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Production and the Real Rate of Interest: A Sample Path Equilibrium

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Abstract. This paper examines a multiperiod production economy where investors do not observe the realizations of productivity factors or security expected returns. Unlike previous work, which expresses the equilibrium conditions as functions of unobservable (to both real-world investors and empiricists) moments of the distributions of returns, we express the equilibrium real rate as a function of the observable sample paths of realizations of returns. We provide a framework for empirically testing this and other asset pricing models without outside-the-model econometric assumptions needed for producing the unobservable moments of returns. We construct versions of the restrictions for any time interval between observations.

Key words: Incomplete information, equilibrium interest rates, asset pricing, sample path equilibrium.

JEL classification codes: E43, G12, D92, D80, D51.

1. Introduction

This paper examines a multiperiod production economy where investors do not observe the realizations of productivity factors or security expected returns. Instead, they use security returns to make inferences about their investment opportunities. Unlike previous work that expresses equilibrium conditions as functions of moments of the distributions of returns (for example, Merton (1973), Breeden (1979), Cox, Ingersoll, and Ross (1985a)), or of conditional moments of returns (for example, Dothan and Feldman (1986), Detemple (1986), and Gennotte (1986)), we express the equilibrium real rate of interest as a direct function of the sample paths of realizations of returns.

The paper has two objectives:

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(1) *To explain the role of the instantaneous spot real rate of interest as a market-clearing link between security returns and the unobservable productivity factors*. In contrast to complete information models (see, for example, Cox et al. (1985b)), where the equilibrium real rate is a direct function of the productivity factors, here the rate becomes a function of security returns.[1] The ability to explicitly solve for the equilibrium interest rate in terms of these returns enables us to examine its role as a market-clearing link.

(2) *To provide a new framework for empirically testing a class of asset pricing models*. The models in this class enable us to solve investors' estimation problems within the theoretical framework. Because economic productivity factors are not directly observable, testing *restrictions* that involve realizations of these variables, or security expected returns, normally requires econometric assumptions outside the model. These assumptions are necessary to produce the moments of security returns that, in reality, are not observable to either investors or empiricists. In this paper, however, investors form these (conditional) expectations endogenously as functions of observable prices and possibly other observable macroeconomic and firm-specific variables. We are, then, able to bypass the dependency on the conditional moments and express the real interest rate as an explicit function of these observations. This constitutes a testable restriction, with implications on preferences, aggregate wealth, and the relation between the real rate and security returns.[2] We develop the restriction within a continuous time framework. As a result, we can construct versions of the restriction for any time interval between observations ("holding period").

Often, researchers who test asset pricing theories must estimate expectations outside the model they test, as part of the empirical procedure. This requires specific econometric approaches such as latent or instrumental variables. A few examples are Hansen and Hodrick (1983), Gibbons and Ferson (1985), Ferson et al. (1992), Campbell (1987), Shanken (1990), and Harvey (1991), Bekaert et al. (2001), and Harvey (2001).[3] In this paper, however, investors form their expectations within the model. Thus, we are able to deliver closed-form expressions for the

[1] Ultimately, all aspects of the economic equilibrium (portfolio demands, the term structure, consumption, prices, etc.) are functions of the fundamentals as "seen" through observable variables (outputs, security returns, macroeconomic variables, etc.). When investors observe the moments of their investment opportunities (here the productivity factors are these moments), however, they use them to make their portfolio/consumption decisions. In this context, the realized returns information becomes redundant.

[2] One of the more successful asset pricing works in terms of inspiring empirical studies, Breeden's (1979) "Consumption CAPM", identified restrictions "along the equilibrium path" between the observable security returns and aggregate consumption. Here, we construct the equilibrium in terms of observables, thus, the "reduction in form" from unobservable productivity factors to observable returns is independent of preferences.

[3] One of the main differences between these works and ours is that, here, all the coefficients of the testable restriction are constrained by the underlying theory. Typically, these coefficients are endogenously determined functions of a few underlying parameters. Please see the example in Footnote 5.

conditional expectations as functions of model parameters and variables observable to the empiricist. The restriction that we produce consists only of realizations of the real rate and security returns, model parameters, and, possibly, preferences and aggregate wealth.

Models with exogenous expectations (Cox et al. (1985a), for example) obtain equilibrium restrictions of the form

$$r_t = Q_1 + P_3\theta_t, \tag{1.1}$$

where r_t is the realized, market-clearing, real rate of interest at date t, Q_1 and P_3 are functions of model parameters (Q_1 possibly stochastic), and θ_t is an exogenously assumed process of expectations of rates of returns or productivity factors.[4] The process θ_t, however, is unobservable to real-world investors and to empiricists. Therefore, direct testing of Equation (1.1) requires the exogenous choice of an information set and a structural form in which θ_t depends on this information set.[5] Testing the *indirect* effects of Equation (1.1) is also possible. This would require the introduction of financial markets, and the pricing of interest rate claims under strong assumptions about preferences and returns' correlations. (See for example Cox et al. (1985b), Feldman (1989), and Longstaff and Schwartz (1992)].[6] Models

[4] In this equation and the next two, Q, Q_1, Q_2 and P_1 are scalar functionals and θ_t, m_t, η_s and η_t are scalars or vectors. The dimensions of the functionals P_2, P_3, and P_4 are the same as that of η_t, θ_t, and m_t, respectively.

[5] An empiricist could proceed with direct testing of the Cox, Ingersoll, and Ross equilibrium restriction (1.1) by choosing an information set $\{X_1, X_2, X_3, \ldots\}$, and a structure

$$\hat{\theta}_t = b_0 + b_1X_{1t} + b_2X_{2t} + \cdots \tag{F.1}$$

and then estimating the system

$$\hat{\theta}_t = b_0 + b_1X_{1t} + b_2X_{2t} + \cdots, \tag{F.2}$$

$$r_t = Q_1 + P_3\hat{\theta}_t. \tag{F.3}$$

This procedure, however, gives rise to the following questions: Are the chosen information set and the assumed structural form of θ_t (Equation (F.1)) consistent with the underlying equilibrium? Is a rejection of the model a consequence of an inappropriate choice of the information set and the structural form of θ_t? Is an inability to reject the model a consequence of fact that the coefficients b_0, $b_1, b_2, \ldots$, unlike Q_1 and P_3, are not restricted by the underlying equilibrium.

[6] Testing the indirect effects of Equation (1.1) could proceed as follows. First, one needs to make strong assumptions about preferences (logarithmic preferences in many cases) and the correlation structure of realized returns. These could give rise to functional forms of Q_1 and P_3, which ensure that observable variables (such as the short rate and the volatility of the short rate) become invertible functions of the state variable (s). This will be the case, for example, in an economy with one state variable (i.e., where θ_t is a scalar) and where Q_1 and P_3 become deterministic. In this case, r_t in Equation (F.3) becomes an invertible function of θ_t. Then, one needs to introduce financial markets and price multiperiod claims on the short rate. Observations of the short rate process and claim prices

with endogenous expectations (Feldman (1989), for example) obtain equilibrium restrictions of the form

$$r_t = Q_2 + P_4 m_t, \tag{1.2}$$

where Q_2, P_4 are functions of model parameters (Q_2 possibly stochastic), and m_t is an endogenously determined process of expectations. Though the process m_t, which represents the *conditional* means of realized rates of returns, can be calculated by the investors in the model, m_t is not directly observable to real-world investors and empiricists.

Here we derive an equilibrium condition of the form

$$r_t = Q + P_1 r_s + P_2[\eta_t - \eta_s], \tag{1.3}$$

where Q, P_1 and P_2 are functions of model parameters (Q possibly stochastic), and r_3, r_t, η_s, and η_t represent the observable realized interest rates and realized security returns at dates s and t, $s < t$ respectively. Thus, in contrast to the first two equations, the last one consists exclusively of variables that are directly observable to investors in the model, real-world investors, and empiricists.[7]

Technically, the intuition behind the main results involves combining probabilistic and equilibrium analyses. The assumption that investors observe realized returns alone guarantees that all equilibrium demands and prices are functions of these returns. In this context we pursue explicit equilibrium expressions. We first conduct a probabilistic analysis of the information structure and obtain explicit expressions for the evolution of the conditional moments of realized returns. Then, we combine this analysis with results of financial equilibrium with incomplete information. This enables us to produce restrictions on the real rate in terms of particular realizations of returns.

We identify a steady-state restriction on realized security returns and the instantaneous spot real rate of interest. In contrast to analyses that assume complete information, here the real rate is not a function of the unobservable realizations of productivity factors. We show that the real rate of interest is a function of security returns. Under logarithmic preferences, this function is linear, thus implying the existence of a portfolio of securities whose rates of return are identical to the

can then be used to test the term structure induced by the underlying model. See for example Cox et al. (1985b), Feldman (1989), and Longstaff and Schwartz (1992), who also proceeded with empirical tests. A different approach to the indirect testing of the equilibrium restrictions is the analysis of the induced variance bounds. See for example Hansen and Jagannathan (1991).

