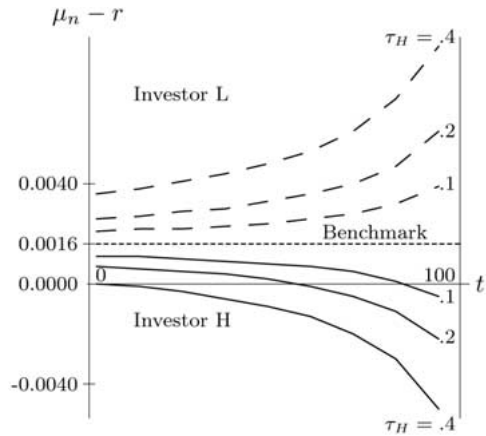


Figure 5. Each investor's stock risk premium versus time for a given τ_H . Investor H 's (L 's) risk premium for $\tau_H = 0.1, 0.2, 0.4$ is given by the solid (dashed) lines. Parameters fixed: $1/(1+\lambda) = 0.5$, $\sigma_\delta = 0.04$, $T = 100$.

1994) and he supports conventional wisdom. The untaxed investor has the opposite behavior; his market price of risk and portfolio weight decrease as he becomes more wealthy and increase as he approaches his horizon. These trends in Figures 3–8 are representative of all parameter values in the numerical trials.

One reason for the analytical intractability of this taxation problem is that there are two interacting effects of the differential taxation. Firstly, investor H faces taxed dividends so he sees the stock as less favorable and will, all other things being equal, divest to some extent out of the stock. Secondly, investor L receives a stochastic “endowment” ϵ_L which causes him to deviate from an instantaneous mean-variance portfolio. These two effects are interlinked because the endowment

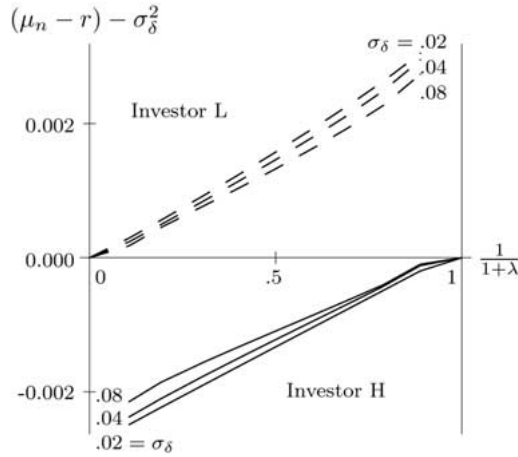


Figure 6. Each investor's stock risk premium over the benchmark versus $1/(1+\lambda)$ for a given σ_δ . Investor H 's (L 's) risk premium over the benchmark for $\sigma_\delta = 0.02, 0.04, 0.08$ is given by the solid (dashed) lines. Parameters fixed: $t = 50$, $\tau_H = 0.2$, $T = 100$.

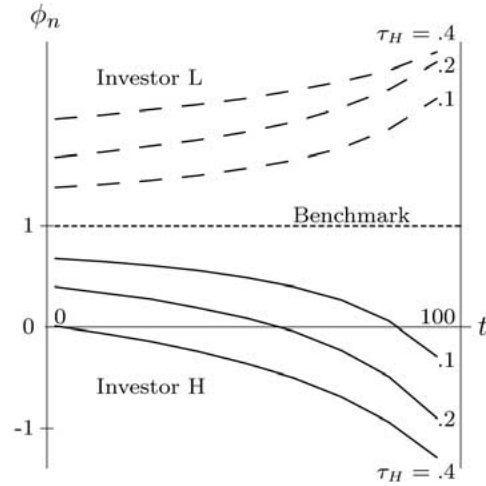


Figure 7. Each investor's proportion of wealth in stock versus time for a given τ_H . Investor H 's (L 's) proportion of wealth in stock for $\tau_H = 0.1, 0.2, 0.4$ is given by the solid (dashed) lines. Parameters fixed: $1/(1+\lambda) = 0.5$, $\sigma_\delta = 0.04$, $T = 100$.

to the untaxed sector (investor L) depends on the taxed sector's portfolio holdings, which in turn depend on how prices are impacted by both sectors.

