European Finance Review **4:** 197–210, 2000.

Stochastic Interest Rates and the Bond-Stock Mix

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Abstract. The optimal bond-stock mix is examined in light of an apparent inconsistency between the Tobin Separation Theorem and the advice of popular investment advisors which has been pointed out by Canner et al. (1997). It is shown that the apparent inconsistency is largely explicable in terms of the hedging demands of optimising long-term investors in an environment in which the investment opportunity set is subject to stochastic shocks. The analysis points to the importance of considering investors' time horizons in analyzing optimal portfolio policies.

Key words: dynamic portfolio policy, hedging, asset allocation puzzle.

The importance of asset allocation for the performance of an investment portfolio is widely recognized; for example, Brinson et al. (1991) show that around 90% of the performance of pension funds is determined by the asset allocation decision.[1] In this paper we are concerned with one aspect of the asset allocation decision, namely the cash-bond-stock mix. This has received increased attention since Canner, Mankiw and Weil (1997) pointed out that the asset mix recommendations of three popular financial advisors are apparently inconsistent with the Tobin (1958) Separation Theorem that, if the distribution of returns belongs to the elliptical class,[2] then the proportions in which different risky assets are held in the optimal portfolio of risky assets are independent of the investor's risk aversion. The authors report that the financial advisors they study recommend that the ratio of bonds to stocks *increase* as the investor's risk aversion increases. They consider several possible explanations for this phenomenon, including the absence of a real riskless asset, non mean-variance preferences, differences between the historical and the subjective distribution of returns, constraints on short sales, and considerations raised by the fact that investors have multi-period horizons, but conclude that these explanations are unsatisfactory, leaving an apparent puzzle. In this paper we show

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** We thank an anonymous referee for helpful comments. The paper was previously circulated under the title 'Resolution of a Financial Puzzle'.

[1] See also Ibbotson et al. (2000).

[2] Tobin originally stated the theorem for mean variance preferences. The necessary condition for investors to have mean-variance preferences for arbitrary utility functions is that the distribution of returns belong to the elliptical class of which the normal distribution is a member.

that the variation in the ratio of bonds to stocks recommended by the financial advisors is consistent with a model of portfolio optimization in a dynamic context. The reason for the violation of the separation principle is that bonds are not just any risky asset but have the particular property that their returns covary negatively with expectations about future interest rates. This covariation, which plays no role in the one period context of the Tobin separation theorem, is important for an investor with a multi-period horizon, as the classic paper of Merton (1973) recognized.

Canner et al. (1997) acknowledge the possible importance of the dynamic considerations that are induced by a multi-period horizon, but attempt to model the dynamic portfolio decision in a *static* fashion by considering only a buy and hold policy, adjusting the mean vector and covariance matrix for the length of the horizon: they conclude that "it appears impossible to reconcile the advice of financial advisors with the textbook CAPM by changing the time horizon". Elton and Gruber (2000) attempt to reconcile the popular asset allocation recommendations with a static mean-variance model by treating cash as a risky asset and imposing no-short sales constraints, so that the Tobin Separation Theorem no longer holds. Our analysis, which does not impose such constraints and treats the Treasury Bill as instantaneously riskless, shows that the bond-stock mix is increasing in the risk-aversion parameter for arbitrary beliefs about the risk premium vector.

Recent papers that discuss the effect of a stochastic interest rate on the optimal portfolio allocation, include Campbell and Viceira (2000), Liu (1999), Sorensen (1999), and Wachter (1999). Campbell and Viceira (2000) use numerical approximation procedures to analyze the optimal allocation of an investor with a recursive utility function and infinite horizon. In contrast, we provide analytical results for different investment horizons. Liu (1999) uses a single factor model of interest rates to develop analytic allocation results in a quite general structure with a stochastic interest rate, stochastic risk premium and stochastic volatility. While our model has constant risk premia and volatilities, the two factor model of interest rates that we analyze provides a more realistic characterisation of the term structure than a one factor model in that it allows for independent variation in long and short term yields. Sorensen (2000) uses a single factor model of the term structure to analyze optimal discrete time portfolio selection. Wachter (1999) provides a nice interpretation of the optimal portfolio allocation when the investor has interim consumption and infinite risk aversion level. We study the investor's optimal stock-bond mix for normal levels of risk aversion, and derive comparative statics with respect to the risk aversion parameter. Finally, although all the above mentioned papers touch on the Canner et. al. financial puzzle, we provide a detailed analysis of this problem for a long but finite horizon investor.

