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Libor Market Models versus Swap Market Models for Pricing Interest Rate Derivatives: An Empirical Analysis

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Abstract. We empirically compare Libor and Swap Market Models for the pricing of interest rate derivatives, using panel data on prices of US caplets and swaptions. A Libor Market Model can directly be calibrated to observed prices of caplets, whereas a Swap Market Model is calibrated to a certain set of swaption prices. For both models we analyze how well they price caplets and swaptions that were not used for calibration. We show that the Libor Market Model in general leads to better prediction of derivative prices that were not used for calibration than the Swap Market Model. Also, we find that Market Models with a declining volatility function give much better pricing results than a specification with a constant volatility function. Finally, we find that models that are chosen to exactly match certain derivative prices are overfitted; more parsimonious models lead to better predictions for derivative prices that were not used for calibration.

Key words: Term structure models, interest rate derivatives, lognormal pricing models, Black formula.

JEL Codes: G12, G13, E43.

1. Introduction

Since the first interest rate swap was traded in 1981, the market for interest rate derivatives has grown enormously. Both the volume and complexity of the products traded has increased. Hence, the modelling and pricing of interest rate derivatives has been an area of research of considerable interest both for academics and practitioners.

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To determine the prices of exotic interest rate derivatives, pricing models are used as an 'extrapolation tool'. Given the prices of liquid instruments available in the market, such as caps and swaptions, pricing models extract information about the distribution of the underlying rates by calibrating to these market prices. The calibrated model is then used to price and hedge the exotic instrument. For the successful pricing of exotic options, it is therefore important to find parsimoniously specified models that provide an accurate fit to the prices of the liquid market derivative instruments.

A recent development in modelling interest rates and pricing interest rate derivatives are the so-called market models. Brace et al. (1997) and Miltersen et al. (1997) present an arbitrage-free interest rate model, the Libor Market Model (LMM), in which forward Libor rates follow lognormal processes, leading to the Black (1976) pricing formula for caps and floors, which is used by market practitioners. A similar model for swap rates and swap rate derivatives was developed by Jamshidian (1997). His so-called Swap Market Model (SMM) leads to the Black formula for swaptions.

There are several advantages of the market models in comparison with the traditional models, such as the instantaneous spot rate models (e.g., Vasicek (1977), Cox et al. (1985), and Hull and White (1990)) and models for instantaneous forward interest rates (Heath et al. (1992) and Ritchken and Sankarasubramanian (1995)). First, the match to the market Black formula for option prices makes calibration of market models very simple. The quoted implied Black volatilities can directly be inserted in the model, avoiding the numerical fitting procedures that are needed for the spot rate or forward rate models. Second, the market models are based on observable market rates, such as Libor rates and swap rates. Hence, one does not need the (unobserved) instantaneous short rate or instantaneous forward rates to price and hedge caps and swaptions.

Given the advantages of the market models, it is not surprising that these models have received a lot of attention recently.[1] There has been little attention however to the empirical performance of the market models. Since the LMM and the SMM are mutually inconsistent approaches, it is an empirical question which model is to be preferred for practical purposes. It is exactly this question we want to address in this paper. We use US panel data on prices of caplets of different option maturities, and prices of the swaption 'matrix' (i.e., prices of swaptions for several option maturities and swap maturities). The paper focuses on three important issues.

The first is an empirical comparison of the LMM with the SMM. We use caplet prices for calibration of the LMM. For calibration of the SMM, we use a subset of the swaption prices. Then, to compare the LMM with the SMM, we use the differences between model prices and actual prices for the derivative instruments that are not used for calibration of the LMM or the SMM. The empirical results

[1] Recent work on the market models includes Andersen and Andreasen (1999), Barton et al. (1998), Glasserman and Zhao (1999), Longstaff et al. (2000a, b), Pedersen (1999), Rebonato (1999), and Schlögl (1999).

show that the LMM in general leads to better prediction of these derivative prices than the SMM. Also, the SMM substantially overprices caplets. In this paper, we provide an explanation for these results.

A second important issue in calibrating the model is the specification of the volatility function. This function plays a crucial role in the model. We show that the usual choice of a constant volatility function for each option maturity date is not a particularly good one, because the LMM with a constant volatility function persistently overprices swaptions and the SMM with a constant volatility function underprices swaptions. Much better results are obtained by specifying an exponentially declining volatility function. This functional form corresponds to mean-reverting behaviour of interest rates. We argue that the use of declining volatility functions decreases the correlation between interest rates at different points in time, which has an important effect on the prices of swaptions.

Third, we consider two different calibration methodologies: exact calibration and non-exact calibration. Interest rate derivatives traders often use exact calibration to match the model as closely as possible to observed market prices. In case of exact calibration, the model under consideration has as many parameters as calibration instruments, and these parameters are chosen such that the model exactly fits caplets (in case of the LMM) or 10-year total maturity swaptions (in case of the SMM). With non-exact calibration, the model has only a small number of parameters and exact fitting of derivative prices is not possible in general. However, 'overfitting' of the model is potentially avoided, which may lead to better pricing of other derivatives. We find that the exactly calibrated models overfit the derivative price data, as in almost all cases non-exact calibration leads to smaller pricing errors for the derivatives that are not used for calibration.

For all models and calibration methodologies, a specification test reveals that the market models are rejected statistically, but given the large bid-ask spreads on these derivative instruments, the size of the pricing errors does not seem to be economically very large.

This paper is related to previous empirical work on pricing interest rate derivatives. Flesaker (1993) and Amin and Morton (1994) analyze the pricing of Eurodollar futures options with Heath, Jarrow, and Morton (HJM, 1992) models. In both articles one-factor models are examined. The effect of the shape of the volatility function on derivative prices is analyzed by Amin and Morton (1994), but since they only use short-maturity options on short-maturity futures, they are not able to precisely estimate the volatilities of both short-maturity and long-maturity forward rates. Thus, they conclude that the constant volatility Ho and Lee (1986) model satisfactorily describes their data. As our dataset contains a wide range of option and swap maturities, we are able to precisely estimate the decrease in the volatility structure and the effect on the pricing of swaptions. Amin and Morton (1994) also examine whether models are overfitted to derivative prices by analyzing whether model-based trading strategies are profitable. They conclude that models with only two calibration parameters are overfitted to the derivative prices. In this paper, we provide a more direct analysis of model overfitting, by studying the prediction of derivative prices that are not used for calibration.

Buhler et al. (1999) use data on German government bond options to compare several one- and two-factor interest rate models. Buhler et al. (1999) estimate the model parameters from historical interest rate data, which leads to very large pricing errors for the bond options. Such an estimation procedure can be useful for the calculation of risk measures, such as VaR. For the accurate pricing of exotic interest rate derivatives however, it seems inevitable to also use the information in derivative prices to estimate the model parameters.

Longstaff, Santa-Clara, and Schwartz (LSS, 2000b) also analyze models based on Libor rates. They use a very different estimation procedure, which also uses principal component analysis. They fit a LMM-type model to the prices of swaptions, and subsequently price caplets. Their focus is, however, on another interesting aspect of market models, namely the number of relevant factors.

The remainder of this paper is organized as follows. In Section 2, we briefly review the construction of the market models. Section 3 describes the data. Section 4 first discusses the calibration methodology for the LMM and then presents the results of this calibration. Section 5 presents the calibration methodology and results for the SMM. In Section 6 we summarize and conclude.

2. Libor and Swap Market Models

In this section we describe the market models of interest rates. We first show how a Libor Market Model (LMM) is constructed. Thereafter, we briefly discuss the formulation of a Swap Market Model (SMM). For a complete discussion of the Swap Market Models, we refer to Jamshidian (1997) and Musiela and Rutkowski (1997). For both models, we briefly discuss the pricing of caplets and swaptions.

2.1. THE LIBOR MARKET MODEL

We describe the LMM formulation based on a finite number of bond prices, following Jamshidian (1997).[2] We start with defining a finite set of dates, the so-called *tenor structure*,

$$T_1 < T_2 < \cdots < T_N. \tag{1}$$

We also define $\delta_i = T_{i+1} - T_i$, $i = 1, \ldots, N-1$ as the so-called daycount fractions, which are determined by the maturity of the Libor rate that is used to determine

[2] In Brace et al. (1997), a formulation based on a continuum of bond prices is presented, so that the LMM fits in the framework of Heath et al. (1992). However, as noted by Jamshidian (1997), for the pricing and hedging of caplets and swaptions it is *not* necessary that a continuum of bond prices and a money market account exist. This is an important difference between the market models and the framework of Heath et al. (1992).

caplet payoffs and are most often equal to 3 or 6 months.[3] Associated with each tenor date T_n is a bond that matures at this date, and its time t price is denoted with $P_n(t)$. It is assumed that these bond prices follow Itô processes under the empirical probability measure, i.e.,

$$\mathrm{d}P_n(t) = P_n(t)(\mu_n^P(t)\,\mathrm{d}t + \sigma_n^P(t)\,\mathrm{d}W_t), \quad n = 1, \ldots, N, \tag{2}$$

where W_t is a standard Brownian Motion, and in this paper it is assumed to be one-dimensional. The one-dimensional drift function $\mu_n^P(t)$ and the bond price volatility $\sigma_n^P(t)$, that has the same dimension as W_t, can depend on the bond price $P_n(t)$. The forward Libor rate at time t for the accrual period $[T_n, T_{n+1}]$ is defined as

$$L_n(t) = \frac{1}{\delta_n}\left(\frac{P_n(t)}{P_{n+1}(t)} - 1\right), \quad n = 1, \ldots, N-1. \tag{3}$$

Applying Itô's division rule to Equation (3), it follows that a forward Libor rate satisfies the following Itô process under the empirical probability measure

$$\begin{aligned}
\mathrm{d}L_n(t) &= L_n(t)(\mu_n(t)\,\mathrm{d}t + \gamma_n(t)\,\mathrm{d}W_t), \quad n = 1, \ldots, N-1 \\
\gamma_n(t) &= \frac{P_n(t)}{P_n(t) - P_{n+1}(t)}(\sigma_n^P(t) - \sigma_{n+1}^P(t)) \\
\mu_n(t) &= \frac{P_n(t)}{P_n(t) - P_{n+1}(t)}(\mu_n^P(t) - \mu_{n+1}^P(t)) - \gamma_n(t)\sigma_{n+1}^P(t).
\end{aligned} \tag{4}$$

The function $\mu_n(t)$ is the drift function of the forward Libor rate, and $\gamma_n(t)$ is the volatility function. The idea behind the LMM is to construct an arbitrage-free interest rate model that implies a pricing formula for caplets that has the same structure as the Black pricing formula for a caplet, that is used by market practitioners. As the Black pricing formula for a caplet is based on the assumption that the relevant forward Libor rate follows a lognormal process under the equivalent martingale measure, the LMM has to imply lognormal processes (under the equivalent martingale measure) for the forward Libor rates in (3). It is therefore assumed that the volatility function $\gamma_n(t)$ is a deterministic function of time, which can be different for the different forward Libor rates. For a given forward Libor rate, this volatility function $\gamma_n(t)$, $t \le T_n$, describes the instantaneous volatility of this forward Libor rate over time, until the forward Libor rate matures at time . If this volatility function is constant, i.e., if $\gamma_n(t)$ does not depend on t, the volatility of the forward Libor rate $L_n(t)$ is constant until its maturity date T_n. A volatility function $\gamma_n(t)$ that is *increasing* in time t would imply that the volatility of the forward Libor rate increases as the rate approaches its maturity date. Such a volatility

[3] Because of daycount conventions, the daycount fractions are actually functions $\delta_i = g(T_i, T_{i-1})$, that are only approximately equal to $(T_i - T_{i-1})$.

function would be consistent with mean-reverting behaviour of interest rates, as mean-reversion typically implies that interest rates close to maturity have a larger volatility than interest rates that are far from maturity.

Given these $N-1$ volatility functions of the forward Libor rates, the N volatility functions of the bond prices cannot be recovered. This indeterminacy is solved by choosing one of these bond prices as numeraire. By assuming that there are no arbitrage opportunities[4] amongst the N bonds, it follows (see Jamshidian (1997) and Musiela and Rutkowski (1997)) that numeraire-denominated bond prices are martingales in the probability measure associated with the choice of the numeraire. One convenient choice for the numeraire asset is the bond with the longest maturity $P_N(t)$, and the associated probability measure is often called the *terminal measure* Q^N. Jamshidian (1997) shows that the process of forward Libor rates under the terminal measure is given by

$$\begin{aligned} \mathrm{d}L_n(t) &= L_n(t)\left(-\sum_{i=n+1}^{N-1} \frac{\delta_i L_i(t)\gamma_i(t)\gamma_n(t)}{1+\delta_i L_i(t)}\,\mathrm{d}t + \gamma_n(t)\,\mathrm{d}W_t^*\right), \\ & \quad n = 1,\ldots,N-1, \end{aligned} \tag{5}$$

where W_t^* is a standard Brownian Motion under the terminal measure. Given the Q^N-processes of these $N-1$ forward Libor rates, all numeraire-denominated bond price processes can be determined.