In order to gain some simple intuition for our numerical results, we may heuristically consider the two effects of taxation separately. We first consider the "direct effect" (d) of the taxation τ_H on the taxed investor, assuming all price parameters (except μ_H) are held as in the benchmark. Investor H "observes" a downward shift in the stock return, $(-\tau_H \delta/S)$, proportional to the tax rate and to the importance of today's dividends in the total stock value. As the horizon approaches, today's

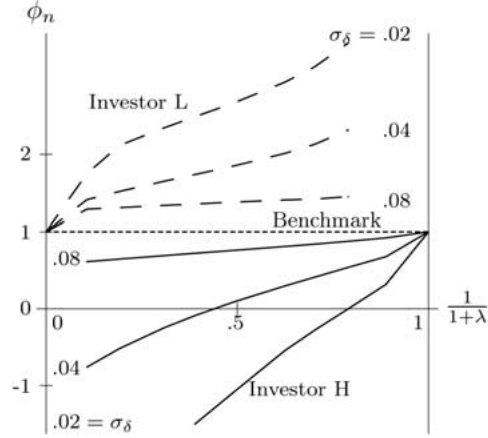


Figure 8. Each investor's proportion of wealth in stock versus $1/(1+\lambda)$ for a given σ_δ . Investor H 's (L 's) proportion of wealth in stock for $\sigma_\delta = 0.02, 0.04, 0.08$ is given by the solid (dashed) lines. Parameters fixed: $t = 50$, $\tau_H = 0.2$, $T = 100$.

dividends become a bigger proportion of the total stock value, so the shift is inversely proportional to the time-to-horizon. Investor H , then, divests out of the stock in proportion to his tax rate and in inverse proportion to the time to horizon and aggregate risk ($\Delta^d \phi_H = \Delta^d(\theta_H)/\sigma_\delta = \Delta^d(\mu_H)/\sigma_\delta^2 = -\tau_H/(T-t+1)\sigma_\delta^2$). In response to the decreased demand for the risky asset, prices must adjust to make the stock more favorable. For simplicity we suppose this occurs via a parallel increase in both investors' expected stock returns μ_n , the "indirect effect" (i) of taxation. To clear markets we require $\Delta^i \phi_H + \lambda \Delta^i \phi_L + \Delta^d \phi_H = 0$ and so deduce that $\Delta^i \phi_H = \Delta^i \phi_L = \Delta^i(\theta_H)/\sigma_\delta = \Delta^i(\theta_L)/\sigma_\delta = \Delta^i(\mu_H)/\sigma_\delta^2 = \Delta^i(\mu_L)/\sigma_\delta^2 = \tau_H/(1+\lambda)(T-t+1)\sigma_\delta^2$. Combining the direct and indirect effects, there is a net decrease in investor H 's stock holdings, of $\Delta \phi_H = \Delta(\theta_H)/\sigma_\delta = -\tau_H \lambda / (1+\lambda)(T-t+1)\sigma_\delta^2$. This first order intuition, then, captures all the basic trends of Figures 3–8. Investor H substitutes out of the stock, so all prices adjust, which leads investor L to hold more stock and investor H to hold less, but more than if prices had not adjusted. The magnitude of this effect is increasing in the tax rate, and decreasing in the appropriate investor's wealth, the time to horizon, and the aggregate consumption risk.

Our analytical results showed the interest rate to be unambiguously decreased relative to the benchmark when investors have identical constant relative risk averse preferences. Since the amount of risk transferred from the taxed to the untaxed investor is increasing in the tax rate, and decreasing in the time-to-horizon and dividend risk, the suboptimality of the risk-sharing and the interest rate decrease would be expected to exhibit the same dependencies. Figures 9 and 10 show the anticipated dependencies, as did all other parameter choices tried. Similarly, when either investor's proportion of the aggregate wealth is very small, the risk transfer

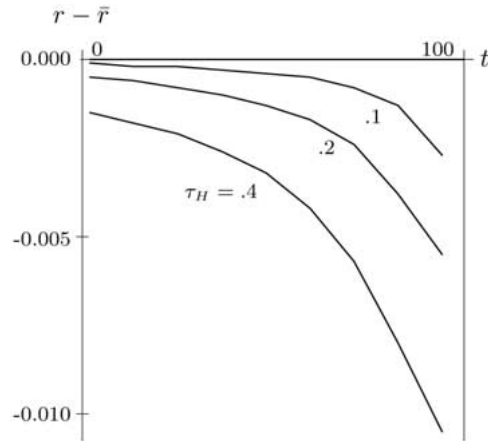


Figure 9. Interest rate over benchmark versus time for a given τ_H . Parameters fixed: $1/(1+\lambda) = 0.5$, $\sigma_\delta = 0.04$, $T = 100$.