[7] Rather than producing a relation of the form $r_t = Q_3 + P_5\eta_t$ which relates observation of the same date, t, we produce the relation in Equation (1.3), which relates observations in two dates, s and t. Ours is more general and captures more information. We take advantage of the model's continuous time framework and allow s and t in Equation (1.3) to be any dates, such that $s < t$. This eventually allows us to construct additional tests that depend on the interval – the "holding period". In addition, by choosing the restriction to be in the form of Equation (1.3), we avoid the problem of testing a continuous time restriction using discrete data.

real rate of interest realization by realization. A similar relation holds for expected values of portfolio rates of return, in complete information economies, where productivity factors or security expected returns are directly observable; that is, there may be portfolios of securities whose expected returns are equal to the real rate of interest. The expression of the real rate of interest as a function of security returns is testable and helps interpret the market-clearing role of the real rate. In this paper, we interpret this relation and suggest a way to test it empirically. Within the model, we determine a forecast of the real rate and identify an explicit solution for the replicating portfolio.

We first look at the equilibrium market-clearing rate of interest in an economy with incomplete information. As demonstrated in the past (see Feldman (1983), Dothan and Feldman (1986), Detemple (1986, 1991), and Gennotte (1986), for example), this equilibrium analysis results in a real interest rate that is a function of preferences, the production function, and the conditional means of the unobservable productivity factors. These conditional means, in turn, are functions of realized security returns. Next, we look at the probabilistic structure of the economy – the stochastic processes that describe the evolution of the productivity factors and security returns. We identify the processes that describe the evolution of the moments of the joint probability distribution and explicitly solve for the moments of the distribution of the unobservable productivity factors conditional on the realizations of security returns. Since both the equilibrium analysis and the probabilistic analysis describe the same variables, we can establish the sought-after restriction by equating the outcomes of both analyses. We obtain restrictions that are functions of any "holding period" cumulative returns.

Under logarithmic preferences, the restriction says that in steady state, when the effects of the initial conditions of the economy become negligible, the changes in the instantaneous spot real rate of interest are proportional to the deviations of realized security returns from their expected values. Moreover, there exists a portfolio of securities that includes all the securities whose returns are informative with respect to the unobservable productivity factors (and possibly a default-free bond) and that has a rate of return equal to the real rate of interest realization by realization. In particular, the equilibrium real rate of interest is equal to a constant, representing its long-term expected value, plus a "beta" type coefficient times the deviation of security returns from their expected values. Therefore, the higher realized returns are, the greater the innovation in the real rate; and a positive (zero, negative) deviation of security returns from their expected values implies an increase of (no change in, decrease of) the innovation in the real rate. Finally, we suggest two procedures for empirically testing the main result.

While the endogenous modeling of the formation of expectations is perhaps not interesting per se, it is useful in motivating particular functional forms and generating testable restrictions within the theoretical framework, as this paper demonstrates.

A special case of the incomplete information economy we investigate here is a complete information economy. This occurs when the error of the estimates of the productivity factors is zero. Thus, an additional empirical test that would be interesting is to determine whether the estimation errors are non-zero, rejecting the hypothesis of a complete information economy.

In our model, investors observe only a vector of realized returns on production processes, but we can interpret part of this vector as observable macroeconomic or firm-specific variables. This would not change the analysis but would allow a broader interpretation of the results.

Section 2 describes the economy; Section 3 develops and interprets the relation that expresses the real rate of interest as a function of security rates of return; Section 4 presents the forecast of the real rate; Section 5 develops a sample path equilibrium; Section 6 looks at empirical implications; Section 7 examines an example; and Section 8 concludes. Proofs of the propositions are in the appendix.

2. An Incomplete Information Production Exchange Economy

The underlying economic structure of the model in this paper is in the spirit of Cox et al. (1985a, 1985b). The model, however, allows the existence of an unobservable state variable. Feldman (1983, 1989, 1992) and Dothan and Feldman (1986) explored this economy; and Detemple (1986, 1991), Detemple and Murthy (1994), and Gennotte (1986) investigated related economies.

A partial sample of works that investigate related incomplete information and interest rate issues includes Coles and Loewenstein (1988), Ketterer (1990), Kuwana (1995), Karatzas and Xue (1991), Wang (1993, 1994), Coles et al. (1995), Jeffrey (1995), Kandel and Stambaugh (1996), Duffie et al. (1997), Naik (1997a, 1997b), Brennan (1998), David (1998), Lakner (1998), Zapatero (1998), Chacko and Das (1999), Kuwana (1999), Laskry and Lions (1999), Treich, (1999), Basak (2000), David and Veronesi (2000), Honda (2000), Lioui and Poncet (2000), Pastor (2000), Riedel (2000), Tzeng et al. (2000), Veronesi (2000), Yan (2000), Riedel (2001), and Xia (2001).

In a competitive frictionless multiperiod economy, individuals observe an $(n \times 1)$ vector of realized returns ξ_t of a single consumption/production numeraire good. We may think of the ξ_t as a vector of firms' outputs or security prices. These constant stochastic returns-to-scale outputs evolve as the Itô diffusions

$$d\xi_t = I(\xi_t)(A_0 + A_1\theta_t)dt + I(\xi_t)B_1 dW_t, \tag{2.1}$$

with initial condition ξ_0. An $[(n+1) \times 1]$ vector of independent Wiener processes, $W = \{W_t, F_t\}$, $W_0 = 0$, describes the underlying uncertainty in the economy. W is defined over a complete probability space (Ω, F, P) with a non-decreasing right continuous family of sub-σ-algebras $\{F_t, 0 \le t \le T\}$. A_0 and A_1 are known $(n \times 1)$ vectors of constants; B_1 is a known $[n \times (n+1)]$ matrix of constants; and

$B \triangleq B_1 B_1'$ – the instantaneous variance-covariance matrix of realized returns – is positive definite. $I(\xi_t)$ is a diagonal matrix with ξ_{ti}'s as elements.

Individuals, however, do not observe the realization of a productivity factor θ_t, the unobservable economic state variable, which evolves as

$$d\theta_t = (a_0 + a_1\theta_t)dt + b_1' dW_t. \tag{2.2}$$

The distribution of the initial condition $\theta + 0$, given ξ_0, is Gaussian with mean m_0 and variance γ_0; b_1 is a known $[(n+1) \times 1]$ vector of constants; $b_1' b_1 \triangleq b$ is positive; and a_0 and a_1 are known constants. We assume that $a_0 > 0$ and $a_1 < 0$ to ensure the stability of θ_t and its reversion to a positive asymptotic mean.

Individuals with identical endowments and preferences maximize expected lifetime time-additive utility of consumption,

$$E_{W_0, \xi_s : 0 \le s \le t} \left\{ \int_t^T h[c(\tau), \tau] d\tau \right\}. \tag{2.3}$$

They continuously allocate their initial wealth w_t (denoted also as w) among consumption c, investment u, and instantaneous borrowing and lending at an endogenously determined market-clearing rate r. Since θ_t is unobservable, neither c, u, nor r is a function of it. The von Neumann–Morgenstern utility function h is strictly concave and increasing, and is twice-differentiable; E is the expectation operator. The expectation in Equation (2.3) is conditional on current wealth and the state of the economy, and is taken with respect to all available information.

Though the utility function is time additive and the underlying stochastic processes are Markovian, at the current formulation, the unobservabilty of the productivity factor results in a non-Markovian structure that does not allow a state vector solution. At each point in time, investors need to update their consumption and portfolio decisions as a function of the moments of future returns. These moments are the state variables, and here they are productivity factors. Because these moments/state variables/productivity factors are not observable, investors must estimate them. Investors form estimates based on their available information and come up with conditional moments, determined endogenously, conditional on their information. In other words, investors engage in a Bayesian revision, using all available historical information, to determine, in each instant, the posterior distributions of the unobservable productivity factors. The dependency on historical information, or, on the past, negates the Markovian property. We can recapture, however, the Markovian structure, under certain conditions that are related to the "efficiency" of the presentation of the historical information through the conditional moments. If the posterior distributions have finite number of moments and if these moments can be updated recursively, we can recapture a Markovian structure and identify a state vector solution. In addition, if the process that updates previous estimates to become current estimates, the "innovation process", is a Brownian motion that generates the original information/σ-algebra, we can proceed using stochastic control techniques used to solve complete information economies. That is, to determine portfolio/consumption rules, we identify a σ-algebra equivalent economy with a finite number of moments that evolve in a Markovian fashion. Thus, we identify a Markovian representation of the original economy. Portfolio/consumption rules are optimal in the Markovian equivalent economy if and only if they are optimal in the original economy. In the Markovian economy, the posterior distribution (nonlinear filter) of θ_t, given the observations ξ_s, $0 \le s \le t$, describe the information flow (see Liptser and Shiryayev (1978) Theorems 12.1 through 12.8, pp. 21–35]:[8]

$$dm_t = (a_0 + a_1 m_t)dt + (D + A_1\gamma_t)' B^{-1/2} d\bar{W}_t, \tag{2.4}$$

$$d\gamma_t = [b + 2a_1\gamma_t - (D + A_1\gamma_t)' B^{-1}(D + A_1\gamma_t)]dt, \tag{2.5}$$

$$d\bar{W}_t = B^{-1/2}[I^{-1}(\xi_t)d\xi_t - (A_0 + A_1 m_t)dt], \tag{2.6}$$

where $m_0 \triangleq E[\theta_0 \mid \xi_0]$, $\gamma_0 \triangleq E[(\theta_0 - m_0)^2 \mid \xi_0]$. The variable $m_t \triangleq E[\theta_t \mid F_t^\xi]$ is the mean; $\gamma_t \triangleq E[(\theta_t - m_t)^2 \mid F_t^\xi]$ is the variance of the conditional distribution; and $\bar{W}_t$ is the innovation process, endogenously determined to be a vector of independent Wiener processes. The innovation process describes the deviations of the observations from the expected values. $D = B_1 b_1$ is the instantaneous covariance between realized returns and the unobservable factor.[9] The conditional mean, the estimate of the unobservable factor, is a sufficient statistic to the posterior distribution and is updated recursively.