The result in (5) implies that, in order to price and hedge interest rate derivatives, only the volatility functions $\gamma_n(t)$ have to be determined. Furthermore, under the terminal measure, the Libor rate $L_{N-1}(t)$ has zero drift. Similarly, under the probability measure Q^n associated with the numeraire $P_n(t)$, the Libor rate $L_{n-1}(t)$ has zero drift. This directly follows from the fact that the Libor rate $L_{n-1}(t)$ is a ratio of bond prices, with the numeraire $P_n(t)$ in the denominator, as shown in Equation (3).

We now turn to the pricing of (European) interest rate derivatives, such as caplets, caps and swaptions. A *caplet* with strike rate k and maturity date T_n pays off $\delta_n(L_n(T_n)-k)^+$ at time T_{n+1}. The LMM-price of this caplet at time t can be calculated using the expectation of the discounted payoff under the Q^{n+1} measure

$$Caplet(t, T_n, k) = P_{n+1}(t)E_t^{n+1}[\delta_n(L_n(T_n)-k)^+], \tag{6}$$

and because $L_n(t)$ follows a driftless lognormal process under the Q^{n+1} measure, Equation (6) leads to the familiar Black pricing formula for a caplet and this caplet price is determined by the conditional variance (under the Q^{n+1}-measure) of the forward Libor rate over the maturity of the caplet, which is equal to $\int_t^{T_n} \gamma_n(s)^2\,\mathrm{d}s$. The price of a cap, which is a sum of caplets of different maturities, can therefore also analytically be determined for the LMM.

[4] The no-arbitrage condition used here is weaker than the usual no-arbitrage condition, because the existence of a continuous savings account is not assumed here, see Jamshidian (1997) for details. In the continuous-tenor case of Brace et al. (1997), the usual no-arbitrage condition is used.

A *swaption* is an option on a swap. Consider a forward swap, with principal 1, where two parties agree to exchange at dates $\{T_{n+1}, \ldots, T_{n+m}\}$ the floating Libor rates $\{L_n(t), \ldots, L_{n+m-1}(t)\}$ for a fixed rate. The forward swap rate is the fixed rate that gives this contract zero initial value and is given by

$$S_{n,m}(t) = \frac{\sum_{j=1}^{m} \delta_{n+j-1} P_{n+j}(t) L_{n+j-1}(t)}{\sum_{j=1}^{m} \delta_{n+j-1} P_{n+j}(t)} = \frac{P_n(t) - P_{n+m}(t)}{\sum_{j=1}^{m} \delta_{n+j-1} P_{n+j}(t)}. \tag{7}$$

A payers swaption with strike rate k, maturity date T_n and m payment dates gives right to enter into a swap at date T_n, where floating Libor payments are received and fixed payments k are paid. Equivalently, a payers swaption gives the right to receive the cash flow $\delta_{n+j-1}(S_{n,m}(T_n) - k)^+$ at dates T_{n+j}, $j = 1, \ldots, m$ (see Musiela and Rutkowski (1997)).

In Equation (7), it is shown that a forward swap rate depends on several forward Libor rates, so that the variance of a swap rate is a function of both the variances and covariances (or correlations) of forward Libor rates. Swaption prices thus depend both on conditional variances and covariances of forward Libor rates of different maturities, whereas caplet prices only depend on the conditional variance of one forward Libor rate. It is easy to show that forward swap rates do not follow lognormal processes in the LMM, as the forward par swap rate in (7) is a linear combination of several forward Libor rates. Swaptions can not be priced analytically by the LMM. We will use simulation to obtain (exact) prices of swaptions, by simulating the Euler discretization of the processes of forward Libor rates in Equation (5) under the terminal measure. The complete simulation procedure is described in Brace (1998).

2.2. THE SWAP MARKET MODEL

The SMM is constructed in a way that is quite similar to the construction of the LMM, but the algebra is more complicated. Therefore we will only briefly discuss the construction of the SMM, and again refer to Jamshidian (1997) for a detailed discussion.

We start with the same tenor structure as given in Equation (1), and we again assume that the N bond prices follow Itô processes. To arrive at the Black-type pricing formula for swaptions, we will now require that forward swap rates follow lognormal processes. More specifically, for the set of forward swap rates that have the same enddate T_N, $S_{n,N-n}(t)$, $n = 1, \ldots, N$, it is assumed that under the empirical probability measure

$$\begin{aligned} \mathrm{d}S_{n,N-n}(t) &= S_{n,N-n}(t)(\mu_{n,N-n}(t)\,\mathrm{d}t + \gamma_{n,N-n}(t)\,\mathrm{d}W_t), \\ & \quad n = 1, \ldots, N-1. \end{aligned} \tag{8}$$

Given the $N-1$ *deterministic* volatility functions $\gamma_{n,N-n}(t)$ of the forward swap rates, the N volatility functions of bond prices cannot be determined and again this indeterminacy is resolved by choosing a bond price as numeraire. For the forward swap rate $S_{n,N-n}(t)$ the most convenient choice of numeraire is the *coupon process* or *present value of a basis point (PVBP)* $\sum_{j=1}^{N-n} \delta_{n+j-1} P_{n+j}(t)$. Equation (7) shows that, assuming no-arbitrage and choosing this coupon process as numeraire, the forward swap rate $S_{n,N-n}(t)$ follows a driftless lognormal process under the equivalent martingale measure induced by this numeraire choice. This directly implies that the SMM-price of a swaption with option maturity T_n and swap maturity $T_N - T_n$ is given by the Black formula. Hence, the $N-1$ swaptions that have a total maturity, defined as the option maturity plus the swap maturity, equal to T_N will have a Black-type pricing formula, as the forward swap rates that determine these swaption prices follow driftless lognormal processes under their equivalent martingale measures. Again, determination of the volatilities $\gamma_{n,N-n}(t)$ is all that is necessary to price derivatives.

The prices of other swaptions and caplets cannot be determined using the closed-form Black formula, as the underlying rates do not follow lognormal processes in the SMM, and we use simulation techniques to obtain prices. Of course, we simulate all forward swap rates under the same equivalent martingale measure, induced by a particular numeraire choice. We choose to simulate under the *terminal* equivalent martingale measure, using an Euler discretization of the processes of forward swap rates under this measure, which are derived by Jamshidian (1997).

The difference between the LMM and the SMM is therefore determined by the set of market rates that follow lognormal processes in the model. For the LMM this is a set of forward Libor rates, and for the SMM this is a set of forward swap rates.

3. Data Description

We use two types of datasets for our analysis: US term structure data to determine the underlying term structure of forward Libor and forward swap rates, and US derivatives data on implied volatilities of caplets and swaptions.[5]

The term structure data consist of daily observations on a spot Libor rate, Eurodollar futures prices, and swap rates of different maturities, for the time period between July 1995 and September 1996. The prices of Eurodollar futures do not directly give information about forward Libor rates, because future prices are different from forward prices. We correct for this difference using an approximation for the price of an Eurodollar future in the LMM, derived by Brace (1998). To calculate this correction, we use the one-factor constant volatility LMM, which is described in the next section. In Table I, we present some summary statistics on these yield-curve instruments. The standard deviations of the rates reveal a humped volatility structure.

To determine forward Libor rates and forward swap rates for all possible (forward) maturities, some assumption on the functional shape of the (forward) interest

[5] The data are provided by ABN-AMRO Bank. The implied volatilities for caplets are obtained from data on cap implied volatilities, using bootstrap methods described in Hull (1997).

Table I. Descriptive statistics Libor rates, Eurodollar futures and swap rates. Summary statistics are calculated from 282 daily observations from July 1995 until September 1996, and all rates are expressed on an annual basis.

Rate	Maturity in years (average)	Average	Standard deviation
Libor	0.25	5.61%	0.21%
100-Eurodollar	0.12	5.60%	0.20%
100-Eurodollar	0.37	5.59%	0.36%
100-Eurodollar	0.62	5.67%	0.48%
100-Eurodollar	0.87	5.79%	0.53%
100-Eurodollar	1.12	5.92%	0.56%
100-Eurodollar	1.37	6.04%	0.58%
100-Eurodollar	1.62	6.13%	0.56%
100-Eurodollar	1.87	6.23%	0.53%
100-Eurodollar	2.12	6.31%	0.51%
100-Eurodollar	2.37	6.38%	0.51%
100-Eurodollar	2.62	6.45%	0.50%
100-Eurodollar	2.86	6.52%	0.47%
Swap	4	6.26%	0.46%
Swap	5	6.37%	0.46%
Swap	7	6.53%	0.45%
Swap	10	6.73%	0.44%
Swap	12	6.81%	0.43%
Swap	15	6.93%	0.41%
Swap	20	7.01%	0.39%
Swap	30	7.04%	0.37%

rates as function of the time to maturity is often made (for example, an exponential spline function) and this shape is fitted to the observed term structure data (spot Libor rates, Eurodollar future rates and swap rates). However, our goal is to price derivatives and therefore we do not want to allow for pricing errors of the underlying term structure instruments at all, because these pricing errors are directly transferred to the pricing errors for derivative prices. We thus choose the forward Libor rate to be a piecewise linear function of the forward maturity of the forward Libor rate,[6] where the nodes are determined by the dates at which the different term structure instruments mature. This piecewise linear function is chosen such that all term structure instruments are fitted exactly. In Figure 1, we plot the behaviour of

[6] The maturity of the Libor rates that is used is equal to 3 months, because caplets are based on these 3-month rates.

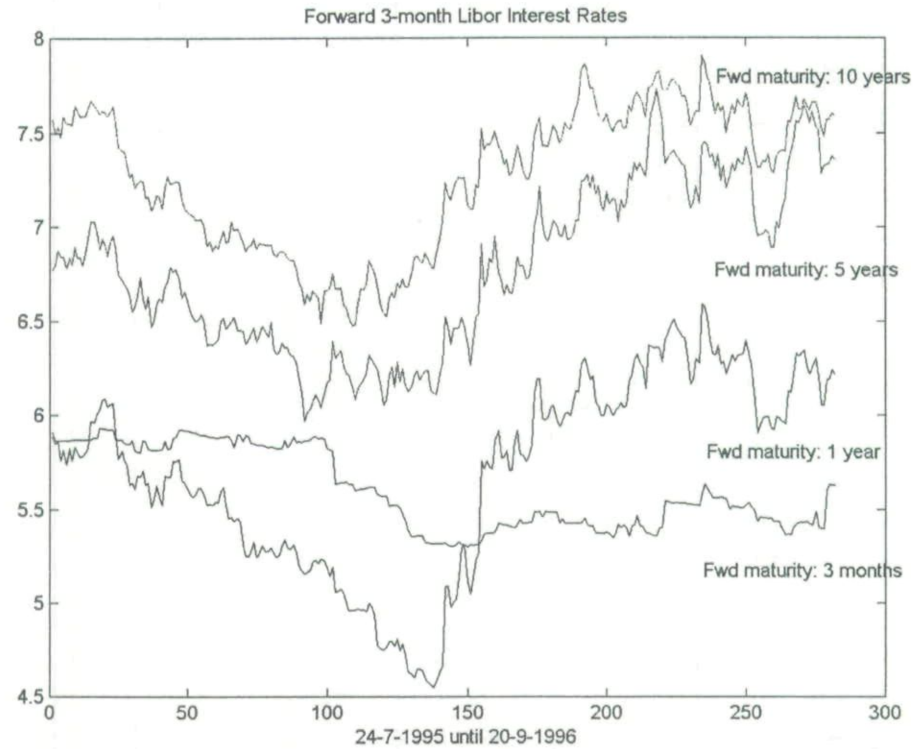

Figure 1. Daily 3-month forward interest rates. Forward rate maturities: 3 months, 1 year, 5 years and 10 years.

some resulting forward Libor rates. It follows that the forward Libor rate curve is mainly an increasing function of maturity, although for very short maturities there are some exceptions.