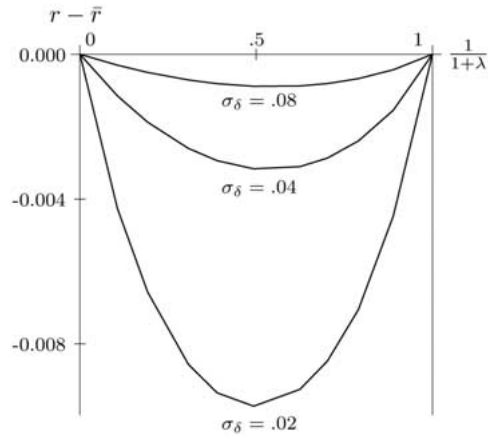


Figure 10. Interest rate over benchmark versus $1/(1+\lambda)$ for a given σ_δ . Parameters fixed: $t = 50$, $\tau_H = 0.2$, $T = 100$.

is small, with higher transfer when investors are more balanced. The interest rate inherits this same behavior as seen in Figure 10.

Some of these dependencies, although consistent with the intuition, seem somewhat unexpected. For example, the impact of taxation does not grow monotonically with the size of the taxed sector of the economy. Also when there is less time remaining in the economy, there is a bigger impact of taxation. And finally, while the interest rate decrease is caused by suboptimal risk sharing, the impact is more pronounced when there is less aggregate risk to share.

In order to explain the impact of taxation on the stock price we now need to consider the second effect, the tax redistribution to investor L . The stock itself does not now provide all the aggregate consumption to the economy, so

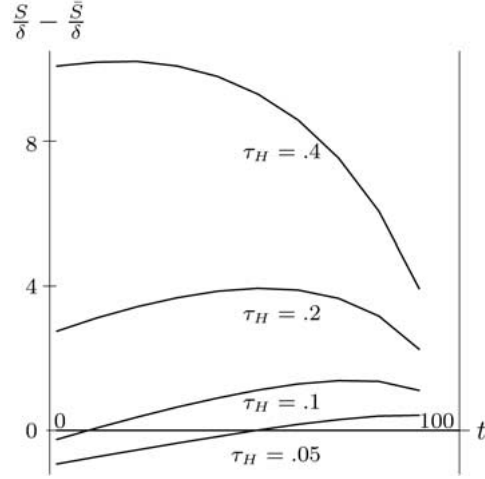


Figure 11. Price-to-dividend ratio over benchmark versus time for a given τ_H . Parameters fixed: $1/(1+\lambda) = 0.5$, $\sigma_\delta = 0.04$, $T = 100$.

$$\begin{aligned}
 S(t) &= \delta(t)(T - t + 1) - \frac{1}{\xi_L(t)} E \left[\int_t^T \xi_L(s) \epsilon_L(s) ds | \mathcal{F}_t \right] \\
 &= \delta(t)(T - t + 1) - \frac{1}{\xi_L(t)} E \left[\int_t^T \xi_L(s) \tau_H \alpha_H(s) \delta(s) ds | \mathcal{F}_t \right]. \quad (4.14)
 \end{aligned}$$

From (4.14) and the earlier intuition, we anticipate non-monotonicities in S ; for small taxation (small τ_H , high σ_δ and/or far from the horizon), the taxed investor should still hold positive weight in the stock and so pay positive taxes, which are then transferred to the untaxed investor thus reducing the stock price. At some point, as the taxation is increased further, the increasing tax rate will be overcome by the reduction in the taxed investor's holding, so the tax reallocation will start to decrease and the stock price returns toward the benchmark. For extremely high taxation, the taxed investor will prefer to short-sell the stock, “deducting” the dividend paid, so that the untaxed investor receives a negative reallocation. The stock's price will then rise above the benchmark value. Figures 11–12 plot the stock price and show the expected dependencies and non-monotonicity. Again the qualitative features of these figures are robust to the various parameter values we employed.