The model of time varying interest rates and expected stock returns is presented in Section 1 and some representative calculations are offered in Section 2. A brief summary of the paper is given in Section 3.

1. The Model

Uncertainty about future interest rates is represented by the Hull and White (1996) two-factor extension of the Vasicek (1977) model of the term structure of interest rates, while the equity risk premium in excess of the instantaneously riskless interest rate is assumed to be a constant. Thus, the instantaneously riskless interest rate, $r(t)$, is assumed to follow a bivariate Markov process:

$$dr = [\theta(t) + \lambda_r + u - ar]dt + \sigma_r dz_r \tag{1}$$

$$du = [\lambda_u - bu]dt + \sigma_u dz_u \tag{2}$$

where dz_r and dz_u, are increments to standard Brownian motions, and λ_r and λ_u are the risk premia per unit exposure to innovations in r and u. In this model of interest rates u affects the stochastic "target" towards which the riskless rate r is adjusting; this second element in the state vector allows for independent variation in the short and long end of the yield curve which is apparent in the changing slope of the yield curve.[3] given the stochastic processes (1) and (2), the price at time t of a bond paying \$1 at time T is given by:

$$P(t, T) = A(t, T)e^{-B(t,T)r-C(t,T)u} \tag{3}$$

where

$$B(t, T) = \frac{1}{a}[1 - e^{-a(T-t)}], \tag{4}$$

$$C(t, T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}, \tag{5}$$

$$\frac{\partial A/\partial t}{A} = \theta(t)B(t, T) - \frac{1}{2}\sigma_r^2 B^2(t, T) - \frac{1}{2}\sigma_u^2 C^2(t, T) - \sigma_{ru} B(t, T)C(t, T).$$

$$\text{with boundary condition} \quad A(T, T) = 1 \tag{6}$$

$A(t, T)$ is a complicated function of $\theta(t)$, and $\theta(t)$ can be chosen to fit any given term structure. Under simplifying assumptions about $\theta(t)$, $A(t, T)$ can be solved explicitly. However, it is not necessary for our purposes to specify $\theta(t)$. Equation (3) implies that the continuously compounded yield to maturity at time t on a bond maturing at time T is $\frac{-\ln A(t,T)+B(t,T)r+c(t,T)u}{T-t}$.

Let P_j denote the price at time t of a bond maturing at time T_j, and let π_S denote the risk premium on the equity security which is assumed to be constant.

[3] Litterman and Scheinkman (1991) show hat two factors capture around 98% of the variance of bond returns.

Then, using (3), the joint stochastic process for the stock price, S, and the bond prices, $P_j (j = 1, 2)$, is given by:

$$\frac{dS}{S} = (r + \pi_S)dt + \sigma_S dz_s \tag{7}$$

$$\frac{dP_j}{P_j} = (r + \pi_j)dt - B(t, T_j)\sigma_r dz_r - C(t, T_j)\sigma_u dz_u \tag{8}$$

where

$$\pi_j \equiv -\lambda_r B(t, T_j) - \lambda_u C(t, T_j) \tag{9}$$

is the risk premium of a bond with maturity T_j.

We consider the problem of an investor who is concerned with maximizing the expected value of an iso-elastic utility function defined over period T wealth, W_T:

$$\text{Max } E\left[\frac{W^{1-\gamma}}{1-\gamma}|\mathfrak{F}_t\right] \tag{10}$$

where $\mathfrak{F}_t \equiv \{r_\tau, u_\tau, S_\tau, \tau \leq t;\ W_t\}$ denotes the information available at time t. The investor is assumed to be able to invest in the equity security and in two (or more) bonds of different maturities.