The second dataset we use consists of daily quotes for caplets and swaptions. More precisely, on every trading day between July 1995 and September 1996 we observe quotes for implied Black volatilities for at-the- money forward caplets of different maturities and for at-the-money forward swaptions with different option and swap maturities. In total, we have 282 daily observations. The caplet maturities range from 3 months to 10 years, the option maturities for the swaptions range from 1 month to 5 years, and the swap maturities range from 1 year to 10 years. We will denote the observed caplet implied Black volatility for maturity T by $IV^C(T)$, and the observed swaption implied Black volatility with option maturity T_1 and swap maturity T_2 by $IV^S(T_1, T_2)$. All options are at-the-money forward, i.e., the strike rate of each caplet is the forward Libor rate corresponding to the maturity date and the strike rate of each swaption is given by the corresponding forward swap rate.[7] There is a one-to-one relation between the implied Black volatility of each

[7] The market for in- and out-of-the-money options has not been liquid enough to obtain reliable historical price data.

Table II. Descriptive statistics caplet implied Black volatilities. Averages and standard deviations are calculated from 282 daily observations on implied volatilities of caplets, from July 1995 until September 1996. The Black volatilities are measured in annualized percentages (i.e., volatility points).

Average maturity In years	Average implied volatility	Standard deviation implied volatility
0.13	10.5	2.8
0.38	14.9	2.1
0.63	18.2	2.1
0.88	20.4	2.1
1.75	23.7	2.3
2.75	19.4	1.7
3.75	22.9	2.4
4.75	18.0	2.8
6.75	18.3	2.3
9.75	14.9	2.3

instrument and its price, so that we are able to construct prices of all caplets and swaptions.

In Tables II–IV, we give the averages and standard deviations of the implied Black volatilities of the caplets and swaptions. It follows both from the caplet and swaption data that the volatility structure, as function of the maturity of forward Libor or swap rates, is first increasing with maturity and then decreasing, i.e., it is humped-shaped. The maximum of the hump for forward Libor and swap rates seems to be somewhere between 1 and 2 years. This observation is in line with other empirical studies on interest rates and interest rate derivatives, such as Amin and Morton (1994) and Moraleda and Vorst (1997).

4. Results for Libor Market Models

In this section we estimate and analyze one-factor Libor Market Models. We start with a description of the calibration methodology and the different choices that we make for the volatility function. Thereafter, we present the pricing results for caplets and swaptions. We end the section with a specification analysis of the pricing errors of the LMM.

4.1. CALIBRATION METHODOLOGY

Throughout the remainder of this paper, we analyze three complementary sets of derivatives:

Table III. Averages of swaption implied volatilities. Averages are calculated from 282 daily observations on implied Black volatilities of swaptions, from July 1995 until September 1996. Each row contains swaptions with a fixed option maturity and different swap maturities. All maturities are expressed in years. The Black volatilities are measured in annualized percentages (i.e., volatility points).

Maturity *swap* *option*	1	2	3	4	5	7	10
0.085	18.3	19.4	19.2	19.0	18.7	17.6	16.7
0.25	18.9	19.6	19.3	18.9	18.4	17.5	16.4
0.50	19.4	19.7	19.2	18.7	18.3	17.3	16.2
1.00	20.7	20.0	19.2	18.4	17.8	16.9	15.9
1.50	20.5	19.6	18.9	18.1	17.5	16.6	
2.00	20.2	19.2	18.5	17.8	17.2	16.2	
3.00	19.5	18.5	17.8	17.2	16.7	15.9	
4.00	18.7	17.8	17.2	16.7	16.1		
5.00	18.0	17.2	16.7	16.1	15.6		

Table IV. Standard deviations of swaption implied volatilities. Standard deviations are calculated from 282 daily observations on implied volatilities of swaptions, from July 1995 until September 1996. Each row contains swaptions with a fixed option maturity and different swap maturities. All maturities are expressed in years. The Black volatilities are measured in annualized percentages (i.e. volatility points).

Maturity *swap* *option*	1	2	3	4	5	7	10
0.085	1.56	1.75	1.63	1.44	1.41	1.15	0.92
0.25	1.13	1.04	0.95	0.82	0.72	0.61	0.51
0.50	0.87	0.74	0.68	0.60	0.62	0.64	0.51
1.00	1.08	1.00	1.05	0.98	1.00	0.91	0.78
1.50	1.14	1.14	1.21	1.07	1.07	0.97	
2.00	1.22	1.25	1.28	1.15	1.14	1.03	
3.00	1.34	1.31	1.23	1.19	1.23	1.12	
4.00	1.31	1.31	1.20	1.24	1.16		
5.00	1.27	1.22	1.25	1.16	1.14		

- *Caplets.*
- *<10-year Swaptions.* Swaptions with a total maturity smaller than 10 years.
- *10-year Swaptions.* Swaptions with a total maturity of 10 years.

The caplets can be priced using Black's formula in case of the LMM, and the 10-year swaptions can be priced by Black's formula in case of the SMM, if the SMM is based on a total maturity of 10 years.[8] The <10-year swaptions cannot be priced analytically by any of these two models. Our empirical setup is such that the LMM is calibrated using the observed implied Black volatilities of caplets, whereas the SMM is calibrated to implied Black volatilities of 10-year total maturity swaptions. Hence, the pricing errors of the LMM and SMM for the <10-year swaptions can be used to compare the accuracy of the models.

We consider both constant volatility functions and exponential volatility functions, that correspond to mean-reverting interest rates. For these two types of volatility functions, we analyze both *exact calibration* and *non-exact calibration* models. In total we thus analyze four different specifications for the volatility function in the LMM.

In practice, the LMM is often parametrized such that all caplet implied volatilities are fitted exactly by the LMM. To exactly fit implied volatilities of caplets, one needs to assume some particular form for the volatility function of forward Libor rates $\gamma_n(t)$ in Equation (5). There are several choices for the volatility function that lead to exact fitting of caplet prices, because for each forward Libor rate, the dependence of $\gamma_n(t)$ on time t can be chosen in many different ways. Most endogenous term structure models, such as the models in the class of Duffie and Kan (1996), imply *time-homogeneous* volatility functions, i.e., volatility functions that only depend on the time-to-maturity $(T_n - t)$. However, exogenous term structure models, such as the Heath, Jarrow, and Morton (1992) models and the market models, do not necessarily have time-homogeneous volatility functions. Still, one could prefer to have a market model with a time-homogeneous volatility function, in order to have a direct relation with endogenous term structure models. However, it is easy to show that for the LMM, it is only possible to exactly fit caplets with a time-homogeneous volatility function if $IV^C(T_n)\sqrt{T_n}$ is increasing with the maturity of the caplet T_n. For most days in our dataset, the observed implied caplet Black volatilities do not satisfy this property, which is a consequence of the humped volatility structure that is present in the data. Therefore, a *time-inhomogeneous* volatility function is necessary to exactly fit caplet implied volatilities. This need for time-inhomogeneous volatility functions to exactly fit caplets has already been observed by Brace and Musiela (1995).

Then, the simplest choice for $\gamma_n(t)$ that leads to exact fitting of caplets is a flat volatility curve for each forward Libor rate

$$\gamma_n(t) = \gamma_n = IV^C(T_n), \quad n = 1, \ldots, N-1. \tag{9}$$

[8] In the next section we motivate our choice for the total maturity in the SMM.

For this *exact calibration, constant volatility LMM* the volatility function $\gamma_n(t)$ is constant over time t and equals the implied volatility of a caplet with maturity T_n. Because the volatility function is different for each maturity date T_n, this is a time-inhomogeneous volatility function. Given this choice for the volatility function of the one-factor LMM, the model is completely determined, and the prices for swaptions that are implied by this model can be calculated. Note that, because observed implied caplet volatilities change over time, the parameters γ_n of this model also change each trading day.

The choice in Equation (9) implies that forward Libor rates have a constant volatility during their evolution until their maturity date. However, it is a stylized fact that interest rates show declining volatility for longer maturities. Such a volatility function is consistent with mean reverting behaviour of interest rates, and it lowers the correlation between Libor rates at different points in time. To capture this effect, we also consider the following specification for the volatility function

$$\gamma_n(t) = e^{-\kappa(T_n - t)}\gamma_n, \quad n = 1, \ldots, N-1. \tag{10}$$

Given a value for κ, exact fitting of caplet implied volatilities is obtained by choosing[9]

$$\gamma_n = IV^C(T_n)\sqrt{\frac{2\kappa T_n}{1 - e^{-2\kappa T_n}}}, \quad n = 1, \ldots, N-1. \tag{11}$$

As explained in Section 2, mean-reverting behaviour corresponds to a value for the parameter κ that is positive. Since the estimates for κ, which are discussed in the next subsection, are indeed always positive, we will refer to the LMM with this exponential volatility function as the *declining volatility LMM*.

To examine whether exact calibration models are overfitted to noise in the derivative price quotes, we also examine *non-exact calibration* models, and compare the pricing accuracy for swaptions with the exact calibration models. To obtain a non-exact calibration counterpart of the constant volatility model in (9), a natural choice is a volatility function that is constant both over time t and over the forward maturity of the Libor rate $T_n - t$

$$\gamma_n(t) = \gamma, \quad n = 1, \ldots, N-1. \tag{12}$$

This volatility function is time-homogeneous and does not allow for a humped volatility structure. The parameter γ is estimated each trading day by minimizing the sum of squared differences between the parameter γ and the implied volatilities of caplets, i.e., the estimate is equal to the average of caplet implied volatilities of different maturities. Because we observe 10 caplets each day, whereas the model in (12) contains only one parameter, this model will not fit all 10 caplet prices exactly.

[9] This follows directly from the relation $T_n IV^C(T_n)^2 = \int_0^{T_n} \gamma_n(t)^2 \, \mathrm{d}t$. We refer to the Appendix for further details.

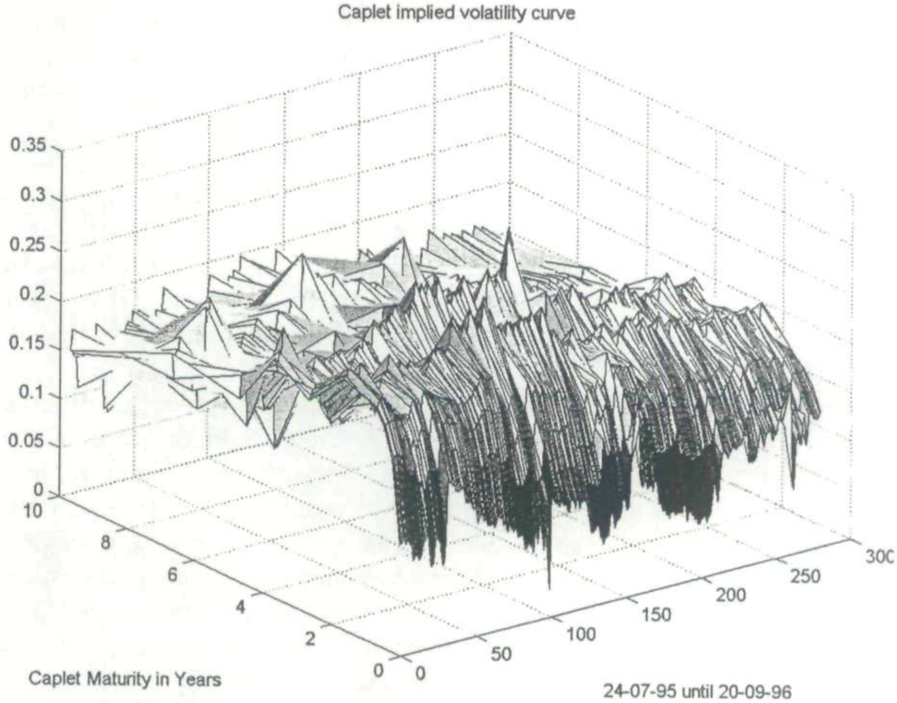

Figure 2. Daily caplet implied Black volatilities.

Similarly, a non-exact calibration version of the declining volatility LMM is obtained by choosing the following volatility function

$$\gamma_n(t) = e^{-\kappa(T_n - t)}\gamma, \quad n = 1, \ldots, N-1. \tag{13}$$

The parameter γ is, given a value for the decay parameter κ, estimated by least squares fitting of caplet implied volatilities. The resulting model is referred to as the *non-exact calibration, declining volatility LMM*.

In Figure 2, we plot the implied Black volatilities of caplets, that directly determine the shape of the volatility function of the exactly calibrated LMMs through Equations (9) or (11). The humped volatility structure is clearly present, and although there is variation in the implied volatilities over time, the shape of the volatility function seems to be more or less the same. Most striking is the steepness of this volatility curve at very short maturities.