The deviation of the stock return volatility from the benchmark is usually positive and shows similar dependencies as all other quantities. When the taxed investor's weight is low, the volatility may fall below its benchmark value. Figures 13 and 14 plot the dependencies. These deviations appear persistent for the parameter values we examined.

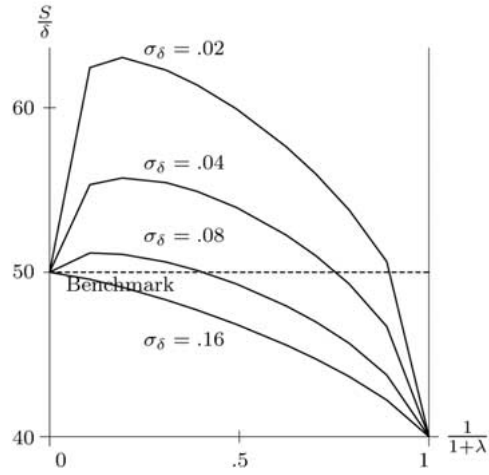


Figure 12. Price-to-dividend ratio versus $1/(1 + \lambda)$ for a given σ_δ . Parameters fixed: $t = 50$, $\tau_H = 0.2$, $T = 100$.

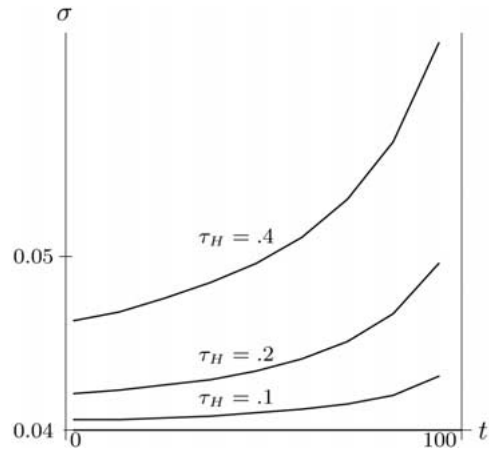


Figure 13. Stock volatility (σ) versus time for a given τ_H . Parameters fixed: $1/(1 + \lambda) = 0.5$, $\sigma_\delta = 0.04$, $T = 100$.

5. Conclusion

We develop a pure-exchange, general-equilibrium model where investors face differential dividend taxation, and investigate the implications of this form of taxation. Differential dividend taxation induces heterogeneity across investors' after-tax investment opportunity sets, leading to Pareto inefficient equilibrium allocations. Using a state-dependent representative investor construction, all equilibrium quantities are characterized. Differentially-taxed investors differ in their pricing of risk, leading to investor-specific consumption-based CAPMs. Additional effects of differential taxation are discussed concerning the equilibrium interest rate and investors' consumption dynamics. Under logarithmic preferences, risk is unam-

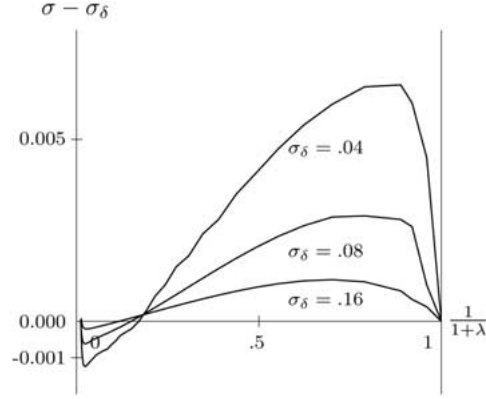


Figure 14. Stock volatility over benchmark ($\sigma - \sigma_\delta$) versus $1/(1 + \lambda)$ for a given σ_δ . Parameters fixed: $t = 50$, $\tau_H = 0.2$, $T = 100$.

biguously transferred from the higher to lower taxed investor. This suboptimal risk sharing induces a decrease in the equilibrium interest rate. In the logarithmic setting, numerical analysis of a quasi-linear partial differential equation characterizing the equilibrium stock price reveals additional features of differential taxation.