Let

$$\sum \equiv \begin{pmatrix} \sigma_S^2 & \sigma_{P_1 S} & \sigma_{P_2 S} \\ \sigma_{P_1 S} & \sigma_{P_1}^2 & \sigma_{P_1 P_2} \\ \sigma_{P_2 S} & \sigma_{P_1 P_2} & \sigma_{P_2}^2 \end{pmatrix}$$

denote the (3×3) variance-covariance matrix of the returns on the three risky assets, where $\sigma_{P_j S} = -\sigma_{Sr} B(t, T_j) - \sigma_{Su} C(t, T_j)$, $\sigma_{P_j, P_k} = \sigma_r^2 B(t, T_j) B(t, T_k) + \sigma_{ru}[B(t, T_j)C(t, T_k) + B(t, T_k)C(t, T_j)] + \sigma_u^2 C(t, T_j)C(t, T_k)$, and σ_{ru} denotes the instantaneous covariance between the innovations in r and u. Let $\underline{\pi}$ be the (3×1) vector of risk premia. $\underline{\sigma_{Rr}}$ and $\underline{\sigma_{Ru}}$, the (3×1) vectors of covariances between the asset returns and the innovations in the state variables, are given by $[\sigma_{Sr}, -B_1\sigma_r^2 - C_1\sigma_{ru}, -B_2\sigma_r^2 - C_2\sigma_{ru}]'$, $[\sigma_{Su}, -B_1\sigma_{ru} - C_1\delta_u^2, -B_2\sigma_{ru} - C_2\sigma_u^2]'$, where $B_1 \equiv B(t, T_1)$ etc.

Then the Bellman equation for optimality may be written as:

$$\begin{aligned} 0 = \max_{\underline{x}} \Bigg[& J_t + W J_W(r + \underline{x}'\underline{\pi}) + J_r(\theta(t) + u - ar) - J_u bu + \frac{1}{2} J_{rr}\sigma_r^2 \\ & + \frac{1}{2} J_{uu}\sigma_u^2 + \frac{1}{2} J_{WW} W^2 \underline{x}'\Sigma\underline{x} + J_{Wr} W\underline{x}'\underline{\sigma_{Rr}} + J_{Wu} W\underline{x}'\underline{\sigma}_{Ru} + J_{ru}\sigma_{ru} \Bigg], \end{aligned} \tag{11}$$

where $J(W, r, u, t, T)$ is the indirect utility of wealth function which satisfies the terminal condition:

$$J(W, r, u, T, T) = \frac{W^{1-\gamma}}{1-\gamma}. \tag{12}$$

The first order condition in (11) implies that the vector of optimal risky asset proportions, $\underline{x}^*$, is given by:

$$\underline{x}^* = \frac{-J_W}{WJ_{WW}}\Sigma^{-1}\underline{\pi} - \frac{J_{Wr}}{WJ_{WW}}\Sigma^{-1}\underline{\sigma}_{Rr} - \frac{J_{Wu}}{WJ_{WW}}\Sigma^{-1}\underline{\sigma}_{Ru}. \tag{13}$$

Equation (13) expresses the investor's optimal portfolio as the weighted sum of three portfolios: the first is the usual mean-variance optimal myopic portfolio which is optimal for an investor with a short horizon or log utility. The other two portfolios are constructed to hedge against innovations in the state variables r and u. Since two bonds (and cash) are sufficient to form a portfolio that is a perfect hedge against these two variables that characterize the stochastic investment opportunity set, the investor's allocation to stock is determined entirely by his myopic demand. The optimal portfolio is as if the investor first chooses holdings of two bonds and cash to hedge the stochastic investment opportunity set, and then makes an additional stock-bond allocation treating the opportunity set as constant. The following proposition, which characterizes the investor's optimal portfolio strategy, may be verified by substitution in Equations (11) and (13):[4]

PROPOSITION 1. (i) The solution to the optimal control problem (11) subject to the terminal condition (12) is:

$$J(W, r, u, t, T) = \frac{F(t, T)(W^*)^{1-\gamma}}{1-\gamma} \tag{14}$$

where $F(t, T)$ is the solution to an ordinary differential equation with boundary $F(T, T) = 1$, and $W^* \equiv W/P(r, u, t, T)$ is the forward value of the investor's wealth measured in time T dollars.