For the declining volatility LMM, the decay parameter κ has to be estimated. As our goal is to accurately price swaptions with the LMM, it is natural to estimate this parameter from the daily cross-section of observed swaption prices.[10] More specifically, we estimate the decay parameter κ at each day by minimizing the

[10] In Moraleda and Pelsser (2000) a similar decay parameter is also estimated from derivative price data.

sum of squared differences between the observed implied Black volatilities for all swaptions and the Black volatilities that correspond to the swaption prices that are implied by the LMM. We choose to use Black volatilities as the way to represent swaption prices, because this is market practice and because our data are in terms of Black volatilities. As the implied Black volatilities of different swaptions are of the same order (between 15% and 20%), this seems to be a fair weighting scheme, which we prefer over fitting the dollar prices, which are quite different for various maturities. Bossaerts and Hillion (1997) also argue that using implied volatilities instead of prices leads to a better weighting scheme. Also, in all tables we present pricing errors in terms of Black implied volatilities, because these are easier to interpret.

For the LMM, prices of swaptions can be calculated using either simulation techniques, as described in Section 2, or the approximation price of Brace et al. (1997), that is described in the appendix. The nonlinear minimization, that is necessary to estimate the decay parameter κ, would be very time-consuming if swaptions are priced using simulation techniques. Therefore we use the analytical approximation of Brace et al. (1997) for a swaption price to obtain an estimate for the decay parameter κ, and, given this estimate, we obtain prices using simulation techniques. The difference between the simulated prices and the prices from the analytical approximation is always smaller than 0.08 volatility points.

4.2. ESTIMATION AND PRICING RESULTS

We start with the *constant volatility LMM*. Recall that we estimate the model parameters for every day in the sample using the daily cross-section of caplet prices. In the first panel of Table V, we present statistics on the parameter estimates for the non-exactly calibrated constant volatility LMM. The standard deviation of the time-series of parameter estimates of γ in Equation (12) is equal to 1.4%. This is smaller than the standard deviations of the caplet implied volatilities, which are above 2%, and this indicates that the parameter of the non-exactly calibrated model is more stable than the parameters of the exactly calibrated model. In Table VI, we give the average fit of the exactly calibrated and non-exactly calibrated LMM. For swaptions, which are the instruments that are not used for calibration, the average absolute pricing error of the exactly calibrated model is just above 2 volatility points, whereas the non-exactly calibrated model has an average absolute pricing error around 1.5 volatility point. This implies that the exactly calibrated model is to some extent overfitted, because the more parsimoniously specified non-exact calibration model has smaller pricing errors on swaptions. Still, although the non-exact calibration model outperforms the exact calibration model, there is clear evidence that the flat volatility function of the non-exact calibration model is in fact too simple; Figure 2 shows that there is a humped-shaped volatility function which is persistent over time. Due to the presence of this volatility hump, the non-exact calibration model does not exactly fit caplet prices, as shown in Figures 5a

Table V. Parameter estimates LMM and SMM. The table contains averages and standard deviations of 282 daily parameter estimates for one-factor LMMs and SMMs, as specified in Equations (9)–(13) and (16), both for the case of exact calibration and non-exact calibration. All parameters are expressed on an annual basis.

	Model	Average γ	St. dev. γ	Average κ	St. dev. κ
Exact calibration					
Constant volatility	LMM	–	–	–	–
Declining volatility	LMM	–	–	0.092	0.032
Constant volatility	SMM	–	–	–	–
Declining volatility	SMM	–	–	0.049	0.025
Non-exact calibration					
Constant volatility	LMM	0.182	0.014	–	–
Declining volatility	LMM	0.194	0.019	0.043	0.022
Constant volatility	SMM	0.163	0.006	–	–
Declining volatility	SMM	0.168	0.006	0.033	0.011

and 5b. Also, the autocorrelations of the caplet pricing errors for the non-exact calibration model in Table VI are very high.

For the exact calibration methodology, we have also examined the following time-inhomogeneous specification for the volatility function: $\gamma_n(t) = e^{-\kappa(T_n-t)}\gamma_n(1+\lambda T)/(1+\lambda t)$. This specification, that can generate a humped volatility structure, is proposed by Moraleda and Vorst (1997). The results, not included in this paper, show that this extended specification does not lead to a much better fit than the specification in Equation (11), and also results in parameter estimates that are not very stable over time. For these reasons, we use the specifications in Equations (9)–(13) in this paper. However, for the non-exact calibration methodology, a more refined volatility function, that is still time-homogeneous and generates a humped volatility structure, might improve the pricing results.

For the constant volatility LMM, the average pricing errors of swaptions are positive, both in case of exact and non-exact calibration. This indicates that swaptions are *overpriced* by the constant volatility LMM. A standard explanation for this result, used by Longstaff et al.(2000a) and Rebonato (1996, 1998), would be a missing second factor. Given the fact that the volatility of forward Libor rates is determined by the caplet implied volatilities, a second factor would lower the variances of swap rates. This is because swaprates are combinations of several forward Libor rates and a second factor would imply nonperfect correlation between these forward Libor rates, thereby lowering the variance of swap rates. This lower swap rate variance in turn lowers the prices of swaptions that are implied by the model.

Although this nonperfect correlation of interest rates by introducing additional factors is potentially an important effect, we would like to address another mech-

Table VI. Pricing results for Libor market model. The table contains summary statistics on pricing errors of one-factor LMMs, for caplets, swaptions with a total maturity less than 10 years and swaptions with a total maturity equal to 10 years. Results for four models are presented: exactly calibrated and non-exactly calibrated models with and without a declining volatility function. All pricing errors are measured in Black implied volatility points, and defined as the Black implied volatility that is implied by the LMM minus the observed Black implied volatility. The Black volatilities are measured in annualized percentages (i.e., volatility points).

Declining Volatility	Exact calibration	Average pricing error	Average absolute pricing error	Average of daily maximal errors	Average Autocorrelation pricing errors
Caplets					
No	No	0.00	3.28	8.03	0.717
No	Yes	–	–	–	–
Yes	No	0.06	3.26	9.06	0.673
Yes	Yes	–	–	–	–
<10-year total maturity swaptions					
No	No	0.45	1.42	3.62	0.930
No	Yes	1.84	2.11	4.41	0.736
Yes	No	−0.27	0.97	3.16	0.904
Yes	Yes	0.30	1.08	3.88	0.716
10-year total maturity swaptions					
No	No	2.15	2.23	3.17	0.941
No	Yes	2.01	2.08	2.99	0.736
Yes	No	0.06	0.59	1.59	0.812
Yes	Yes	−0.80	0.90	1.93	0.715

anism by which the overpricing of swaptions can be explained. This alternative explanation for the overpricing of swaptions by constant volatility models, which we feel has been ignored in the literature, is the absence of a declining volatility structure. As mentioned above, such a volatility structure is consistent with mean-reverting interest rates. This can be explained as follows. The price of a swaption at time t, that has as option maturity date T_n and swap maturity date $T_n + m\delta$, is primarily determined by the following conditional covariance matrix Δ of forward Libor rates

$$\Delta_{ij} = \mathrm{Cov}(L_i(T_n),\ L_j(T_n)|\mathcal{F}_t), \quad i, j = n+1, \ldots, n+m. \tag{14}$$

Because the LMM is always calibrated to caplets, the variance of forward Libor rates over their entire maturity is (on average) fixed. On the other hand, a declining volatility function implies that the volatility of forward Libor rates increases as the rate comes closer to its maturity date, and it follows that this leads to lower volatility for forward Libor rates far from maturity, and higher volatility for forward Libor

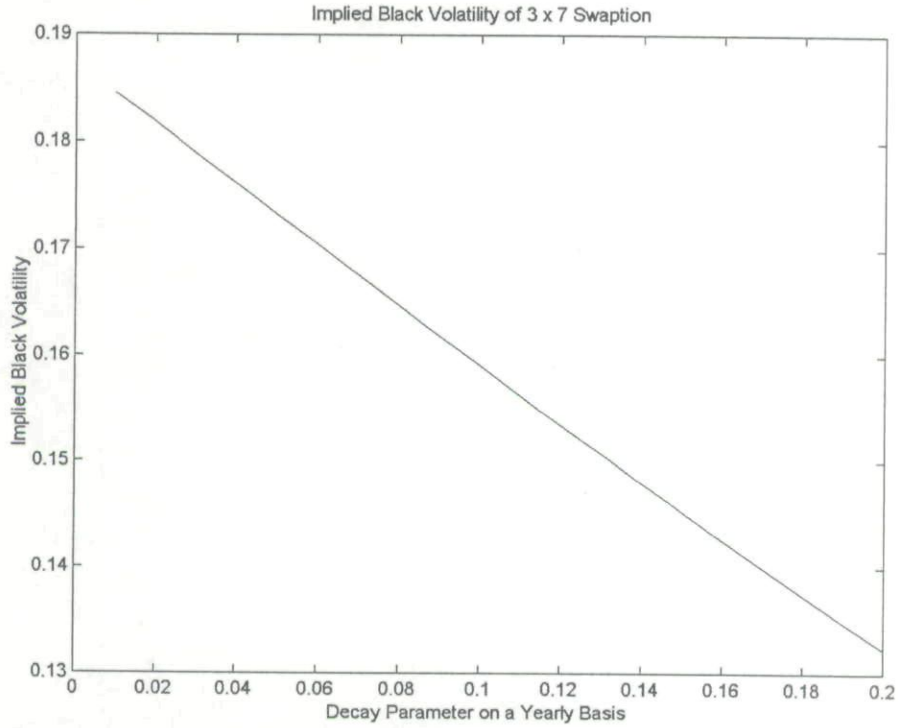

Figure 3. Influence of decay parameter κ on swaption price. For different values of the decay parameter κ, the price and corresponding implied Black volatility of a 3×7 swaption (3 year option maturity, 7 year swap maturity) in the exactly-calibrated one-factor LMM is calculated, using data for the first day in the dataset.

rates close to maturity. For the pricing of swaptions, only the conditional variance of the forward Libor rate up to the option maturity date is relevant, as shown in Equation (14), and thus a declining volatility function implies lower swaption prices, especially for swaptions with short option and long swap maturities. In Figure 3, we illustrate this argument graphically, by plotting the swaption price implied by the LMM for different values of the decay parameter κ.

To investigate the effect of a declining volatility function, we have estimated the declining volatility LMM as described above, by minimizing, at each day in the sample, squared pricing errors (in terms of Black volatilities) for swaptions over the decay parameter κ. In Figure 4, the daily κ estimates are plotted and in Table V we give summary statistics on the parameter estimates. At all days, the estimates for κ are positive, and these estimates are not very unstable over time, although there seems to be a regime-shift in the middle of the time-period. Again, the parameter estimates for the non-exactly calibrated model are more stable over time than for the exactly calibrated model, as shown in table V. As the overpricing of swaptions is larger for the exactly calibrated constant volatility model than for the non-exact

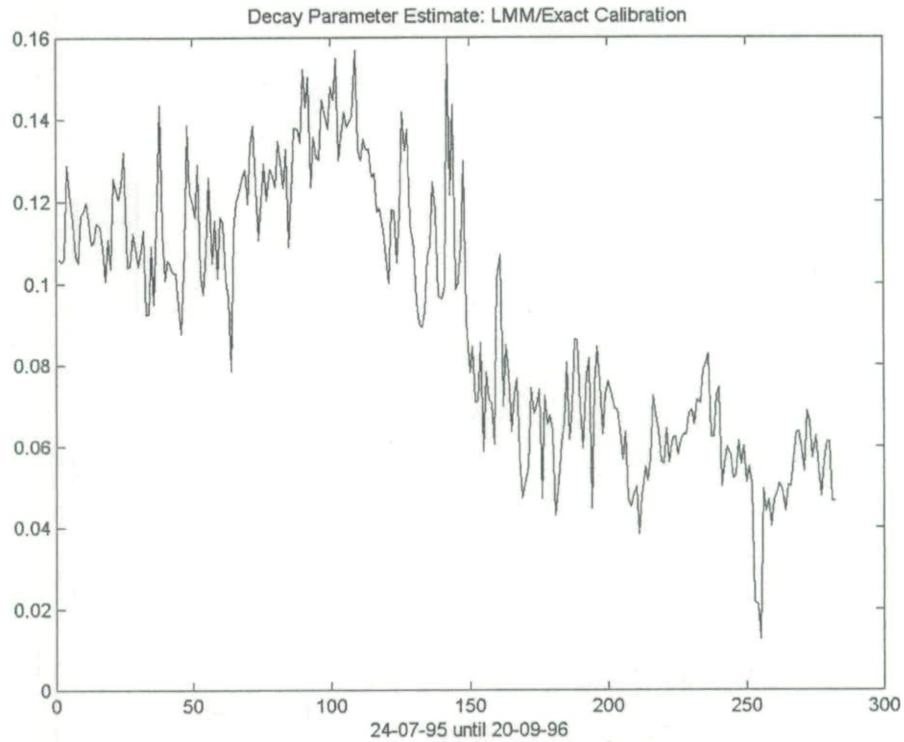

Figure 4a. Decay parameter κ estimates. Daily estimates of decay parameter κ for the exactly calibrated one-factor LMM (upper graph) and SMM (lower graph), from July 1995 to September 1996.

calibration model, the estimates for κ of the model with exact calibration are also larger than for the model with non-exact calibration.