The filter equations define the diffusion coefficient (instantaneous standard deviation) of m_t as

$$\sigma_m(\gamma_t) = B^{-1/2}(D + A_1\gamma_t). \tag{2.7}$$

They define the instantaneous covariance, G, between the rates of return on investment in ξ_t and the realizations of the estimate $m_t$of the unobservable factor, as

$$G \triangleq B^{-1/2}\sigma_m(\gamma_t) = D + A_1\gamma_t \tag{2.8}$$

[8] The underlying structure in the above Liptser and Shiryayev theorems allows the process parameters, a_0, a_1, b_1, A_0, A_1, and B_1 to be functions of ξ_1. This results in a conditionally Gaussian distribution of θ_t givenξ_s, $0 \le s \le t$, and a nonlinear filter. For simplicity, we analyze the case where the process parameters are constants, thus the distribution of θ_t, given ξ_s, $0 \le s \le t$, is (unconditionally) Gaussian and the filter is linear. The case of stochastic parameters, which results in a stochastic estimation error, or quality of information, is an interesting topic for future research.

[9] Liptser and Shiryayev (1978 Theorems 12.5 and 12.7) prove that if the conditional distribution of θ_0 and ξ_0 is Gaussian with mean m_0 and variance γ_0, then$\bar{W}_t$, m_t, and γ_t are the unique, continuous, measurable solutions of the system of filter equations. Moreover, the innovation process $\bar{W}_t$ generates the economy; that is, the σ-algebras $F_t^{\xi_0,\bar{W}}$ and F_t^ξ are equivalent.

where (solving for the case of $n = 1$ to simplify notation)

$$\gamma_t = \gamma_u \left[\frac{\gamma_{ss}}{\gamma_u} - \frac{\gamma_0 - \gamma_{ss}}{\gamma_0 - \gamma_u} e_t\right]\left[1 - \frac{\gamma_0 - \gamma_{ss}}{\gamma_0 - \gamma_u} e_t\right]^{-1}, \tag{2.9}$$

$$e_t \triangleq \exp\left[\left[-\frac{A_1^2}{B}\right](\gamma_{ss} - \gamma_u)^2\right], \tag{2.10}$$

$$\gamma_{ss} \triangleq k_2\{(k_1 - \rho) + [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\}, \tag{2.11}$$

$$\gamma_u \triangleq k_2\{(k_1 - \rho) - [(k_1 - \rho)^2 + (1 - \rho^2)]^{1/2}\}, \tag{2.12}$$

and

$$k_1 \triangleq \frac{a_1/b^{1/2}}{A_1/B^{1/2}}, k_2 \triangleq \frac{(bB)^{1/2}}{A_1}, \rho \triangleq \frac{D}{(bB)^{1/2}}. \tag{2.13}$$

Note that

$$\lim_{t\to\infty} \gamma_t = \gamma_{ss}, \text{ therefore } \lim_{t\to\infty} G = D + A_1\gamma_{ss} \triangleq G_s. \tag{2.14}$$

Note also that the quality of information, γ_t, has one, nonnegative, absorbing steady state (root), γ_{ss}.[10] In equilibrium, G is set to be an average of D and A_1, dynamically weighted by the filtering error. D represents the diffusion source of covariance between the unobservable factor and realized returns, and A_1 represents the drift source of covariance.

Using the filter equations, investors reformulate the problem as a Markovian One[11] (see also Dothan and Feldman (1986), Detemple (1986), and Gennotte (1986)):

$$\max_{c,u} E_{w_0,\xi_t}\left\{\int_t^T h[c(\tau), \tau]d\tau\right\}, \tag{2.15}$$

where

$$dm_t = (a_0 + a_1 m_t)dt + (D + A_1\gamma_t)' B^{-1/2} d\bar{W}_t, \tag{2.16}$$

$$d\gamma_t = [b + 2a_1\gamma_t - (D + A_1\gamma_t)' B^{-1}(D + A_1\gamma_t)]dt, \tag{2.17}$$

$$I^{-1}(\xi_t)d\xi_t = (A_0 + A_1 m_t)dt + B^{1/2} d\bar{W}_t, \tag{2.18}$$

[10] See Feldman (1989) Proposition 2, page 797, and its proof, pages 808–809.

[11] The notation of the expectation operator in Equation (2.15) is different from the one in Equation (2.3). We replaced the conditioning on $\xi_{s:0\le s\le t}$ with conditioning on ξ_t only. Under the Markovian representation, the current state of the economy ξ_t is sufficient for the full history.

s.t,

$$dw = w[u'I^{-1}(\xi_t)d\xi_t + (1 - u'\mathbf{1})r] - cdt, \tag{2.19}$$

$$c \geq 0, u \geq 0, w_0 > 0, w_T \geq 0. \tag{2.20}$$

Assuming all production processes are active, investors determine the equilibrium rate of interest and optimal portfolio/consumption rules as in Cox et al. (1985a):

$$r = u'(A_0 + A_1 m_t) + \frac{wJ_{ww}}{J_w}u'Bu + \frac{J_{wm}}{wJ_w}u'G, \tag{2.21}$$

or

$$\begin{aligned} r &= (1'B^{-1}1)^{-1}\left[1'B^{-1}(A_0 + A_1 m_t) + \frac{wJ_{ww}}{J_w} + \frac{J_{wm}}{wJ_w}1B^{-1}G\right] \\ &= \left[\frac{1'B^{-1}A_0}{1'B^{-1}1} + \frac{1}{1'B^{-1}1}\frac{wJ_{ww}}{J_w} + \frac{1'B^{-1}G}{1'B^{-1}1}\frac{J_{wm}}{wJ_w}\right] + \left[\frac{1'B^{-1}A_1}{1'B^{-1}1}\right]m_t, \end{aligned} \tag{2.22}$$

and

$$u = \frac{-J_w}{wJ_{ww}}B^{-1}[(A_0 + A_1 m_t) - r\mathbf{1}] + \frac{-J_{wm}}{wJ_{ww}}B^{-1}G, \tag{2.23}$$

where J, $J = J(w, m_t, t)$, is the indirect utility function of the optimization problem, $w^2u'Bu$ is the variance of optimally invested wealth, and $wu'G$ is the covariance between the estimate of the unobservable factor m_t and optimally invested wealth. Subscripts denote partial derivatives, and $\mathbf{1}$ is an $(n \times 1)$ vector of ones.

Equation (2.22) has identified the equilibrium, instantaneous, spot rate of interest r as a deterministic function of the estimate m_t of the unobservable productivity factor.

3. The Realized Equilibrium Spot Rate of Interest as a Function of Security Returns

Equation (2.22) shows that the equilibrium stochastic rate of interest is a function of m_t, the estimate of the unobservable productivity factor. This estimate, however, is a function of the observable realized returns. This section uses these properties to directly represent the realized real interest rate as a function of security returns. To attain the steady state relations that avoid the transient effects of the initial conditions of the economy, we seek the asymptotic value of the restriction.

In the first part of the following proposition, we transform the conditionally Gaussian system (θ_t, ξ_t) to a Gaussian system (θ_t, η_t). Then, in the second part of the proposition, we use the well-known properties of Gaussian systems to identify the functions that describe the evolution of the moments of θ_t and η_t.

PROPOSITION 1. (I) The state equations can be written as follows:

$$d\begin{bmatrix}\theta_t\\ \eta_t\end{bmatrix} = \begin{bmatrix}a_0\\ \hat{A}_0\end{bmatrix} + \begin{bmatrix}a_1 & 0\\ A_1 & 0\end{bmatrix}\begin{bmatrix}\theta_t\\ \eta_t\end{bmatrix}dt + \begin{bmatrix}b_1'\\ B_1\end{bmatrix}dW_t, \tag{3.1}$$

where the initial conditionsw are (θ_0, η_0), $\eta_t \triangleq \log \xi_t$ and $\hat{A}_0 \triangleq A_0 - 1/2B$.