(ii) The vector of optimal risky asset allocations is given by:

$$\underline{x}^* = \frac{1}{\gamma}\Sigma^{-1}\underline{\pi} + \left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\begin{pmatrix}\sigma_{SP}\\ \sigma_{P_1P}\\ \sigma_{P_2P}\end{pmatrix} \tag{15}$$

where P denotes the price of a bond which matures at the horizon, T, and σ_{SP} and σ_{P_jP} denote the covariances between the returns on the T period bond and the return on the stock and the T_j period bond respectively.

4 The solution to the portfolio problem may also be obtained constructively using the martingale pricing technique suggested by Cox and Huang (1989). Wachter (1999) solves the single factor model using this method.

(iii) Since the return on the bond that matures at time T is an exact linear combination of the returns on the other two bonds, the portfolio allocation vector, $\underline{x}^*$, may also be written as:

$$\underline{x}^* = \frac{1}{\gamma}\Sigma^{-1}\underline{\pi} + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} 0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \tag{16}$$

where β_1, and β_2 are the coefficients from the multiple regression of the return of the T-period on the other two bonds.[5]

Equations (15) and (16) clearly show, as in Merton (1973), that the vector of optimal portfolio proportions is a linear combination of the myopic mean variance efficient portfolio, $\Sigma^{-1}\underline{\pi}$, and of a hedge portfolio where the relative proportions of the two portfolios depend on the risk aversion parameter, γ. The portfolio weights in the hedge portfolio itself are proportional to the coefficients from a regression of the return of the T period bond on the returns of the two bonds used to form the hedge portfolio.

Equation (14) indicates that the investor's indirect utility function only depends on his investment horizon and the price of the bond with maturity equal to his horizon. Therefore, $P(r, u, t, T)$ is the single factor that matters to the investor, because it summarizes the effect of both r and u on investor's investment opportunity set. Because P is the single factor, the two hedge demands in Equation (13) collapse into one hedge demand characterized by the covariance between P and the three assets available to the investor.

If one of the discount bonds held by the investor matures on the horizon date, T, then assuming without loss of generality that it is bond one, $T_1 = T$, and the hedge portfolio takes on a particularly simple form, since the regression of the return on the T period bond on the returns of the two bonds held by the investor becomes degenerate and all of the weight is placed on the T period bond; the hedge portfolio is then simply the T period bond, and the portfolio holdings are given by:[6]

$$\underline{x}^* = \frac{1}{\gamma}\Sigma^{-1}\underline{\pi} + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{17}$$

As the investor's risk aversion is increased, the weight on the T period bond increases until, in the limit, as $\gamma \to \infty$, the investor allocates the whole of her wealth to this bond and, as a result, bears no risk.

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[5] $\beta_1 = (C_2B_0 - B_2C_0)/(B_1C_2 - C_1B_2)$, $\beta_2 = (-C_1B_0 + B_1C_0)/(B_1C_2 - C_1B_2)$ where $B_0 \equiv B(t, T)$, $B_1 \equiv B(t, T_1)$ etc.

[6] Merton (1971) also analyses a model in which changes in the investment opportunity set are perfectly correlated with an asset price.

While Equations (16) and (17) are inconsistent with the Tobin Separation Theorem, they do place other restrictions on the characteristics of optimal portfolios. It is straightforward to verify the following:

COROLLARY 1. (i) If the stock holding is positive for any level of risk aversion then it is decreasing in the risk aversion coefficient, γ.

(ii) If the stock holding is positive and $T = T_1$ or T_2, then $\alpha(\gamma) \equiv (x_2^* + x_3^*)/x_1^*$, the ratio of bonds to stock held in the optimal portfolio, is increasing in the risk aversion coefficient, γ.

(iii) The proportions of the portfolio allocated to the riskless asset ("cash"), and to each of the bonds, are linear functions of the proportion allocated to stock.