Areas for future work could include exploring more complex taxation strategies, exploring differences in relative pricing across multiple non-redundant stocks, or applying the numerical techniques used here to other asset pricing problems with frictions. One area we have considered is that of nonlinear dividend taxation (Basak et al., 1998). In this setting, an investor's consumption-portfolio problem is complicated by the stock's expected return being policy-dependent. Such a problem is amenable to the techniques developed by Cvitanic and Karatzas (1992), Cvitanic (1995), and Cuoco and Cvitanic (1998). The particular piecewise linear case of one tax rate for positive stock holdings and a lower rate for negative stock holdings is quite tractable. For logarithmic preferences with one taxed investor, the equilibrium resembles our linear taxation model with either the higher or the lower tax rate, or lies in an extended region where the taxed investor has insufficient incentive to hold the stock long or short. For other work related to the current paper, see Basak and Croitoru (2001) who study the equilibrium implications of nonlinear taxation with redundant securities and Belfer and Benninga (2002) who study the equilibrium effects of introducing options in a discrete-time version of our model.

Appendix: Proofs

Proof of Lemma 2.1. Under appropriate regularity conditions, the discounted gains process $\xi_n(t)S_n(t) + \int_0^t (1 - \tau_n)\xi_n(s)\delta(s)ds$ is a martingale, implying (2.6). $\square$

Proof of Proposition 3.1. From clearing in the consumption good market and the final wealth market, (3.4), (3.5), (3.9), (3.10), and (3.6) follow from the represent-

ative investor construction and by substituting into the solution to each investor's consumption-portfolio problem. Equation (3.8) follows by substituting (3.4) and (3.5). To show (3.7), apply Itô's lemma to $\lambda(t) = y_H \xi_H(t)/y_L \xi_L(t)$ and substitute the market price of risk for investor H given in Proposition 3.3.

To show the converse, assume that there exists ξ_H , ξ_L , λ , and S satisfying (3.4)–(3.8). Clearing in the consumption good market (3.1) and the final wealth market (3.2) follows from the solutions to the investors' maximization problems (2.8) with (3.4) and (3.5) substituted.

To show clearing in the bond market, define a *tax-free* state price density process $\xi(t)$ by

$$d\xi(t) = -\xi(t) [r(t)dt + \theta(t)dW(t)], \quad (\text{A.1})$$

where

$$\theta(t) \equiv \theta_n(t) + \frac{\tau_n \delta(t)}{S(t)\sigma(t)} = \theta_H(t) + \frac{\tau_H \delta(t)}{S(t)\sigma(t)} = \theta_L(t) + \frac{\tau_L \delta(t)}{S(t)\sigma(t)}.$$

This state price density process is the appropriate set of Arrow-Debreu security prices for a fictitious untaxed investor in the economy. Define the stock price plus the value of accumulated dividends under this state price density by

$$\bar{S}(t) \equiv S(t) + \frac{1}{\xi(t)} \int_0^t \xi(s) \delta(s) ds.$$

An application of Itô's lemma to $\xi \bar{S}$ shows that $\xi \bar{S}$ is a martingale and hence the analogue of Lemma 2.1 holds, yielding

$$S(t) = \frac{1}{\xi(t)} E \left[\int_t^T \xi(s) \delta(s) ds + \xi(T) \delta(T) | \mathcal{F}_t \right]. \quad (\text{A.2})$$

Applying Itô's lemma to $\xi(X_H + X_L)$ yields

$$d(\xi(t)(X_H(t) + X_L(t))) = -\xi(t)(c_H(t) + c_L(t))dt + [(\alpha_H(t) + \alpha_L(t))S(t)\sigma(t) - (X_H(t) + X_L(t))\theta(t)]\xi(t)dW(t), \quad (\text{A.3})$$

from which it can be deduced that

$$\xi(t)(X_H(t) + X_L(t)) + \int_0^t \xi(s)(c_H(s) + c_L(s))ds$$

is a martingale, implying

$$X_H(t) + X_L(t) = \frac{1}{\xi(t)} E \left[\int_t^T \xi(s)(c_H(s) + c_L(s))ds + \xi(T)(X_H(T) + X_L(T)) | \mathcal{F}_t \right]. \quad (\text{A.4})$$