(II) Let

$$A \triangleq \begin{bmatrix}a_1 & 0\\ A_1 & 0\end{bmatrix}, \text{ and } S \triangleq \begin{bmatrix}b_1'\\ B_1\end{bmatrix};$$

then the distribution of (θ_t, η_t) conditional on (θ_0, η_0) is a joint normal distribution,[12] i.e.,

$$\begin{bmatrix}\theta_t\\ \eta_t\end{bmatrix} \sim N[H(t), K(t)], \tag{3.2}$$

where

$$H(t) = e^{At}\left\{\begin{bmatrix}\theta_0\\ \eta_0\end{bmatrix} + \left[\int_0^t e^{-As}ds\right]\begin{bmatrix}a_0\\ \hat{A}_0\end{bmatrix}\right\}, \tag{3.3}$$

$$K(t = \int_0^t e^{A(t-s)}SS'e^{A'(t-s)}ds. \tag{3.4}$$

□

The particular representations of the results in Proposition 1, Part II, are taken from Arnold (1974), and were previously used in a finance context, for example, by Feldman (1983) and Wang (1993).

PROPOSITION 2. The realized spot rate of interest is function of security rates of return. Asymptotically, and in steady state

$$r = r^* + A^* \times \text{ASYCOVAR} \times \text{ASYVARR}^{-1}[\text{RRR} - \text{ASYEXRR}], \tag{3.5}$$

[12] To simplify the notation and exposition, we assume here that θ_0 is a constant rather than assuming that θ_0 is a normally distributed random variable. Our assumption does not change any of the steady-state results. However, if θ_0 had been a random variable before steady state, the results would have had the following minor changes: in Equation (3.3), θ_0 would have been replaced by its mean, and the right-hand side of Equation (3.4) would have had an additional term $e^{At}S_0S_0'e^{A't}$ where S_0S_0' is the variance-covariance matrix of the initial conditions (θ_0, η_0). This additional term would have resulted in the addition of a constant to $K_3(t)$, and in no changes to $K_1(t)$ and $K_2(t)$ (see the solution for e^{At} in Equation (A.12)). We define $K_1(t)$, $K_2(t)$, and $K_3(t)$ in Equation (A.20) as the elements of $K(t)$. The change of $K_3(t)$ by a constant is of no consequence in our analysis. This constant will vanish anyway when we calculate the asymptotic variance rate.

where

$$A^* \triangleq \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}} \tag{3.6}$$

$$r^* \triangleq \frac{\mathbf{1}B^{-1}A_0}{\mathbf{1}'B^{-1}\mathbf{1}} + \frac{1}{\mathbf{1}'B^{-1}\mathbf{1}}\frac{wJ_{ww}}{J_w} + \frac{\mathbf{1}'B^{-1}G_s}{\mathbf{1}'B^{-1}\mathbf{1}}\frac{J_{wm}}{J_w} - \frac{\mathbf{1}'B^{-1}A_1}{\mathbf{1}'B^{-1}\mathbf{1}}\frac{a_0}{a_1}. \tag{3.7}$$

ASYCOVAR, is the asymptotic covariance between the unobservable productivity factor and security rates of returns (a $(1 \times n)$ vectorl); $A^* \times$ ASYCOVAR, is the asymptotic covariance between security rates of returns and the real rate of interest (A^* is a scalar), ASYVARR, is the asymptotic variance rate of security rates of return (an $(n \times n)$ matrix; RRR, is the realized rates of return on securities (an $(n \times 1)$ vector); ASYEXRR, is the asymptotic expected rates of return on securities (an $(n + 1)$ vector).

In particular, for $n = 1$ Equation (3.5) becomes

$$r = r^* + \frac{A_1\left[\frac{bA_1}{2a_1^2} - \frac{D}{a_1}\right]}{B + 2A_1\left[\frac{bA_1}{2a_1^2} - \frac{D}{a_1}\right]}\left[\frac{\eta_t - \eta_0}{t} - \frac{E(\eta_t - \eta_0)}{t}\right]. \tag{3.8}$$

□

The dynamic stochastic nature of the economy results in an ever-evolving equilibrium. Thus, neither side of Equation (3.5) or (3.8) is normally constant over time. When we use the terms *asymptotic* and *steady state*, we refer to the equilibrium in an economy that evolved long enough that expected returns, variances, covariances, and the quality of information, are independent of their initial conditions.

We first examine the asymptotic relation in Equation (3.5) of Proposition 2. This choice frees the relation from the impact of the initial conditions of the model. In this case, the exogenous choice of the initial conditions does not enhance the structure or interpretation of the model. Moreover, because the quality of information is in steady state, intermediate returns are irrelevant when expressing the real rate as a function of dates 0 and t returns. In other words, there is some kind of path independence that generally disappears when the quality of information changes in time because it weighs differently returns of different dates. In this case, one could either characterize an error or express time t rate with time t information only (see footnote 7). The latter method, in fact, solves for the path dependency at the cost of obtaining a less general empirical restriction.

The expression r^*, in the equation for r, is the asymptotic expected return of the real rate r^* consisting of four addends. The first and last are risk-adjusted expected rates of return; the second and third are preferences dependent. The first addend is a risk-adjusted average of the elements of A_0, the value representing the firms' productivity that is independent of the productivity factor. The last addend is a product. The multiplier is a risk-adjusted average of A_1, the value representing firms' productivity that is dependent on the unobservable productivity factor, and the multiplicand is the asymptotic mean $-a_0/a_1$ of this factor. The second addend involves the second partials of the indirect utility function; it represents investors' relative risk aversion. The third term involves the cross partials of the indirect utility function and represents investors' sensitivity to changes in the investment opportunity set. Depending on the direct utility function, the preferences-dependent terms might or might not be functions of wealth and the conditional mean of the productivity factor.

In the expression for r in Equation (3.5), the second addend is a product of a ratio and a difference. The ratio is a "beta" type expression. The denominator is the variance of security returns, and the numerator is the covariance of the real rate and security returns. This covariance is a function of the covariance of the unobservable productivity factor and security returns. The difference is the deviation of security returns from their expected values. Markowitz's Portfolio Theory and Sharp, Mossin, Lintner's CAPM beta type relations are frequently used in finance. This relation is new and original, however, for two reasons. First it uses sample paths realizations and not moments or expectations, and second, it explains current equilibrium interest rates in terms of past and asymptotic equilibrium interest rates and realized security returns.

Equation (3.5) shows how, in equilibrium, r is set to clear the market. When the economy is doing well (poorly) – when realized rates of return are higher (lower) than their expected value – the innovation in the second addend is positive (negative). For logarithmic preferences, where $-WJ_w/J_w = 1$, and $J_{wm} = 0$ the substitution and the income effects offset each other, r^* is constant, and the innovation in r is due to the innovation in the second addend only. For other preferences, the substitution and the income effects do not offset each other, and r^* changes to reflect the net effect. Unlike Equation (2.22), where r is expressed in terms of expected averages, r is expressed here as a function of actual realized outputs. Note that the asymptotic mean for r is identical to the one determined by taking the mean of the expression in Equation (2.22), characterizing r as a function of m_t. In Section 7 below, we specialize Proposition 2 for the case of logarithmic preferences.

4. The Forecast of the Real Rate

PROPOSITION 3. Let $m(t,s) = E(\theta_t \mid F_s^{xi})$, $s \le t$. Then $m(t,s)$ is the time s mean square optimal forecast of θ_t, and

$$m(t,s) = -\frac{a_0}{a_1}[1 - e^{a_1(t-s)}] + m_s e^{a_1(t-s)}. \tag{4.1}$$

□

According to Equation (4.1), the optimal forecast of the productivity factor is a weighted average of current and expected future values. The weights are time-dependent exponentials. The one that weighs the current value starts at one and decays to zero, and the one that weighs the asymptotic value starts at zero and goes to one. Recall that the parameter a_1 is negative.

Denote the realization of the real interest rate at time u as r_u, and the forecast of r_t conditional on information available at time s as $n(t, s)$. Then, $n(t, s) = E(r_t \mid F_s^{\xi})$, $s \leq t$. Since r is a deterministic function of m_t and w, as specified in Equation (2.22), we can determine $n(t, s)$ by substituting in Equation (2.22) the optimal forecast of m_t, $m(t, s)$ as found in Proposition 3, and the forecast of w as implied by the forecast of realized returns in Equation (A.18).