Note that the bond-stock ratio increases with the risk aversion parameter when one of the bonds available has a maturity equal to the investment horizon. Similar results are obtained when there is only a single interest rate factor. However, when there are two factors and there is no bond with maturity equal to the investment horizon, the ratio of bond to stock holdings is no longer necessarily increasing in risk aversion: in this case, β_1 and/or β_2 in Equation (16) could be negative, depending on the maturities of T_1 and T_2 relative to T and the relative sizes of mean reversion parameters a and b. In general, it is necessary to impose additional assumptions on the model parameters, a, b, T_1, T_2, to ensure that the bond-stock allocation is increasing in risk aversion for a given horizon, T. Thus a two-factor stochastic interest rate model is qualitatively different from a single factor model in its predictions about the relation between the risk aversion and the bond-stock ratio.

2. Some Illustrative Calculations

Table I repeats the evidence on the advisors' recommendations reported by Canner et al. (1997). Note first that, consistent with Corollary (i), the recommended allocations to stock are positive and decreasing in the level of risk aversion. Corollary (ii) predicts that if the maturity of one of the bonds coincides with the investor's horizon, then the ratio of bonds to stock held in the optimal portfolio will be increasing in the investor's risk aversion. This is precisely the pattern observed for all four advisors, and the pattern that Canner et al. regarded as a puzzle. Finally, since the Corollary predicts that the bond allocation will be a linear function of the stock allocation, the fourth column reports the bond allocation for the moderate risk aversion investor that would be obtained by linear interpolation from the bond and stock allocations for the other two investors. In three cases the interpolated allocation which is consistent with the theory falls below the actual recommendation. However, *The New York Times*' recommendation does correspond precisely to the linearity restriction imposed by the theory.

Table I. Asset allocations recommended by financial advisors and evidence on the linearity hypothesis

Advisor investor type	Cash	Stock	Bond	Linearly interpolated bond allocation	Ratio of bonds to stock
Fidelity					
Conservative	50%	20%	30%		1.50
Moderate	20%	40%	40%	30.00%	1.00
Aggressive	5%	65%	30%		0.46
Merrill Lynch					
Conservative	20%	45%	35%		0.78
Moderate	5%	55%	40%	30.00%	0.73
Aggressive	5%	75%	20%		0.27
Jane Bryant Quinn					
Conservative	50%	20%	30%		1.50
Moderate	10%	50%	40%	18.75%	0.80
Aggressive	0%	100%	0%		0.00
New York Times					
Conservative	20%	40%	40%		1.00
Moderate	10%	60%	30%	30%	0.50
Aggressive	0%	80%	20%		0.25

Source: Canner et al. (1997).

To illustrate the effect of dynamic hedging considerations on optimal portfolio allocations, optimal portfolios were constructed for investors with decision horizons of one, five, ten, and twenty years when the set of available assets included a riskless asset, pure discount bonds with maturities of ten and twenty years, and an equity portfolio. The variance-covariance matrix of asset returns was estimated from the time series of monthly returns on the value weighted stock market index and on constant maturity Treasury bonds of these maturities for the period January 1942 to December 1997 provided by CRSP.[7] It is not possible to estimate risk premia precisely with available time series data. Therefore optimal portfolios are reported using two sets of risk premia estimates. The first set assumes that the pure expectations theory of the term structure holds so that $\lambda_u = \lambda_r = 0$, and that the

[7] The covariance matrix of the theory contains covariances with pure discount bonds of different maturities. We have approximated this with covariances of constant maturity coupon bonds.

equity market risk premium is 4%.[8] The second set assumes that the risk premia for stock, ten year and twenty year bonds are given by their historical premia: $\pi_S = 9\%$, $\pi_1 = 1.18\%$ and $\pi_2 = 1.24\%$. Although the allocations are quite different under the two sets of parameters, the qualitative predictions are similar.

Tables II and III report the results for investors with different horizons and risk aversions, when the bond portfolio is constructed from ten and twenty year bonds; Table II is based on the risk premia derived from the pure expectations hypothesis and the 4% equity premium which we refer to as the "conservative risk premia"; Table III is based on the historical risk premia. For the ten and twenty year horizons the investor can invest in a bond with a maturity equal to his horizon, and therefore the risky asset allocations are computed using Equation (17). For the one and five year horizons the investor is assumed to still invest in the ten and twenty year bonds, and Equation (15) is used to calculate the risky asset allocation. The myopic portfolio corresponds to the first term in expressions (15) and (16), and is the mean-variance efficient portfolio that would be held by an investor with an instantaneous investment horizon who has no need to hedge against future changes in the investment opportunity set.