Given the fact that the decay parameter κ is estimated from swaption prices, it is clear that the declining volatility LMM will always have a better fit than the constant volatility LMM. However, as shown in Table VI, the increase in the fit of swaption prices is large, both for the exactly and non-exactly calibrated models. Also, the average errors are close to zero now. The exactly calibrated model still has a worse fit on swaptions than the non-exactly calibrated model, although the differences are smaller in this case.

These results show the importance of including declining volatility functions in the specification of market models. These results also imply that studies in which the number of factors is estimated on the basis of constant volatility functions are likely to overestimate the number of factors. We have also analyzed a two-factor version of the LMM. The results, that are available on request, show that the two-factor model gives only slightly smaller pricing errors. Also, the empirical results on exact calibration versus non-exact calibration and the effect of a declining volatility structure are very similar to the results in case of the one-factor

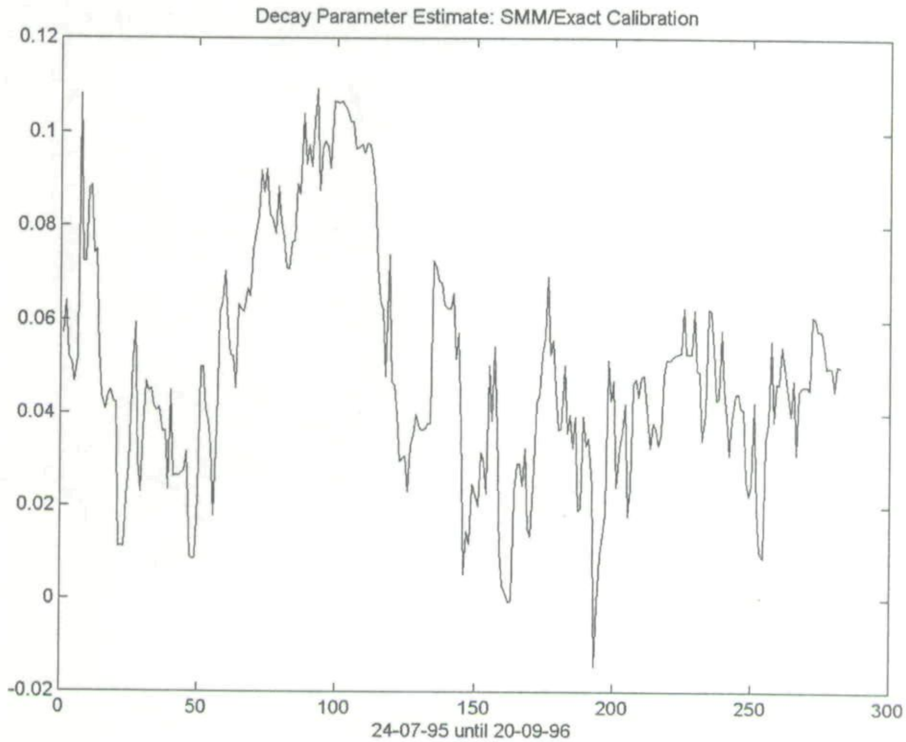

Figure 4b.

model. Therefore, we have not included these two-factor results in the paper. Of course, to fit other aspects of the term structure of interest rates, for example, the correlation matrix of contemporaneous changes in interest rates of different maturities, including multiple factors is very important. Also, for the pricing of options on the spread of term structure using a multi-factor term structure model is inevitable. In De Jong et al. (2001) we estimate multi-factor market models using both interest rate data and option price data.

4.3. ANALYSIS OF THE PRICING ERRORS

In Table VI, we also report the autocorrelation of daily pricing errors for caplets and swaptions. These autocorrelations are quite high for all models, which is an indication of the systematic nature of the pricing errors. There are also maturity effects in the pricing errors for the caplets and swaptions, as is shown in Figures 5a–d. For the exactly calibrated model, the pricing errors decrease with swap maturity, and swaptions with long swap maturities are underpriced. Also, swaptions with very short or very long option maturities are underpriced. In case of non-exact calibration, swaptions with short and long total maturities are overpriced, whereas swaptions with intermediate total maturities are underpriced. This pricing error

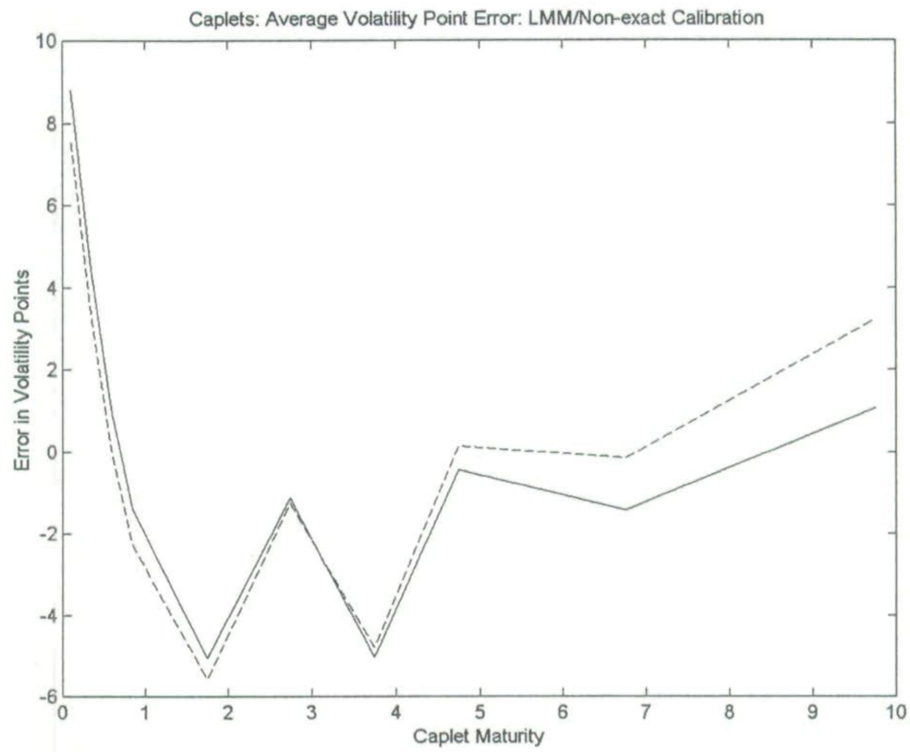

Figure 5a. Pricing errors caplets for LMM. Volatility point error caplets for non-exact calibration LMM, in case of constant volatility (dashed line) and declining volatility (solid line). The upper graph contains the average volatility point errors, the lower graph contains the average absolute volatility point errors.

pattern is caused by the absence of a humped volatility structure in the non-exact calibration model, which also explains the maturity pattern in the pricing errors for the caplets in case of non-exact calibration (see Figures 5a–b).

To formally test whether the models on average correctly price caplets and swaptions, we estimate the following regression equation for every swaption separately

$$\begin{aligned} IV_t^S(T_i, T_j) - IV_t^C(T_i) &= \alpha_{ij} + \beta_{ij}(IV_t^{S,\mathrm{LMM}}(T_i, T_j) \\ &\quad - IV_t^{C,\mathrm{LMM}}(T_i)) + \epsilon_{ij,t}, \quad t = 1, \ldots, 282, \quad (15) \end{aligned}$$

where $IV_t^{S,\mathrm{LMM}}$ is the implied volatility at day t of the swaption price that is implied by the LMM; similar notation is used for caplets. Using regression Equation (15), we thus investigate to what extent the LMM explains the observed difference between swaption and caplet implied volatilities. If the model is correct, α_{ij} is equal to zero and β_{ij} is equal to one. Amin and Morton (1994) estimate the same regression equation for their analysis of Eurodollar futures options, but they use prices of options instead of implied volatilities. This is an important difference, because

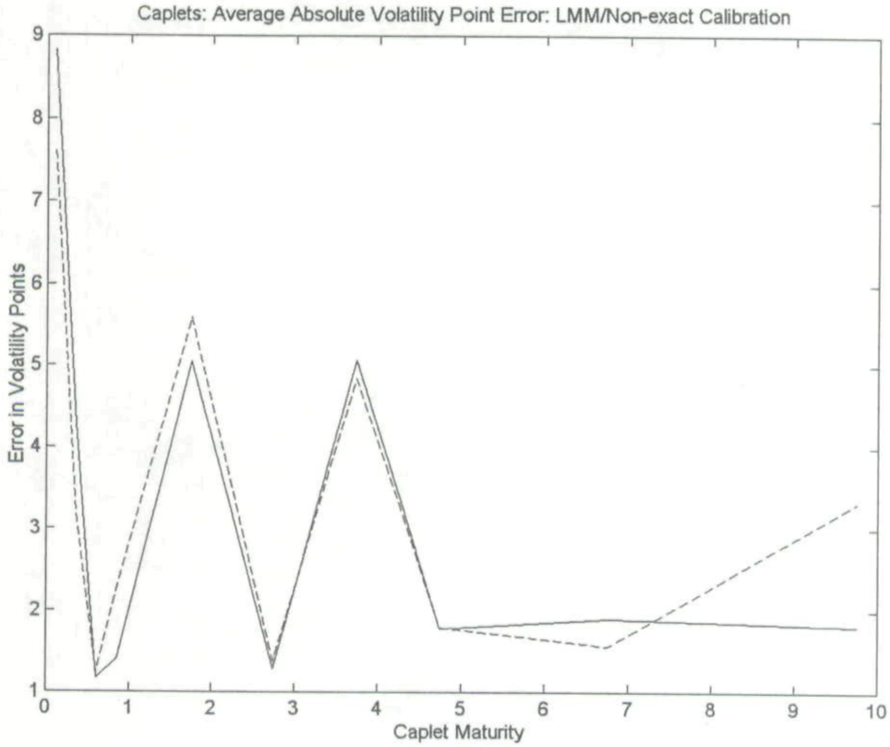

Figure 5b.

prices of options are partially determined by their intrinsic value and the initial term structure, which are the same for the model price and the observed price. This explains why Amin and Morton (1994) find very high R^2 values, although the models they analyze are still rejected statistically. We prefer to use implied volatilities, because the implied volatility of an option is a measure that corrects for the price effects of the intrinsic value and the initial term structure. We choose to use the difference between swaption and caplet volatilities as this difference has a lower autocorrelation than the caplet and swaption implied volatilities itself. In other words, we eliminate a possible 'trend' in the implied volatilities. This decreases biases in the parameter estimates that are associated with regressions of this type, see e.g., Bekaert et al. (1997). The regression setup is also such that both exact and non-exact calibration models can be analyzed with this regression.