5. A Sample Path Equilibrium

Asset pricing theories that assume complete information show how, in equilibrium, the market-clearing, instantaneous, spot real rate of interest is a function of realizations of productivity factors (see, for example, Cox et al. (1985b)). Because, in reality, these realizations and security expected returns are unobservable, one needs to develop special procedures for testing restrictions that involve these variables. In our model, however, the endogenous modeling of the unobservability of productivity factors makes it possible to identify relations that do not include the unobservable realizations of productivity factors. The empirical difficulty of measuring unobservable realizations of productivity factors is now replaced by the need to measure the deviations of security returns from their, endogenously determined, conditional expected values.

The restriction for the real rate in Equation (3.5) is in terms of a reference time point. If s is this point, then, RRR $-$ ASYEXRR measures the deviations at times $t, t > s$ from expectations formed at s. As the time interval $t - s$ grows, estimating these expectations might become difficult.

Alternatively, we can estimate the one-period-ahead relation and move the reference point forward for each measurement. This, perhaps, leads to the simplest empirical implementation. The expression that we need to estimate in this case is a non-asymptotic version of Equation (3.5). As before, we consider an economy that is far from its initiation. We assume, therefore, that the initial conditions no longer affect the equilibrium and that the quality of information is in a steady state.

PROPOSITION 4. Assume that the initial conditions have no effect (the economy is in “steady state”). Then, the time t realization of the real rate, conditional on information available at time s, is

$$r_t = n(t, s) + \frac{A^* \times \mathrm{COVAR}(t, s)}{\mathrm{VARR}(t, s)}[\mathrm{RRR}(t) - \mathrm{EXRR}(t, s)] \tag{5.1}$$

where $n(t, s)$ is as defined in Section 4 and COVAR(t, s), is the time t covariance between the unobservable productivity factor and security rates of returns, conditional on information available at time s (a $(1 \times n)$ vector); $A^* \times$ COVAR(t, s), is the time t covariance between security rates of returns and the real rate of interest, conditional on information available at time s (A^* is a scalar); VARR(t, s), is the time t variance rate of security rates of return, conditional on information available at time s (an $(n \times n)$ matrix). RRR(t), is the time t realized rates of return on securities (an $(n \times 1)$ vector); EXRR(t, s), is the time t expected rates of return on securities conditional on information available at time s (an $(n \times 1)$ vector).

In addition, the dependency of the moments on t and s is only through$t - s$. In particular, for $n =$, Equation (5.1) becomes

$$r_t = n(t, s)$$

$$+\frac{A_1\left\{\frac{bA_1}{2a_1^2}[1 - e^{a_1(t-s)}] - \frac{D}{a_1}\right\}[1 - e^{a_1(t-s)}]}{\left\{\frac{bA_1^2}{2a_1^3}[e^{a_1(t-s)} - 3] + 2\frac{DA_1}{a_1^2}\right\}[e^{a_1(t-s)} - 1] + \left(B + \frac{bA_1^2}{2a_1^2} - \frac{2A_1D}{a_1}\right)(t - s)}$$

$$\times\left\{\eta_t - \eta_s - (r^* - r_s)\frac{1}{a_1}[1 - e^{a_1(t-s)}] - \left(\hat{a}_0 - \frac{a_0A_1}{a_1}\right)(t - s)\right\}. \tag{5.2}$$

□

The discussion relevant to the transient state in the two paragraphs following Equation (3.8) is relevant here as well, so we will not repeat it. If the quality of information is still changing over time, Equations (5.1) and (5.2) will have changes additional to those required for Equations (3.5) and (3.8) and discussed there. To account for the affect of changes in the quality of information, instead of using G_5 in the definition of r^*, we will use G while substituting for γ_t from Equation (2.9). In this case, the conditional moments will be changing over time.

There is probably no sufficient economic rationale to explore the impact of the initial conditions. It might be interesting, however, to look at the equilibrium conditions where the quality of information γ_t changes. In our model a changing γ_t corresponds to the transient period of the economy, but changes in γ_t also corres-pond to equilibrium conditions in a more complex economy where γ_t is a function of realized security returns. While we suppressed the notation, the filtering results of Section 2 are general enough to allow this case. Quality of information shocks also cause γ_t to change. Although our model does not allow for these shocks, we may look at them as "resetting" the initial conditions and "restarting" the economy. Thus, the transient conditions here resemble "regular" conditions in an economy with quality of information shocks.

We can write Equation (5.2) as

$$r_t = Q + P_1r_s + P_2[\eta_t - \eta_s], \tag{5.3}$$

where P_1 and P_2 consist of model parameters and Q is a function of model parameters and preferences. While Q and P_1 are scalars, P_2 is an n dimensional row vector. Thus we obtained the sample path equilibrium restriction described in Equation (1.3).

6. Empirical Implications

The right-hand side of Equation (5.2) consists of model parameters, the observable time s realization of the real rate r_s, r^*, and conditional expected returns. We can estimate the model parameters and r^*. As for the conditional expected returns, we can take two approaches. One is to write explicitly these expected returns as in Equation (5.2) and obtain Equation (5.3). Within our model Equation (5.3) holds without an error term; but in reality there will be an additional error term because we directly observe nominal rather than real rates of interest and because we do not model market microstructure effects. Moreover, because any practical test of this model will not include all the informative prices and economic variables, Equation (5.3) will hold with an error.

We can test Equation (5.3) using, for example, the generalized method of moments. Since n (the number of securities, macroeconomic variables, and firm-specific variables that are informative about the state of the economy) is large, there are more than enough restrictions to estimate the relevant model parameters. Of course, we can further increase the number of restrictions by using components of the information set that are orthogonal to the errors generated by Equation (5.3). One informative procedure is to test Equation (5.3) for various values of $t - s$.

A second approach is to test Equation (5.2) using a procedure we describe below. Works that examined Rational Expectations Equilibria in Macroeconomics developed procedures that could be used to conduct empirical tests of the restriction we have obtained. Mishkin (1983) contains one example. First, we could run a regression of the current period's realized returns explained by last period's variables; these variables are those which the market uses in predicting returns on risky investments. Alternatively, we can use last period's returns to predict this period's returns. Then, we could run a second regression explaining the realized rate of interest by realized returns on risky investments and the variables or returns used earlier to predict these returns. After running another regression where the coefficients are unrestricted, we could test the restriction on the coefficients (of these two regressions) implied by the model, say, by a maximum likelihood ratio. Since the model has specific suggestions about the nature of the regression coefficients, we could conduct further tests. In particular, the coefficient of the returns on risky investments is a "beta" type coefficient. Therefore, we could perform tests to identify the productivity factors correlated with realized returns. It will be most interesting to identify the market variables that satisfy additional "beta" type restrictions imposed on the coefficients. Identifying these variables will enhance the understanding of the formation of interest rates as well as the determination of realized returns on risky investments. We can also search for the portfolio of securities that has the highest covariance with the real rate.

The following example of this procedure is taken from Mishkin (1983) and can be applied to Equation (5.2) before substituting for $E(\eta_t)$. Consider the equation

$$y_t = y_t^* + \beta(X_t - X_t^e) + \varepsilon_t, \tag{6.1}$$

where $y_t \triangleq$ value of a market-clearing variable or price. $y_t^* \triangleq$ when using Equation (3.5) – asymptotic or "normal" value of this variable; when using Equation (5.1) – the variable's one-period-ahead forecast. $\varepsilon_t \triangleq$ a disturbance with the property $E(\varepsilon_t \mid \Phi_{t-1}) = 0$. Thus, ε_t is serially uncorrelated and uncorrelated with X_t^e. $X_t \triangleq$ the vector containing variables relevant to the pricing of y_t. $X_t^e \triangleq$ the vector of one-period-ahead rational forecasts of X_t; that is, $X_t^e = E_m(X_t \mid \Phi_{t-1}) = E(X_t \mid \Phi_{t-1})$. $\Phi_t \triangleq$ the set of information available at time t. $E_m(\cdot \mid \Phi_t) \triangleq$ the subjective expectation assessed by the market. $E(\cdot \mid \Phi_t) \triangleq$ the objective expectation conditional on Φ_t. β vector of coefficients.

Now consider the forecast equation

$$X_t = Z_{t-1}\gamma + u_t, \tag{6.2}$$

where $Z_{t-1} \triangleq$ a vector of variables available at time $t-1$, used to forecast X_t. $\gamma \triangleq$ a vector of coefficients. $u_t \triangleq$ an error term which is assumed to be uncorrelated with any information available at $t-1$.

Therefore,

$$X_t^e = E_m(X_t \mid \Phi_{t-1} = E(X_t \mid \Phi_{t-1}) = Z_{t-1}\gamma. \tag{6.3}$$

Substituting Equation (6.3) into Equation (6.1) gives

$$y_t = y_t^* + \beta(X_t - Z_{t-1}\gamma) + \varepsilon_t. \tag{6.4}$$

We now form a test of the market's rationality. First we estimate the restricted system

$$\begin{aligned} X_t &= Z_{t-1}\gamma + u_t, \\ y_t &= y_t^* + \beta(X_t - Z_{t-1}) + \varepsilon_t, \end{aligned} \tag{6.5}$$

and then the unrestricted one

$$\begin{aligned} X_t &= Z_{t-1}\gamma + u_t, \\ y_t &= y_t^* + \beta(X_t - Z_{t-1}\gamma^*) + \varepsilon_t. \end{aligned} \tag{6.6}$$

Finally, we use the likelihood ratio test (X^2 statistic) to compare the constrained and unconstrained systems and provide a test of market rationality $\gamma = \gamma^*$.