Note first that the stock allocations are determined solely by the myopic portfolio because the hedge portfolio does not contain equity; consequently, the stock allocation is independent of the investment horizon. Moreover, although not shown here, the stock allocations are also independent of the maturities of the two bonds that the investor uses to implement his portfolio strategy, because all bond returns are linear functions of the same two stochastic shocks as shown in Equation (8); thus any pair of bonds spans the same set of returns. Of course, the amount that is optimally allocated to bonds does depend on the maturity of the bonds chosen, because different bonds have different loadings on r and u.

Secondly, as long as one bond has a maturity equal to the investment horizon, the optimal allocations to stock, bond and cash are independent of the horizon. This is illustrated in Tables II and III: since the investor is assumed to be investing in 10 and 20 year bonds, the bond allocation is the same for the 10 year horizon as for the 20 year horizon; in both cases the hedge portfolio consists entirely of the bond of the horizon maturity as seen in Equation (17). When neither bond has maturity equal to the investment horizon, the total bond and cash allocations are horizon dependent. The composition of the hedge portfolio depends on the covariances between the three assets and a hypothetical bond with maturity equal to the horizon. As the horizon changes, the covariances also change, which makes the allocation horizon-dependent.

[8] While this is significantly below the historical average risk premium, Brown et al. (1995) observe that survival bias would make the historical data consistent with an equity risk premium of 4% if the probability of a stock market surviving over the long run is 80%.

Table II. Portfolio allocations for investors with different horizons and risk aversion investing in 10 and 20 year bonds

		Conservative risk premium assumption				
γ	Horizon	Myopic	1 year	5 years	10 years	20 years
2	Cash	38%	28%	7%	−12%	−12%
2	Bonds	−45%	−35%	−14%	5%	5%
2	Stock	107%	107%	107%	107%	107%
2	Bonds-stock ratio	n.a.	n.a.	n.a.	0.05	0.05
3	Cash	59%	47%	17%	−8%	−8%
3	Bonds	−30%	−18%	12%	37%	37%
3	Stock	71%	71%	71%	71%	71%
3	Bonds-stock ratio	n.a.	n.a.	0.16	0.52	0.52
4	Cash	69%	56%	23%	−6%	−6%
4	Bonds	−22%	−9%	24%	53%	53%
4	Stock	53%	53%	53%	53%	53%
4	Bonds-stock ratio	n.a.	n.a.	0.46	1.00	1.00
6	Cash	80%	65%	28%	−3%	−3%
6	Bonds	−15%	0%	37%	68%	68%
6	Stock	35%	35%	35%	35%	35%
6	Bonds-stock ratio	n.a.	0.00	1.06	1.92	1.92

The investor is assumed to be allowed to invest in cash, an equity security, and ten and twenty year discount bonds. The investor's investment horizon is one, five, ten or twenty years. The equity risk premium (π_S) is assumed to be 4%, and the pure expectations hypothesis is assumed to hold so that $\lambda_u = \lambda_r = 0.0$. The variance-covariance matrices of asset returns is derived from monthly returns on the value-weighted equity market index and constant maturity bonds for the period January 1942 to December 1997 from CRSP, and the elements of the matrix are $\sigma_S^2 = 0.01996$, $\sigma_{SP_1} = 0.00222$, $\sigma_{SP_2} = 0.00301$, $\sigma_{P_1}^2 = 0.00469$, $\sigma_{P_1P_2} = 0.00530$, $\sigma_{P_2}^2 = 0.00716$. When the investment horizon is ten or twenty years, one of the bonds has maturity equal to the investment horizon T. In this case, the optimal allocation is calculated using Equation (17). When the investor cannot invest in a bond with maturity equal to his investment horizon, the optimal allocation is calculated using Equation (15). The hedge demand depends on the covariance between the three assets and the bond with maturity equal to T. When $T = 1$, $\sigma_{SP} = 0.00035$, $\sigma_{P_1P} = 0.00087$ and $\sigma_{P_2P} = 0.00103$. When $T = 5$, $\sigma_{SP} = 0.00127$, $\sigma_{P_1P} = 0.00297$ and $\sigma_{P_2P} = 0.00352$. The myopic portfolio corresponds to the first term in expressions (15) and (17).