By substituting consumption good clearing and final wealth clearing into (A.4) and comparing (A.2) with (A.4), clearing in the bond results. To show clearing in the stock market, apply Itô's lemma to $\xi(t)S(t)$ yielding

$$d\xi(t)S(t) = -\xi(t)\delta(t)dt + [\sigma(t) - \theta(t)]S(t)\xi(t)dW(t). \quad (\text{A.5})$$

Then, by substituting $X_H(t) + X_L(t) = S(t)$ and $c_H(t) + c_L(t) = \delta(t)$ into (A.3) and matching diffusion coefficients of (A.4) and (A.5), clearing in the stock results. $\square$

Proof of Proposition 3.2. Under appropriate regularity conditions,

$$U'(\delta(t); \lambda(t))F(\delta(t), \lambda(t), t) + \int_0^t U'(\delta(s); \lambda(s))I_H(U'(\delta(s); \lambda(s)))ds$$

and

$$U'(\delta(t); \lambda(t))P(\delta(t), \lambda(t), t) + \int_0^t U'(\delta(s); \lambda(s))(1 - \tau_H)\delta(s)ds$$

are martingales under $\mathbf{P}$; hence, their drifts must be 0, resulting in the two partial differential equations given by (3.13) and (3.14). Furthermore, applying Itô's lemma to $P(\delta(t), \lambda(t), t)$, the volatility of the stock must satisfy (3.17). The dynamics of the weighting process follow from Proposition 3.1. $\square$

Proof of Proposition 3.3. Applying Itô's lemma to each investor's first order conditions, (2.9) or $u'_n(\hat{c}_n(t)) = y_n \xi_n(t)$, and matching diffusion terms we arrive at

$$\sigma_{c_n}(t) = \frac{\theta_n(t)}{A_n(t)}, \quad n \in \{H, L\}, \quad (\text{A.6})$$

where $\hat{c}_n(t)$ satisfies $d\hat{c}_n(t) = \mu_{c_n}(t)dt + \sigma_{c_n}(t)dW(t)$. From market clearing in the consumption good market (which implies $\sigma_{c_H}(t) + \sigma_{c_L}(t) = \delta(t)\sigma_\delta(t)$) and $1/A(t) = 1/A_H(t) + 1/A_L(t)$, equations (3.18) and (3.19) follow. $\square$

Proof of Proposition 3.4. Applying Itô's lemma to each investor's first order conditions, (2.9) or $u'_n(\hat{c}_n(t)) = y_n \xi_n(t)$, and matching deterministic terms we arrive at

$$\frac{r(t)}{A_n(t)} = \mu_{c_n} - \frac{1}{2} B_n(t) \sigma_{c_n}(t)^2, \quad n \in \{H, L\}, \quad (\text{A.7})$$

where $\hat{c}_n(t)$ satisfies $d\hat{c}_n(t) = \mu_{c_n}(t)dt + \sigma_{c_n}(t)dW(t)$. Using market clearing in the consumption good market (which implies $\mu_{c_H}(t) + \mu_{c_L}(t) = \delta(t)\mu_\delta(t)$) and

$1/A(t) = 1/A_H(t) + 1/A_L(t)$, the interest rate expression results by summing across (A.7) for both investors. $\square$

Proof of Proposition 3.5. An investor's consumption volatility follows from (A.6). An investor's consumption growth follows from (A.7). $\square$

Proof of Proposition 4.1. Substituting the logarithmic preferences assumption into Proposition 3.1 gives the desired result. $\square$

Proof of Proposition 4.2. Substituting the logarithmic preferences assumption into Propositions 3.3 and 3.4 gives the desired result. $\square$

Proof of Proposition 4.3. The stock price given by (4.9) satisfies the partial differential equation for the stock price subject to the boundary conditions given in Proposition 3.2. Note that the optimal wealth of investor L is given by $\bar{X}_H(t) = \delta(t)(T - t)/(1 + \lambda(t))$, so its PDE need not be solved. $\square$

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