We can also use a modified "Mishkin approach", where we restrict the forecast equation, Equation (6.2), with relations from Equation (5.3). This, in fact, would combine the first, direct approach, to Mishkin's two-stage approach.

7. Example: Logarithmic Preferences

If we further specialize the results for $n = 1$ to the case of the Bernoulli logarithmic preferences,

$$h[c(t), t] = e^{-\zeta t} \log c(t), \quad \zeta > 0, \tag{7.1}$$

the equilibrium restrictions are somewhat simpler,

$$r = A_0 - B + A_1 m_t, \tag{7.2}$$

and

$$r^* = A_0 - B + A_1 \left(-\frac{a_0}{a_1}\right). \tag{7.3}$$

Thus r^* becomes a constant (please see the discussion). The asymptotic restriction, Equation (3.5), becomes

$$r = \left[A_0 - B - A_1 \frac{a_0}{a_1}\right] + \frac{A_1 \left[\dfrac{bA_1}{2a_1^2} - \dfrac{D}{a_1}\right]}{B + 2A_1 \left[\dfrac{bA_1}{2a_1^2} - \dfrac{D}{a_1}\right]} \left[\frac{\eta_t - \eta_0}{t} - \frac{E(\eta_t - \eta_0)}{t}\right] \tag{7.4}$$

The forecast of the real rate is ,

$$n(t, s) = r^*[1 - e^{a_1(t-s)}] + r_s e^{a_1(t-s)}, \tag{7.5}$$

and the one-period-ahead restriction, Equation (5.2), is

$$\begin{aligned} r_t &= r^*[1 - e^{a_1(t-s)}] + r_s e^{a_1(t-s)} \\ &\quad \frac{A_1 \left\{\dfrac{bA_1}{2a_1^2}[1 - e^{a_1(t-s)}] - \dfrac{D}{a_1}\right\} [1 - e^{a_1(t-s)}]}{\left\{\dfrac{bA_1^2}{2a_1^3}[e^{a_1(t-s)} - 3] + 2\dfrac{DA_1}{a_1^2}\right\} [e^{a_1(t-s)} - 1] + \left[B + \dfrac{bA_1^2}{a_1^2} - \dfrac{2A_1 D}{a_1}\right](t - s)} \\ &\quad \times \left\{\eta_t - \eta_s - (r^* - r_s)\frac{1}{a_1}[1 - e^{a_1(t-s)}] - \left(r^* + \frac{1}{2}B\right)(t - s)\right\}. \end{aligned} \tag{7.6}$$

The equilibrium spot rate of interest is perfectly correlated with risk-adjusted average returns on the realized outputs. In particular, when an increase in m_t indicates an increase (decrease) in risk-adjusted average returns on the realized outputs, r increases (decreases). Also, the higher (lower) the risk-adjusted average returns that are independent of m_t, the higher (lower) the r. This correspondence implies that when individuals expect high (low) risk-adjusted average returns on their investment, the market-clearing rate of interest is high (low).

When the economy is doing well, investors enjoy higher wealth levels and face high returns for investment in risky assets. Therefore, investors demand higher return on their risk-free borrowing and lending. To clear the market, the equilibrium spot rate of interest is set high. When the economy is doing poorly, individuals observe lower wealth levels. They want to hedge their risky investment; thus, lower returns on risk free borrowing and lending are sufficient to clear the market.

In reality, of course, other forces interact in setting the level of interest rates, forces that this model does not directly capture. However, some of the influence of these forces can be accounted for indirectly, through the effect of the stochastic productivity factor on realized returns.

The nature of the equilibrium within the economy resembles the effects of the demand for capital by firms. Firms that face investment opportunities of high returns are willing to incur higher costs of capital to take advantage of these investment opportunities. On the other hand, firms that face investment opportunities of low returns have lower demand for capital and cannot afford to guarantee high returns on this capital.

Using the results of this section, we now specialize earlier results to the case of logarithmic preferences.

PROPOSITION 5. Under Bernoulli logarithmic preferences [as defined in Equation (7.1)]

(I) r^* is a constant.

(II) The instantaneous spot interest rate is a linear function of an index of investments in the realized returns.

(III) There is a portfolio of realized returns that is perfectly locally correlated with the instantaneous spot interest rate. □

The results stated in Proposition 5 are implications of the property of the Bernoulli logarithmic preferences that cause its indirect utility function to $-WJ_{ww}/J_w = 1$, and $J_{wm} = 0$. The proof is made by straightforward substitutions in the equilibrium conditions, Equation (3.7) and (5.2) for example.

The results of Proposition 5, are the sample path equilibrium parallels to the well-known results regarding logarithmic preferences in the familiar moments equilibrium results, see, for example, Mossin (1968), and Cox, Ingersoll, and Ross (1985b). While result (I) of Proposition 5 is similar in both sample path and moments equilibria, results (II) and (III) hold only in sample path equilibria. Only under sample path equilibria is there a portfolio of securities whose returns are equal to the equilibrium real rate of interest realization by realization. Under the familiar moments equlibria, there is a portfolio whose expected return is equal to the equilibrium interest rate (the “zero beta” portfolio).

8. Summary and Conclusion

Known equilibrium models in finance are moments equilibria; they specify relations between risk (usually the second moment) and return (usually first moment) of security rates of return. This poses a dilemma, both to empiricists and real-world investors: these moments are unobservable to both, so either implementation or testing of these models requires "outside the model" assumptions and procedures. We introduce a sample path equilibrium that is a function of observable realized returns, thus observable to empiricists and real-world investors alike. While this paper focused on the real rate of interest, the first price determined in equilibrium, future research might explore, theoretically and empirically, other aspects of the general equilibrium.

Appendix

8.1. PROOF OF PROPOSITION 1

We obtain part (I) by a straightforward application of ItŞ's formula. To simplify the notation and exposition, we will prove the proposition for $n = 1$; the proof of the case where $n > 1$ is similar. We first find the stochastic differential equation that describes the evolution of $\eta_t = \log \chi_t$. The evolution of χ_t is given by

$$d\chi_t = (A_0 + A_1\theta_t)\chi_t dt + B_1\chi_t dW_t. \tag{A.1}$$

We express η_t as $\eta_t(\chi_t, t)$, and by a basic notation convention of stochastic calculus,

$$d\eta_t = \frac{\partial \eta_t}{\partial t}dt + \frac{\partial \eta_t}{\partial \chi_t}d\chi_t + \frac{1}{2}\frac{\partial^2 \eta_t}{\partial \chi_t^2}d\chi_t^2. \tag{A.2}$$

Using the relations, $dtd\chi_t = 0$, $dt^2 = 0$ and $d\chi_t^2 = B\chi_t^2 dt$,

$$d\eta_t = \left[\frac{\partial \eta_t}{\partial t} + (A_0 + A_1\theta_t)\chi_t\frac{\partial \eta_t}{\partial \chi_t}\right]dt + B_1\chi_t\frac{\partial \eta}{\partial \chi_t}dW_t \tag{A.3}$$

We substitute

$$\frac{\partial \eta_t}{\partial t} = 0, \quad \frac{\partial \eta_T}{\partial \chi_t} = \frac{1}{\chi_t}$$

and

$$\frac{\partial^2 \chi_t}{\partial \chi_t^2} = -\frac{1}{\chi_t^2}$$

to get

$$d\eta_t = (A_0 - 1/2B + A_1\theta_t)dt + B_1 dW_t. \tag{A.4}$$

This completes the proof of part (I). The proof of (II) is in Arnold (1974), Theorems 8.2.6 and 8.2.10. □

PROOF OF PROPOSITION 2

Proof Overview. Proposition 1 shows that (θ_t, η_t) are jointly normal. Therefore, we can express the conditional mean of the unobservable factor θ_t, given the information available at time s, $s \leq t$, as a function of the realization η_T:

$$E_s[\theta_t \mid \eta_t] = E_s[\theta_t] + \frac{\mathrm{COV}_s[\theta_t, \eta_t]}{\mathrm{VAR}_s[\eta_t]}(\eta_t - E_s[\eta_t]), \tag{A.5}$$

where $E_u[\cdot]$ denotes the expectation with respect to information available at time u. We then solve the stochastic filter equations and determine the expressions for A and K as defined in Proposition 1. We use this expressions to compute $E_s[\theta_t \mid \eta_t]$ explicitly. Next, we compare the result with $E_t[\theta_t] = m_t$ while expressing m_t as a function of r (as found in Section 2). Since $E_s[\theta_t \mid \eta_t]$ is the mean-square optimal estimate of m_t, in general,

$$m_t = E_t[\theta_t] = E_s[\theta_t \mid \eta_t] + \varepsilon(s, t), \tag{A.6}$$

where the Gaussian disturbances $\varepsilon(s, t)$ are serially uncorrelated and orthogonal to $E_s[\theta_t \mid \eta_t]$. Here, however, the stochastic processes, including those that describe the conditional expectations, are Markov. Thus, we are able to show that all the relevant information about the path of η_u over the interval $[s, t]$ is captured by the aggregate innovation $\eta_t - E_s[\eta_t]$, and hence $\varepsilon(s, t) = 0$ (see the discussion regarding path independence following Equation (3.8)). As a consequence, the equality in Equation (A.6) holds without an error. In fact, whether Equation (A.6) holds with or without error bears no theoretical or empirical significance to our results.