In Table VII, we present a summary of the results for this regression analysis. It follows that all models are clearly rejected; the F-test for $\alpha_{ij} = 0$ and $\beta_{ij} = 1$ rejects much more often than the 5%-confidence level that is used. If we compare the difference of the parameter estimates with the 'correct' parameter values (zero intercept and slope equal to one) and the R^2 values of the constant volatility LMMs and the declining volatility LMMs, the performance of the declining volatility LMMs is clearly better. Also, because of the large errors on caplet implied volatil-

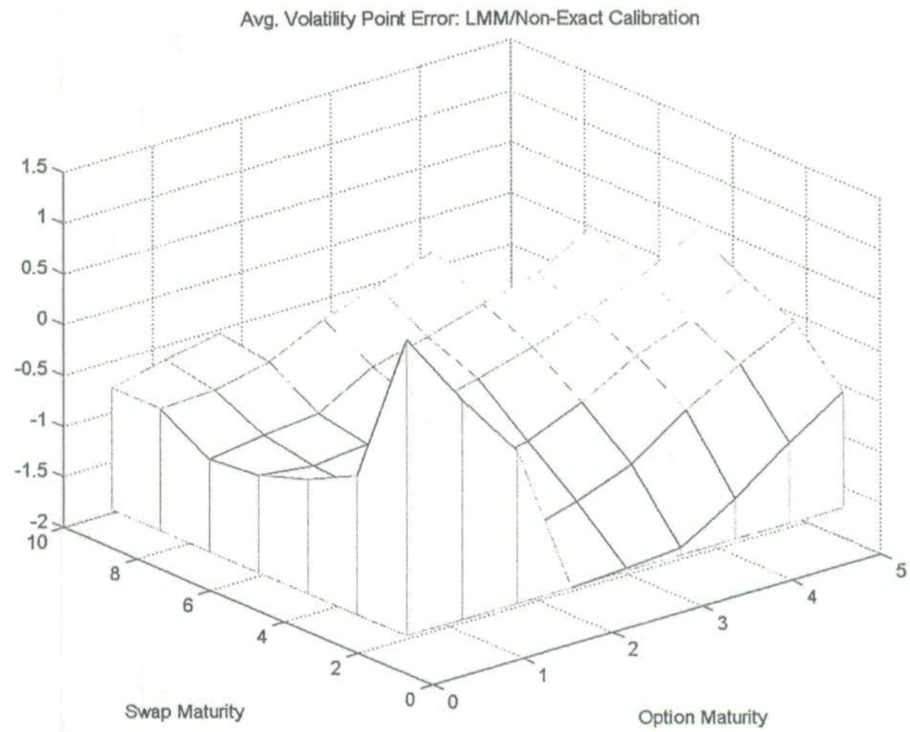

Figure 5c. Pricing errors swaptions for LMM. Average volatility point error of swaptions for declining volatility LMM, in case of non-exact calibration (upper graph) and exact calibration (lower graph).

ities for the non-exactly calibrated models, the regression results for these models are worse than for the exactly calibrated models.

The relatively low values for the R-squared are remarkable. Apparently, there is a lot of variation in the swaption and caplet prices that cannot be explained by the model, although a part of this variation could be due to measurement errors in the data, caused by the fact that we use *quotes* for the implied Black volatilities.

5. Results for Swap Market Models

In this section we will perform a similar analysis for the SMM as in the previous section for the LMM, and compare the results for the different models. We will calibrate the SMM to the set of 10-year total maturity swaptions and investigate the pricing implications of this model for caplets and swaptions with total maturity less than 10 years.

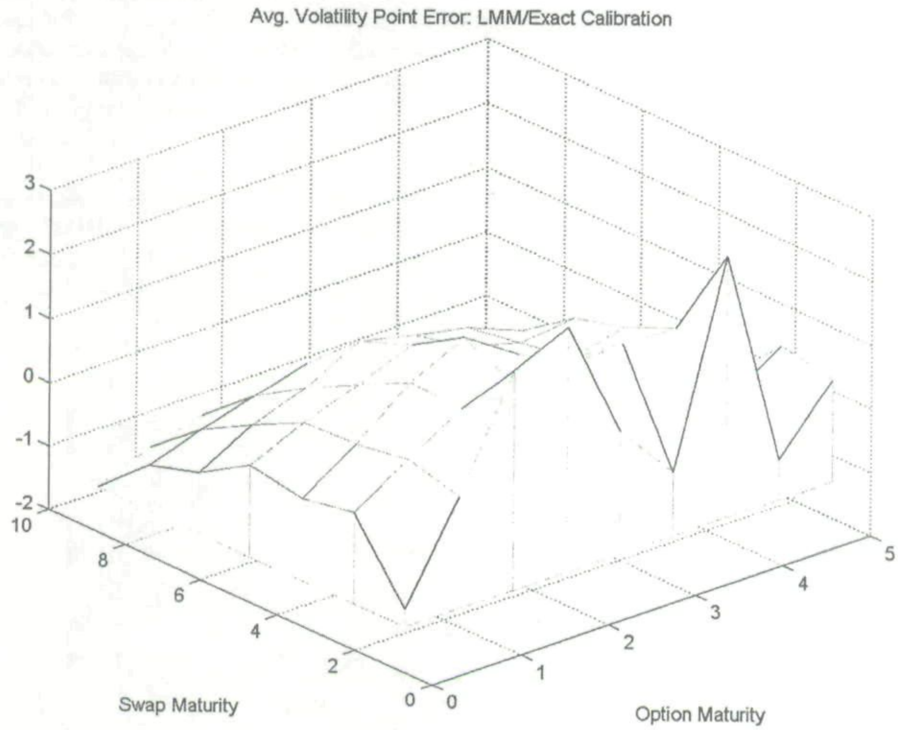

Figure 5d.

Table VII. Difference between swaption vols and caplet vols. Regression results for LMM. For every swaption with a certain option maturity and swap maturity, the regression parameters in Equation (15) are estimated:

$$IV_t^S(T_i, T_j) - IV_t^C(T_i) = \alpha_{ij} + \beta_{ij}(IV_t^{S,\mathrm{LMM}}(T_i, T_j) - IV_t^{C,\mathrm{LMM}}(T_i)) + \epsilon_{ij,t} \quad t = 1, \ldots, 282$$

The table reports the average (absolute) coefficient estimates, average (absolute) t-values in brackets and the average of the R^2. The column 'Percentage Rejections' shows for how many swaptions the F-test for the joint hypothesis $\alpha_{ij} = 0$ *and* $\beta_{ij} = 1$ is rejected at the 5% confidence level, relative to the total number of swaptions. The t-values and F-tests are corrected for autocorrelation using the Newey–West (1987) method with 15 lags. The bold numbers indicate average t-values larger than 1.96 or smaller than -1.96.

Declining Volatility	Exact calibration	Average α	Average $\lvert\alpha\rvert$	Average $\beta - 1$	Average $\lvert\beta - 1\rvert$	Average R^2	Percentage rejections
No	No	0.01 (1.6)	**0.02 (2.2)**	**−4.94 (2.3)**	**4.94 (2.3)**	0.06	79.0%
No	Yes	0.00 (1.3)	**0.01 (2.3)**	**−0.68 (4.7)**	**0.68 (4.7)**	0.27	98.4%
Yes	No	**0.01 (3.3)**	**0.01 (4.5)**	−0.84 (0.4)	**1.93 (3.4)**	0.22	79.0%
Yes	Yes	0.00 (−0.8)	**0.01 (3.1)**	**−0.21 (−3.8)**	**0.38 (4.5)**	0.55	88.7%

Table VIII. Pricing results for swap market model. The table contains summary statistics on pricing errors of one-factor SMMs, for caplets, swaptions with a total maturity less than 10 years and swaptions with a total maturity equal to 10 years. Results for four models are presented: exactly calibrated and non-exactly calibrated models with and without a declining volatility function. All pricing errors are measured in Black implied volatility points, and defined as the Black implied volatility that is implied by the SMM minus the observed Black implied volatility.

Declining Volatility	Exact calibration	Average pricing error	Average absolute pricing error	Average of daily maximal errors	Average Autocorrelation pricing errors
Caplets					
No	No	−0.55	3.09	7.21	0.713
No	Yes	2.87	6.99	26.11	0.707
Yes	No	1.93	3.96	11.55	0.704
Yes	Yes	5.16	7.36	28.09	0.732
<10-year total maturity swaptions					
No	No	−1.81	2.28	5.08	0.910
No	Yes	−1.22	2.26	12.05	0.796
Yes	No	−0.22	1.24	4.78	0.897
Yes	Yes	0.32	1.77	12.48	0.798
10-year total maturity swaptions					
No	No	0.00	0.59	1.39	0.951
No	Yes	–	–	–	–
Yes	No	0.0	0.43	1.10	0.939
Yes	Yes	–	–	–	–

5.1. CALIBRATION METHODOLOGY

The SMM is constructed such as to obtain analytical pricing formulas for a certain set of swaptions, and we will estimate the volatility function of the SMM using this set of swaptions, that all have the same total maturity. Then, we can analyze how well the resulting model prices caplets and swaptions that have a smaller total maturity.

For the SMM, the choice for the total maturity is not unique, because, given a choice for this total maturity, only swaptions and caplets with a total maturity smaller than this given total maturity can be priced with the SMM. Therefore, an obvious choice for this total maturity is 10 years, so that all caplets and (almost) all swaptions can be priced with the SMM.[11]

[11] We have also examined a SMM with a total maturity of 5 years. The empirical results are similar to the results of the 10-year SMM.

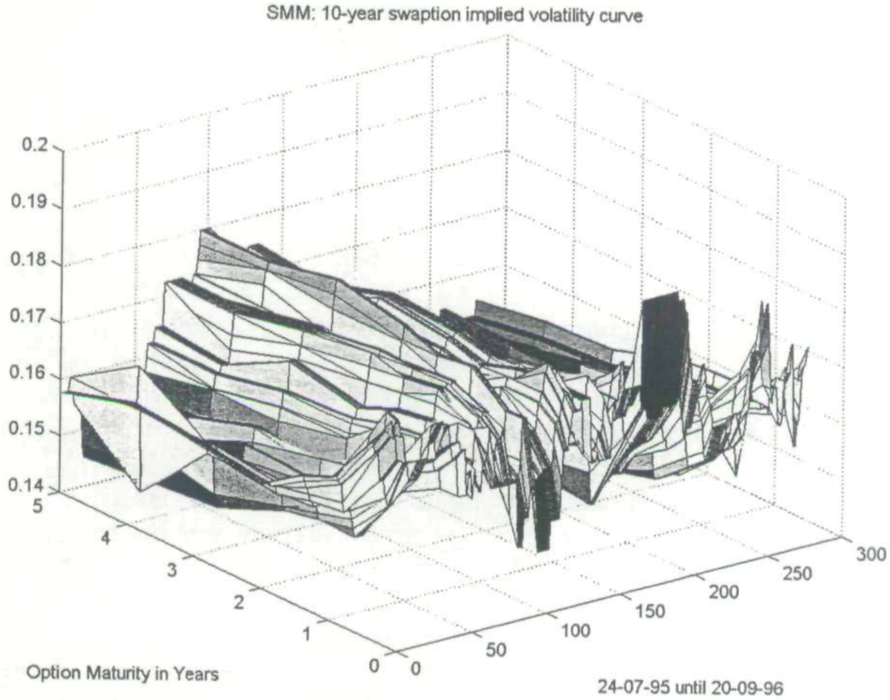

Figure 6. 10-year total maturity swaptions. Daily swaption implied Black volatility curve for 10-year SMM. All swaptions have a total maturity equal to 10 years. Data from July 1995 to September 1996.

In the data, it is not always the case that we observe swaptions with a total maturity of exactly 10 years. For example, we observe the implied Black volatility of a swaption with an option maturity of 1 month and swap maturities of 7 and 10 years. In this case, we follow market practice and linearly interpolate between the 1-month/7-year swaption implied volatility and the 1-month/10-year swaption implied volatility to obtain the implied volatility for the swaption with option maturity 1 month and total maturity 10 years. In Figure 6, we plot the resulting implied Black volatilities of the swaptions with a total maturity of 10 years. It follows that this volatility curve takes different shapes through time, both increasing, decreasing and humped shapes.

Similar to the LMM, it is only possible for a SMM to exactly fit swaption implied Black volatilities with a time-homogeneous volatility function if $IV^S(T_n, 10 - T_n)\sqrt{T_n}$ is increasing with the option maturity T_n of the swaption. Again, for the swaption implied Black volatilities with total maturities of 10 years, this is not always the case in the data. Therefore, we choose the same form for the volatility function of the SMM as the choice in Equations (9)–(13) for the LMM

$$\begin{aligned} \gamma_{n,N-n}(t) &= e^{-\kappa(T_n-t)}\gamma_{n,N-n}, \quad n = 1, \ldots, N-1 \\ \gamma_{n,N-n} &= IV^S(T_n, T_N - T_n), \quad \text{if } \kappa = 0 \end{aligned} \tag{16}$$

$$\gamma_{n,N-n} = IV^S(T_n, T_N - T_n)\sqrt{\frac{2\kappa T_n}{1-\exp(-2\kappa T_n)}}, \quad \text{if } \kappa \neq 0.$$

This leads to exact fitting of the implied Black volatilities of the 10-year total maturity swaptions. The non-exactly calibrated Swap Market Models are also specified similar to the non-exactly calibrated LMMs. Hence, the 10-year SMM is calibrated, either exact or non-exact, to 10-year total maturity swaptions with option maturities of 1, 3 and 6 months, and 1, 1.5, 2, 3, 4 and 5 years.

We both consider a constant volatility SMM and a declining volatility SMM. For the declining volatility SMM, the decay parameter κ is estimated daily by minimizing the sum of squared differences between observed implied Black volatilities and Black volatilities that correspond to the prices implied by the SMM, where the sum is taken over all instruments that are not priced directly by the SMM, i.e., swaptions with a total maturity smaller than 10 years, and all caplets.

The prices of these caplets and swaptions cannot be determined analytically for the SMM, and thus we use simulation to price these instruments.[12] However, for computational reasons, we use approximate analytical formulas for the prices of caplets and swaptions to estimate the decay parameter κ. In the appendix, these approximation formulas are given. The difference between the approximation and simulation prices is larger than for the LMM, but is at maximum equal to 0.5 volatility point. Given the estimated value for κ, we use simulation to obtain prices for the caplets and swaptions.

5.2. ESTIMATION AND PRICING RESULTS

In Table VIII, we give the pricing results for the constant volatility SMM. Most remarkable are the large pricing errors for caplets in case of exact calibration. In Figur 7, it is shown that especially short-maturity caplets are overpriced substantially. In the next section, we will provide a detailed explanation of this result. In case of non-exact calibration, the pricing errors for caplets are much smaller and of the same size as the caplet pricing errors of the non-exactly calibrated LMMs. We can therefore again conclude that exact calibration leads to overfitting of the model, and in this case exact calibration results in very large pricing errors for caplets. Furthermore, the average absolute pricing error on 10-year swaptions for the non-exactly calibrated model is around 0.6 volatility points, so that the flat volatility function quite reasonably fits the 10-year swaption implied volatilities.

For <10-year swaptions, the effect of differences in the implied volatilities of 10-year swaptions is not as dramatic, because the swaption prices are determined by the variances of several forward Libor rates, so that the impact of the high variances of forward Libor rates with short forward maturities is much smaller. Hence, the differences between the exactly and non-exactly calibrated models are

[12] We use the swaptions that can be priced analytically as control variates.

Table IX. Difference between swaption vols and caplet vols. Regression results for SMM. For every swaption with a certain option maturity and swap maturity, the regression parameters in the following equation are estimated:

$$IV_t^S(T_i, T_j) - IV_t^C(T_i) = \alpha_{ij} + \beta_{ij}(IV_t^{S,\mathrm{SMM}}(T_i, T_j) - IV_t^{C,\mathrm{SMM}}(T_i)) + \epsilon_{ij,t} \quad t = 1, \ldots, 282$$

The table reports the average (absolute) coefficient estimates, average (absolute) t-values in brackets and the average of the R^2. The column 'Percentage Rejections' shows for how many swaptions the F-test for the joint hypothesis $\alpha_{ij} = 0$ *and* $\beta_{ij} = 1$ is rejected at the 5% confidence level, relative to the total number of swaptions. The t-values and F-tests are corrected for autocorrelation using the Newey–West (1987) method with 15 lags. The bold numbers indicate average t-values larger than 1.96 or smaller than −1.96.

Declining Volatility	Exact calibration	Average α	Average $\lvert\alpha\rvert$	Average $\beta - 1$	Average $\lvert\beta - 1\rvert$	Average R^2	Percentage rejections
No	No	−0.01 (−0.3)	**0.02 (2.8)**	**−0.95 (−3.3)**	**0.95 (3.4)**	0.18	95.2%
No	Yes	0.00 (1.1)	**0.01 (4.8)**	**−1.07 (−2.3)**	**1.07 (2.3)**	0.15	100%
Yes	No	0.00 (1.1)	**0.02 (2.6)**	**−(0.82 (−2.6)**	**0.82 (2.6)**	0.22	93.6%
Yes	Yes	0.00 (−0.1)	**0.01 (4.2)**	**−0.97 (−2.4)**	**0.97 (2.4)**	0.17	98.4%

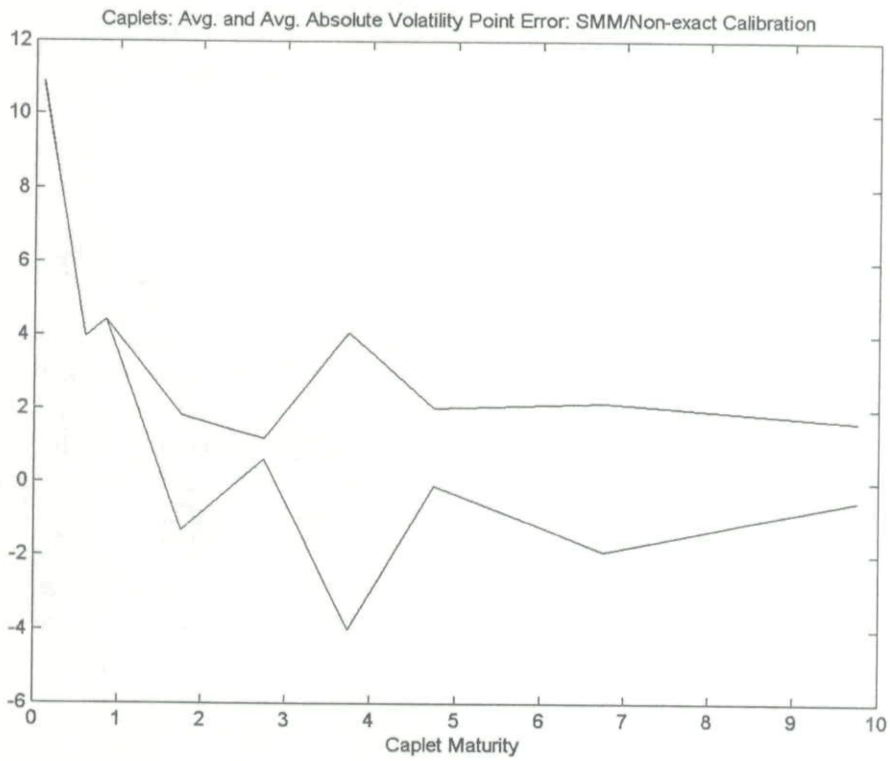

Figure 7a. Pricing errors caplets for SMM. Average and average absolute volatility point errors caplets for one-factor declining volatility SMM, with no exact calibration (upper graph) and exact calibration (lower graph).

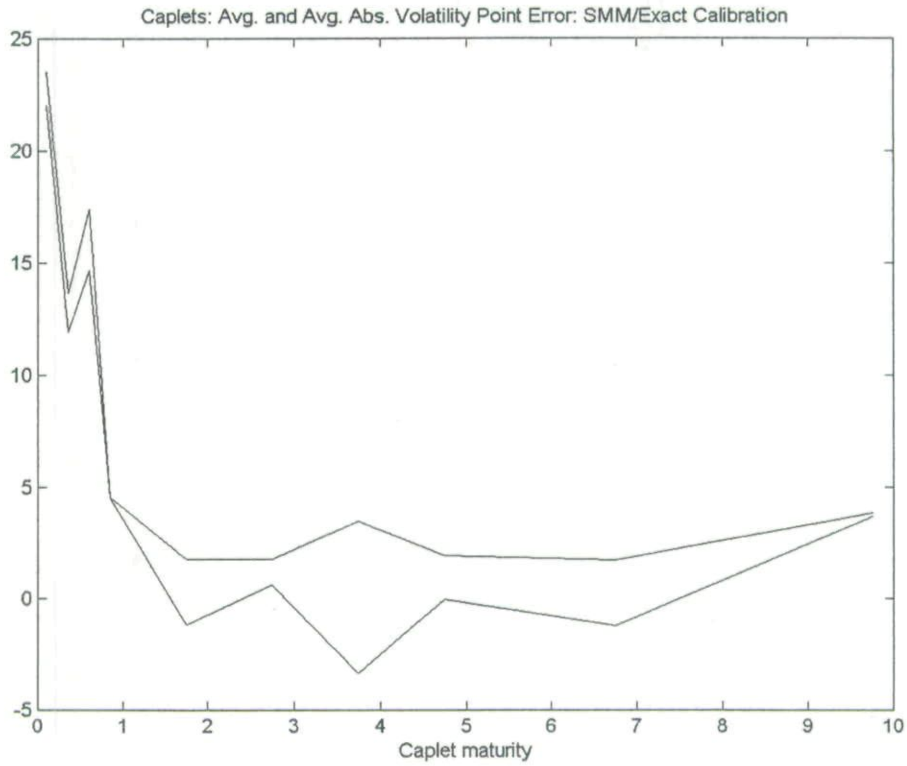

Figure 7b.

small for swaptions, although the maximal pricing errors are larger in case of exact calibration. Also, the parameter estimate for the non-exactly calibrated constant volatility SMM is less variable over time than the implied volatilities of the swaptions that are used for the exactly calibrated SMM, as can been seen in Tables IV and V.

For the constant volatility SMM, the fit on <10-year swaptions is a little worse than the constant volatility LMM, both in the case of exact and non-exact calibration. The constant volatility SMM typically underprices swaptions with total maturities less than 10 years. An explanation for this result is the absence of a declining volatility function. This can be seen as follows. Consider the 10-year SMM and suppose we are pricing a swaption with option maturity and swap maturity of 1 year, in other words, a 1×1 swaption. The conditional variance $\mathrm{Var}(S_{1,2}(T_1)|\mathcal{F}_t)$ of the 1×1 forward swaprate that determines this swaption price is roughly determined by the conditional variance, over the first year, of the 1×9 swap rate minus the 2×8 swap rate, $\mathrm{Var}(S_{1,9}(T_1) - S_{2,8}(T_1)|\mathcal{F}_t)$. A declining volatility function will lower the conditional variance over the first year of the 2×8 swap rate, whereas the conditional variance of the 1×9 swap rate is fixed and determined by the observed implied volatility of the 1×9 swaption. Therefore, the

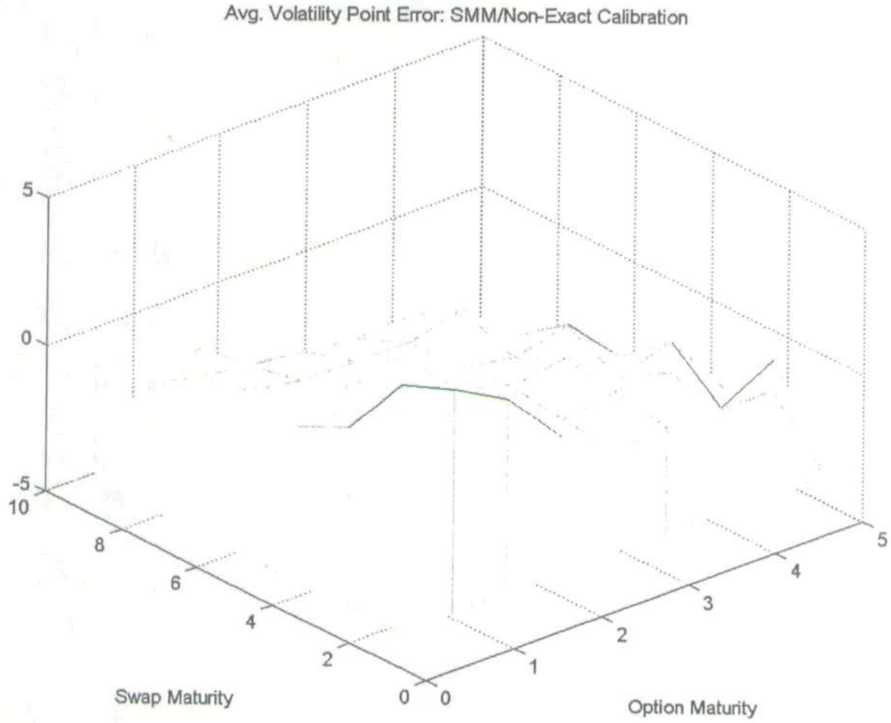

Figure 7c. Pricing errors swaptions for SMM. Average volatility point errors swaptions for one-factor declining volatility SMM, with no exact calibration (upper graph) and exact calibration (lower graph).

conditional variance of the 1×1 swap rate will increase if a declining volatility function is introduced. Hence, the absence of a declining volatility function implies that the SMM underprices swaptions.

In Figure 4b we plot the estimates for the decay parameter κ, for every day in the dataset, and in Table V we summarize the parameter estimates. As expected, the average κ estimates are positive, but a little smaller than the κ estimates for the LMM. Table IX shows that including a declining volatility function leads to a large decrease in absolute pricing errors for swaptions, although the influence of the declining volatility function is somewhat smaller than for the LMM. Similar to the constant volatility SMMs, the declining volatility SMMs have larger absolute pricing errors on the $<$10-year swaptions than the declining volatility LMMs. We can thus conclude that the Libor Market Models outperform the Swap Market Models in pricing caplets and swaptions. Barton et al. (1998) and Rebonato (1999) show in simulation studies that if the LMM (or the SMM) is the true model, the prices for caps and swaptions implied by the SMM (or the LMM) are quite close to the true prices. In contrast, we show that empirically, calibrating the LMM to caplets leads to very different results than calibrating the SMM to swaptions.