Proof. To simplify the notation and exposition, we will prove the proposition for $n = 1$; the proof of the case where $n > 1$ is similar. We first compute the expected values, variances, and covariances of (θ_t, η_t), i.e., the matrices $H(t)$ and $K(t)$.

Denote $\mathbf{L}$ as the unilateral Laplace transform operator, so $\mathbf{L}\{f(t)\} = \bar{F}(s)$. I is the identity matrix, and $e^{At} = \phi(t)$. Hence, $\frac{d\phi(t)}{dt} = A\phi(t)$, and $\phi(0) = I$. Taking the Laplace transform,

$$sL\{\phi(t)\} - I = AL\{\phi(t)\}, \tag{A.7}$$

or

$$(sI - A)\mathbf{L}\{\phi(t)\} = I, \tag{A.8}$$

and

$$\mathbf{L}\{\phi(t)\} = (sI - A)^{-1}. \tag{A.9}$$

Hence,

$$\phi(t) = \mathbf{L}^{-1}\{(sI - A)^{-1}\}, \tag{A.10}$$

where $\mathbf{L}^{-1}$ denotes the inverse Laplace transform. Now,

$$\begin{aligned}(sI - A)^{-1} &= \begin{pmatrix} s - a_1 & 0 \\ -A_1 & s \end{pmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s - a_1} & 0 \\ \dfrac{A_1}{s(s - a_1)} & \dfrac{1}{s} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{s - a_1} & 0 \\ \dfrac{A_1}{a_1)\left[\dfrac{1}{s - a_1} - \dfrac{1}{s}\right]} & \dfrac{1}{s} \end{bmatrix}. \end{aligned} \tag{A.11}$$

Therefore,

$$\phi(t) = e^{At} = \mathbf{L}^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} e^{a_1 t} & 0 \\ \dfrac{A_1}{a_1}(e^{a_1 t} - 1) & 1 \end{bmatrix}. \tag{A.12}$$

and

$$\phi^{-1}(t) = e^{-At} = \begin{bmatrix} e^{a_1 t} & 0 \\ \dfrac{A_1}{a_1}(e^{a_1 t} - 1) & 1 \end{bmatrix}. \tag{A.13}$$

From Proposition 1,

$$H(t) \triangleq \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix} = e^{At}\left[\begin{bmatrix} \theta_0 \\ \eta_0 \end{bmatrix} + \left(\int_0^t e^{-Az} dz\right)\begin{bmatrix} a_0 \\ \hat{A}_0 \end{bmatrix}\right] \tag{A.12}$$

$$= \begin{bmatrix} e^{a_1 t} & 0 \\ \dfrac{A_1}{a_1}(e^{a_1 t} - 1) & 1 \end{bmatrix}\left[\begin{bmatrix} \theta_0 \\ \eta_0 \end{bmatrix} + \left\{\int_0^t \begin{bmatrix} e^{a_1 t} & 0 \\ \dfrac{A_1}{a_1}(e^{a_1 z} - 1) & 1 \end{bmatrix} dz\right\}\begin{bmatrix} a_0 \\ \hat{A}_0 \end{bmatrix}\right] \tag{A.15}$$

$$\begin{bmatrix} e^{a_1 t} & 0 \\ \dfrac{A_1}{a_1}(e^{a_1 t} - 1) & 1 \end{bmatrix}\left[\begin{bmatrix} \theta_0 \\ \eta_0 \end{bmatrix} + \left\{\begin{matrix} \dfrac{1}{a}(1 - e^{-a_1 t}) & 0 \\ \dfrac{A_1}{a_1}\left[\dfrac{1}{a_1}(1 - e^{a_1 z} - 1) - t\right] & t \end{matrix}\right\}\begin{bmatrix} a_0 \\ \hat{A}_0 \end{bmatrix}\right] \tag{A.16}$$

$$= \begin{bmatrix} e^{a_1 t} & 0 \\ \frac{A_1}{a_1}(e^{a_1 t} - 1) & 1 \end{bmatrix} \begin{bmatrix} \theta_0 + \frac{1}{a_1}(1 - e^{-a_1 t}) \\ \eta_0 + \frac{a_0 A_1}{a_1}\left[\frac{1}{a_1}(1 - e^{-a_1 t} - t)\right] + \hat{A}_0 t \end{bmatrix}. \tag{A.17}$$

$$H(t) = \begin{bmatrix} \theta_0 e^{a_1 t} + \frac{a_0}{a_1}(e^{a_1 t} - 1) \\ \eta_0 + \theta_0 \frac{A_0}{a_1}(e^{a_1 t} - 1) + \frac{a_0 A_1}{a_1^2}(e^{a_1 t} - 1) + \left[\hat{A}_0 - \frac{a_0 A_1}{a_1}\right] t \end{bmatrix}. \tag{A.18}$$

We proceed by computing K(t). Note that here

$$SS' = \begin{pmatrix} b & D \\ D * B & \end{pmatrix}. \tag{A.19}$$

Therefore,

$$K(t) \triangleq \begin{bmatrix} K_1(t) & K_2(t) \\ K_2(t) & K_3(t) \end{bmatrix} = \int_0^t e^{A(t-z)} SS' e^{A'(t-z)} dz \tag{A.20}$$

$$= \int_0^t \begin{Bmatrix} e^{a_1(t-z)} & 0 \\ \frac{A_1}{a_1}[e^{a_1(t-z)}] & 1 \end{Bmatrix} \begin{bmatrix} b & D \\ D & B \end{bmatrix} \begin{Bmatrix} e^{a_1(t-z)} & \frac{A_1}{a_1}[e^{a_1(t-z)}] \\ 0 & 1 \end{Bmatrix} dz. \tag{A.21}$$

Hence,

$$K_1 = \int_0^t b e^{2a_1(t-z)} dz, \tag{A.22}$$

$$K_2(t) = \int_0^t \left\{ \frac{bA_1}{a_1}[e^{2a_1(t-z)} - e^{a_1(t-z)}] + D e^{a_1(t-z)} \right\} dz, \tag{A.23}$$

$$K_3(t) = \int_0^t \left\{ \frac{bA_1^2}{a_1^2}[e^{2a_1(t-z)} - 2e^{a_1(t-z)} + 1] + 2\frac{DA_1}{a_1}[e^{a_1(t-z)}] + B \right\} dz. \tag{A.24}$$

After we calculate the integrals,

$$K_1(t) = \frac{b}{2a_1}(e^{a_1 t} - 1), \tag{A.25}$$

$$K_2(t) = \left[\frac{bA_1}{2a_1^2}(e^{a_1 t} - 1) + \frac{D}{a_1}\right](e^{a_1 t} - 1). \tag{A.26}$$

$$K_3(t) = \left[\frac{bA_1^2}{2a_1^3}(e^{a_1 t} - 3) - 2\frac{DA_1}{a_1^2}\right](e^{a_1 t} - 1) + \left(B + \frac{bA_1^2}{a_1^2} - \frac{2A_1 D}{a_1}\right)t. \tag{A.27}$$

These values are conditional on (θ_0, η_0). The realized returns at time t on investment in the securities at time 0 is

$$\frac{\eta_t}{\eta_0} = e^{\eta_t - \eta_0} = e^{\frac{\eta_t - \eta_0}{t} t}. \tag{A.28}$$

Hence, the corresponding rate of return is

$$\frac{\eta_t - \eta_0}{t}.$$

We denote

$$\frac{\eta_t - \eta_0}{t} \triangleq RRR \qquad \text{(realized rate of return).}$$

$$\frac{E(\eta_t - \eta_0)}{t} \triangleq EXRRR \qquad \text{(expected value at time 0, of } RRR \text{ at time } t).$$

$$\frac{\mathrm{VAR}(\eta_t)}{t} \triangleq VARR \qquad \text{(variance rate of } \eta_t).$$

$$\mathrm{COV}(\theta_t, \eta_t) \triangleq COVAR.$$

. Now, we interpret the prefix *ASY* as restricting the value of the following expression to its asymptotic value in time: $ASYVARR \triangleq \lim_{t\to\infty} VARR$. Since (θ_t, η_t) has a joint normal distribution[13]

$$\begin{aligned} E[\theta_t, \eta_t) &= H_1(t) + K_2(t)K_3^{-1}(t)[\eta_t - H_2(t)] && \text{(A.29)} \\ &= E[\theta_t] + \frac{\mathrm{COV}[\theta_t, \eta_t]}{\mathrm{VAR}[\eta_t]}(\eta_t - E[\theta_t]), && \text{(A.29)} \end{aligned}$$