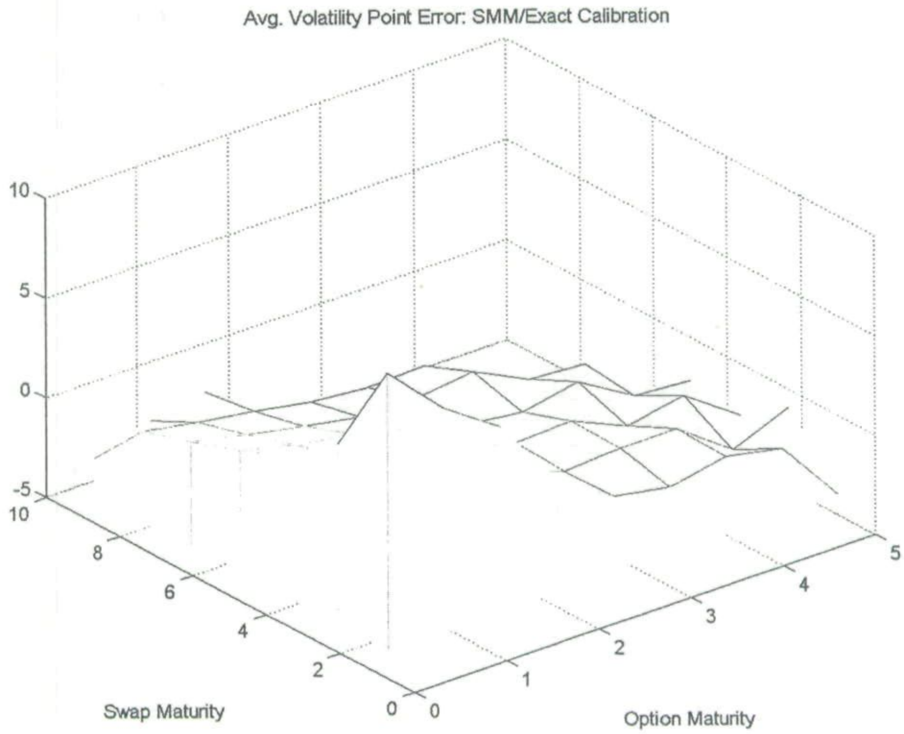

Figure 7d.

5.3. EXPLANATION OF POOR PERFORMANCE OF SWAP MARKET MODEL

The most remarkable feature of the analysis above is the large pricing errors for caplets generated by the SMM in case of exact calibration. Once the SMM is calibrated to the prices of swaptions, it does a very poor job pricing caplets. This effect is persistent whether or not a declining volatility structure is included.

An intuitive explanation for this phenomenon can be given as follows. In a swap contract various floating and fixed payments are exchanged. As the floating payments are based on LIBOR rates, we can view the swap contract as a portfolio of LIBOR payments. Hence, the volatility of the forward swap rate is determined by the volatility of the forward LIBOR rates. To first order, we can say that the volatility of a forward swap rate is the volatility of a weighted average of forward LIBOR rates (see also Rebonato (1996)).

If we observe a set of LIBOR volatilities, we can impute the swaption volatilities (given additional assumptions on the correlations of the LIBOR rates) by averaging over the LIBOR volatilities. Due to the averaging, individual differences in the observed LIBOR volatilities are cancelled out and we typically obtain reasonable swaption volatilities. If, on the other hand, we observe a set of swaption volatilities we can invert the calculation and obtain the individual LIBOR volatilities by differencing. However, small errors in the swaptions volatilities (measurement errors and bid-ask spreads in the data) will get magnified when calculating the LIBOR volatilities and we often obtain very unrealistic LIBOR volatilities. It is exactly this effect which explains the poor performance of the Swap Market Model. It is also clear that this effect will play a role irrespective of the number of factors used in a Swap Market Model and irrespective of the strike rate of the caplets and swaptions.

5.4. ANALYSIS OF PRICING ERRORS

In Figure 7, we plot the average pricing error per caplet and swaption for the declining volatility SMMs The maturity effects in the pricing errors of the exact calibration model are similar to the maturity effects in the pricing errors of the non-exact calibration model. We already mentioned above that short-maturity caplets are overpriced by all SMMs. It also follows that swaptions with short total maturities have the largest pricing errors, and these pricing errors are positive on average. The swaptions with longer total maturities are underpriced by the declining volatility SMM.

We again perform the regression in (15) of the observed difference between swaption and caplet implied volatilities on the difference between swaption and caplet implied volatilities that follows from the SMM and the results are given in Table IX. In almost all cases the model is statistically rejected. The regression results for the case of non-exact calibration are a little better than for exact calibration. Also, the R^2 values are in most cases smaller than for the LMMs.

6. Concluding Remarks

In this paper, we empirically examine a recently developed class of models to price interest rate derivatives, the so-called market models of interest rates. This class of models has several advantages over the traditional approach of Heath et al. (1992). First, these models are based directly on observable market rates, such as Libor rates and swap rates, instead of instantaneous (forward) interest rates. Second, the models yield pricing formulas for caplets or swaptions that correspond to the Black pricing formulas that are used in practice. And third, as a consequence, these models can easily be calibrated to market prices of caps or swaptions.

We analyze and compare two one-factor market models: the Libor Market Model (LMM), developed by Miltersen et al. (1997) and Brace et al. (1997), and the Swap Market Model (SMM), introduced by Jamshidian (1997) and Musiela and Rutkowski (1997). For both models, we analyze a constant volatility specification and a declining volatility specification, that corresponds to mean reverting behaviour of interest rates. We also consider two calibration methodologies, exact calibration, that leads to exact fitting of a set of derivative prices, and non-exact calibration, that does not lead to such an exact fit.

We show that a constant volatility specification for the LMM leads to overpricing of all swaptions. An exponentially declining volatility specification (corresponding to mean reverting interest rates) typically leads to a better fit of the LMM, and the estimates for the decay parameter are quite stable over time. For this model, the average absolute pricing error for all swaptions is around 1 volatility point, which is, given the size of the bid-ask spreads for swaptions, not very large. For the SMM with a constant volatility specification, we find that the model underprices swaptions on average. Again, if we specify and estimate a declining volatility function, this average underpricing disappears, although there are still some maturity effects in the pricing errors. Overall, the average fit of the LMM is better than the fit of the SMM.

We also show that the exact calibration methodology, often used by derivatives traders, can lead to overfitting to derivative prices, because using this methodology most often leads to higher pricing errors than the non-exact calibration methodology. In other words, due to the presence of noise in derivative prices, a better fit on some set of derivatives can lead to a worse fit of other derivative instruments. The non-exact calibration methodology therefore (partially) eliminates the noise in the derivative price quotes.

We have also analyzed two-factor versions of the Libor Market Model. The empirical results on exact calibration versus non-exact calibration and the effect of a declining volatility structure are very similar to the results in case of the one-factor model. Therefore, we have not included these two-factor results in the paper.

In general, the relation between LIBOR and swap rate movements is influenced by, at least, two effects: the correlation of interest rates at different points in time (as captured by the declining volatility function/mean-reversion) and the correlation of different interest rates at the same point in time (as captured by multi-factor models). To accurately price exotic interest rate derivatives other than European caps and swaptions, such as spread options, one needs a model that is capable of describing both types of correlations. Then, to precisely estimate such multi-factor models, it seems wise to also use information on historical movements of interest rates of different maturities. This is a topic of current research (De Jong et al. (2001)). Another interesting extension of this paper would be to analyze the hedging performance of market models.

Finally, another extension of this paper is to include in-the-money and out-of-the-money options in the analysis. The LMM implies a flat volatility structure as function of the strike rate for caps. For swaptions, the LMM does not imply a flat strike volatility function. The opposite is true for the SMM. It remains to be investigated which of the two models generates the most realistic strike volatility functions for caps and swaptions. Very likely, modifications of the market models to incorporate non-lognormal interest rate distributions are necessary to fit the strike volatility functions of caps and swaptions (see Andersen and Andreasen (1999)).

Appendix

In this appendix, we describe pricing formulas for caplets and swaptions for the LMM and SMM. For the LMM, as it is described in Section 2, the price of a caplet is equal to

$$\begin{aligned} Caplet(t, T_n, k) &= \delta_n P_{n+1}(t)[(L_n(t)N(h) - kN(h-\xi_n)]. \\ h &= \tfrac{\log(L_n(t)/k)+0.5\xi_n^2}{\xi_n}, \quad \xi_n^2 = \textstyle\int_t^{T_n} \gamma_n^{\mathrm{LMM}}(s)^2\,\mathrm{d}s. \end{aligned} \tag{A.1}$$

This follows from calculating the expectation in Equation (6). This pricing formula is the same as the Black pricing formula for a caplet, where the term ξ_n is replaced by $\sigma\sqrt{T_n - t}$, where σ is the Black volatility.

Swaptions cannot be priced analytically in the LMM. Brace et al. (1997) derive an approximation for the swaption price. They make a one-factor approximation to the covariance matrix Δ in Equation (14)

$$\Delta \approx \Gamma\Gamma^T, \quad \Gamma \in \mathbb{R}^{mxl}. \tag{A.2}$$

Then, they show that an approximation price is given by (for notational convenience we consider a swaption that matures at T_1 and has m payment dates)

$$\begin{aligned} Swptn(t, T_1, m, k) &= \textstyle\sum_{j=1}^{m} \delta_j P_{j+1}(t)[L_j(t)N(-s-d_j+\Gamma_j) \\ &\quad -kN(-s-d_j)] \\ d_i &= \textstyle\sum_{j=1}^{i} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Gamma_j, \quad i = 1, \ldots, m, \end{aligned} \tag{A.3}$$

where s solves

$$\sum_{k=2}^{m+1} c_k \left[\prod_{j=1}^{k-1} (1 + \delta_j L_j(t) \exp(\Gamma_j(s+d_j) - 0.5\Gamma_j^2)) \right]^{-1} = 1 \tag{A.4}$$

$$c_j = k\delta_{j-1},\ j = 2, \ldots, m,\ c_{m+1} = 1 + k\delta_m.$$

We have compared the swaption values based on this approximation with the values computed using simulation. We find that the differences are always smaller than 0.08 volatility points.

For the SMM, the price of a swaption with a total maturity that is equal to the total maturity of the SMM is given by the following formula

$$\begin{aligned} Swptn(t, T_n, N-n, k) &= \left(\textstyle\sum_{j=n+1}^{N} \delta_{j-1} P_j(t)\right)[S_{n,N-n}(t)N(h) \\ &\quad -kN(h - \xi_{n,N-n})], \\ h &= \frac{\log(S_{n,N-n}(t)/k)+0.5\xi_{n,N-n}^2}{\xi_{n,N-n}}, \\ \xi_{n,N-n}^2 &= \textstyle\int_t^{T_n} \gamma_{n,N-n}^{\mathrm{SMM}}(s)^2\,\mathrm{d}s. \end{aligned} \tag{A.5}$$

Again, the Black pricing formula also has this form, where the term $\xi_{n,N-n}$ is replaced by $\sigma\sqrt{T_n - t}$, where σ is the Black volatility of the swaption.

Caplets and swaptions (with smaller total maturities than the total maturity of the SMM) cannot be priced analytically with the SMM. Brace et al. (1998) derive the following approximate relation between swap rate volatilities and Libor rate volatilities, assuming driftless lognormal processes for both forward Libor and swap rates

$$\gamma_{n,m}^{S}(t) \approx \frac{\sum_{j=n+1}^{n+m} \delta_{j-1} P_j(t) L_{j-1}(t) \gamma_{j-1}^{C}(t)}{\sum_{j=n+1}^{n+m} \delta_{j-1} P_j(t) L_{j-1}(t)}. \tag{A.6}$$

Then, by assuming bond prices and Libor rates in (A.6) are fixed at their current values, and given the swap rate volatilities for 10-year total maturity forward swap rates, the volatilities of other forward swap rates and forward Libor rates can be determined. Given these forward Libor and swap rate volatilities, the Black formula is used to obtain approximate prices for caplets and swaptions. Compared to the LMM, more approximation assumptions are made for the SMM, and therefore the difference between simulated and approximated prices is larger and at maximum equal to 0.5 volatility points.

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