[13] In the more general case where θ_t is a k dimensional vector and χ_t is an n dimensional one, Equation (A.29) becomes

$$\begin{aligned} E(\theta_t, \eta_t) &= E(\theta_t) + \mathrm{COV}(\theta_t, \eta_t)[\mathrm{VAR}(\eta_t)]^{-1}[\eta_t - E(\eta_t)] \\ &= H_1(t) + K_2'(t)K_3^{-1}(t)[\eta_t - H_2(t)], \end{aligned}$$

where $H_1(t)$ is a $(k \times 1)$ vector; $K_2'(t)$, the transpose of $K_2(t)$, is a $(k \times n)$ matrix; K_3^{-1}, the inverse of $K_3(t)$, is an $(n \times n)$ matrix; and $H_2(t)$ is a $(n \times 1)$ vector. In this case, $K(t)$ is

$$K(t) \triangleq \begin{bmatrix} K_1(t) & K_2'(t) \\ K_2(t) & K_3(t) \end{bmatrix}$$

or (see explanation in the proof outline and denote $\varepsilon(0, t) = \varepsilon_t$)

$$m_t = E[\theta_t] + \frac{\text{COV}[\theta_t, \eta_t]}{\frac{\text{VAR}[\eta_t]}{t}} \left(\frac{\eta_t - \eta_0}{t} - \frac{E[\eta_t - \eta_0]}{t} \right) + \varepsilon_t. \tag{A.30}$$

The term ε_t represents the difference between the mean in the left-hand side, which is conditional on the full sample path of the observations η_s, $0 \le s \le t$, and the mean in the right-hand side, which is conditional only on the aggregate over this interval, $\eta_t - E[\eta_t]$. Examination of the right-hand side of Equation (A.30) shows that here, $\varepsilon_t = 0$ (recall that the quality of information is at steady state). The terms $E[\theta_t]$, $\text{COV}[\theta_t, \eta_t]$, and $\text{VAR}[\eta_t]$ are not functions of the sample path. An analysis (eliminated for brevity) of the explicit solutions for the aggregate innovation shows that for any intermediate date s, $0 \le s \le t$, and realization θ_s,

$$E[\eta_t \mid \eta_s] = E[\eta_t] + (\eta_s - E[\eta_s]), \quad \text{or} \quad E[\eta_t - \eta_s \mid \eta_s] = E[\eta_t - \eta_s]. \tag{A.31}$$

The left-hand version of Equation (A.31) shows that conditioning on an intermediate realization η_s affects the mean only by the addition of time s innovation. Of course, this innovation is part of the aggregate innovation at time t. The right-hand version of Equation (A.31) highlights the property that conditioning on an intermediate realization does not alter future expectations. Therefore, the intermediate realizations η_s, affect the conditional mean only through the aggregate innovation. In fact, since ε_t is a Gaussian disturbance, orthogonal (to the conditional mean in the right-hand side of Equation (A.30)), and serially uncorrelated, whether or not ε_t vanishes ($\varepsilon_t = 0$ for every t) does not bear any theoretical or empirical significance to our results.

Since m_t is a function of r, we use Equation (2.22) to substitute for m_t in Equation (A.30). Asymptotically,

$$r = r^* + A^* \times ASYCOVAR \times ASYVARR^{-1}[RRR - ASYEXRR], \tag{A.32}$$

where

$$A^* = A_1, \tag{A.33}$$

and

$$r^* = A_0 + B\frac{wJ_{ww}}{J_w} + G_s\frac{J_{wm}}{J_w} - A_1\frac{a_0}{a_1}. \tag{A.34}$$

Note that Equations (A.33) and (A.34) are the special case for $n = 1$ of Equations (3.6) and (3.7). We will now show that for logarithmic preferences, $A^* \times \text{COV}[\theta_t, \eta_t] = \text{COV}[r_t, \eta_t]$. First we rewrite Equation (2.22) as

$$r = \hat{A} + A^* \times m_t. \tag{A.35}$$

Note that for logarithmic preferences, $\hat{A}$ is a constant. Then,

$$\begin{aligned} A^* \times \mathrm{COV}[\theta_t, \eta_t] &= A^*(E[\theta_t \eta_t] - E[\theta_t]E[\eta_t]) \\ &= A^*\{E[E[\theta_t \eta_t \mid \eta_t]] - E[\theta_t]E[\eta_t]\} \\ &= A^*\{E[\eta_t E[\theta_t \mid \eta_t]] - E[\theta_t]E[\eta_t]\} \\ &= A^*\{E[\eta_t m_t] - E[\theta_t]E[\eta_t]\} \\ &= A^*\{E[\eta_t(r - \bar{A})/A^*] - E[\theta_t]E[\eta_t]\} \\ &= E[\eta_t r] - \bar{A} \times E[\eta_t] - A^* \times E[\theta_t]E[\eta_t] \\ &= \mathrm{COV}[r, \eta_t] + E[r]E[\eta_t] - \bar{A} \times E[\eta_t] - A^* \times E[\theta_t]E[\eta_t] \\ &= \mathrm{COV}[r, \eta_t] + E[\eta_t](E[r] - \bar{A} - A^* \times E[\theta_t]) \\ &= \mathrm{COV}[r, \eta_t] + E[\eta_t](E[r] - \bar{A} - A^* \times E[\theta_t \mid \eta_t]) \\ &= \mathrm{COV}[r, \eta_t] + E[\eta_t](E[r] - \bar{A} - A^* \times m_t) \\ &= \mathrm{COV}[r, \eta_t]. \end{aligned} \tag{A.36}$$

In the previous equation, we used the definition of covariance, the law of iterated expectations, the definition of m_t, the equality of the mean and the conditional mean of θ_t, and Equation (A.35). A more formal statement of Equation (A.32) is

$$r^* = \lim_{t\to\infty} \left[r - \frac{A^* \times \mathrm{COV}(\theta_t, \eta_t)}{\dfrac{\mathrm{VAR}(\eta_t)}{t}} \left[\frac{\eta_t - \eta_0}{t} - \frac{E(\eta_t - \eta_0)}{t} \right] \right]. \tag{A.37}$$

This completes the development of the first part of the proposition.

To obtain the explicit relation for $n = 1$, remember that in this case, $E(\theta_t) = H_1(t)$, $\mathrm{COV}(\theta_t, \eta_t) = K_2(t)$, $E(\eta_t) = H_2(t)$, $\mathrm{VAR}(\eta_t) = K_3(t)$, and that Equation (2.22) defines the relation between m_t and r. By direct substitution and omitting the terms that vanish for large t (recall that $a_1 < 0$),

$$ASYCOVAR = \frac{bA_1}{2a_1^2} - \frac{D}{a_1}, \tag{A.38}$$

$$ASYVARR = B + 2A_1 \left(\frac{bA_1}{2a_1^2} + \frac{D}{a_1} \right), \tag{A.39}$$

$$\lim_{t\to\infty} E(\theta_t) = -\frac{a_0}{a_1}. \tag{A.40}$$

Therefore,

$$r = r^* + \frac{A^* \left(\dfrac{bA_1}{2a_1^2} - \dfrac{D}{a_1} \right)}{B + 2A_1 \left(\dfrac{bA_1}{2a_1^2} - \dfrac{D}{a_1} \right)} \left[\frac{\eta_t - \eta_0}{t} - \frac{E(\eta_t - \eta_0)}{t} \right]. \tag{A.41}$$

When we further assume logarithmic preferences, we have

$$r^* = A_0 - A_1 \frac{a_0}{a_1} - B,$$

and

$$r = A_0 - A_1 \frac{a_0}{a_1} - B + \frac{A_1 \left(\dfrac{bA_1}{2a_1^2} - \dfrac{D}{a_1} \right)}{B + 2A_1 \left(\dfrac{bA_1}{2a_1^2} - \dfrac{D}{a_1} \right)} \left[\frac{\eta_t - \eta_0}{t} - \frac{E(\eta_t - \eta_0)}{t} \right]. \tag{A.42}$$

PROOF OUTLINE OF PROPOSITION 3

We use Theorem 12.12 in Liptser and Shiryayev (1978) to establish $m(t, s)$ as the mean square optimal forecast and to identify the forward differential equation that describes its evolution. Equation (4.1) is the solution to this equation.

PROOF OUTLINE OF PROPOSITION 4

The proof is similar to that of Proposition 2 and thus is omitted. We obtain the expression for using Equations (2.22) and (A.18).

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