Table III. Portfolio allocations for investors with different horizons and risk aversion investing in 10 and 20 year bonds

		Historical risk premia				
γ	Horizon	Myopic	1 year	5 years	10 years	20 years
2	Cash	−164%	−173%	−194%	−214%	−214%
2	Bonds	37%	46%	68%	87%	87%
2	Stock	227%	227%	227%	227%	227%
2	Bonds-stock ratio	0.16	0.20	0.30	0.38	0.38
3	Cash	−76%	−88%	−117%	−142%	−142%
3	Bonds	25%	37%	66%	91%	91%
3	Stock	151%	151%	151%	151%	151%
3	Bonds-stock ratio	0.16	0.24	0.44	0.60	0.60
4	Cash	−31%	−45%	−78%	−106%	−106%
4	Bonds	18%	32%	65%	93%	93%
4	Stock	113%	113%	113%	113%	113%
4	Bonds-stock ratio	0.16	0.28	0.58	0.82	0.82
6	Cash	13%	−3%	−39%	−71%	−71%
6	Bonds	12%	28%	64%	96%	96%
6	Stock	75%	75%	75%	75%	75%
6	Bonds-stock ratio	0.16	0.37	0.86	1.26	1.26

The investor is assumed to be allowed to invest in cash, an equity security, and ten and twenty year discount bonds. The investor's investment horizon is one, five, ten or twenty years. The equity risk premium (π_S), the risk premia for the ten and twenty year bonds (π_1 and π_2) are set equal to be their historical estimates, 9%, 1.18% and 1.24%. The variance-covariance matrices of asset returns is derived from monthly returns on the value-weighted equity market index and constant maturity bonds for the period January 1942 to December 1997 from CRSP, and the elements of the matrix are $\sigma_S^2 = 0.01996$, $\sigma_{SP_1} = 0.00222$, $\sigma_{SP_2} = 0.00301$, $\sigma_{P_1}^2 = 0.00469$, $\sigma_{P_1P_2} = 0.00530$, $\sigma_{P_2}^2 = 0.00716$. When the investment horizon is ten or twenty years, one of the bonds has maturity equal to the investment horizon T. In this case, the optimal allocation is calculated using Equation (17). When the investor cannot invest in a bond with maturity equal to his investment horizon, the optimal allocation is calculated using Equation (15). The hedge demand depends on the covariance between the three assets and the bond with maturity equal to T. When $T = 1$, $\sigma_{SP} = 0.00035$, $\sigma_{P_1P} = 0.00087$ and $\sigma_{P_2P} = 0.00103$. When $T = 5$, $\sigma_{SP} = 0.00127$, $\sigma_{P_1P} = 0.00297$ and $\sigma_{P_2P} = 0.00352$. The myopic portfolio corresponds to the first term in expressions (15) and (17).

Thirdly, the stock allocation is decreasing, and the cash and bond allocations are increasing in the investor's risk aversion.[9] The portfolio allocations shown in Tables II and III are qualitatively consistent with the advisors' recommendations

[9] Although not shown in the tables, the optimal allocation between the two bonds also changes with the investor's horizon as shown in Equation (17).

shown in Table I; in particular, the ratio of bonds to stocks and the cash holding are both increasing in the investor's risk aversion. Thus the puzzle that Canner et al. identified can be resolved by supposing that the portfolio recommendations that they cite are intended for investors with long horizons. For such investors, the need to hedge against shifts in the investment opportunity set as represented by the stochastic long and short interest rates creates an additional demand for bonds that increases with risk aversion, leading to the positive relation between the recommended bond-stock ratio and risk aversion that Canner et al. found: this is simply an example of the hedging phenomenon that was first discussed by Merton (1973).

However, the allocations reported in Tables II and III are quantitatively different from those in Table I. Most strikingly, the cash allocations in Table III which is based on the historical risk premia are all negative, and the stock and total bond allocations are much larger than recommended by the financial advisors. These results are quite sensitive to the assumed risk premia for the three assets. For example, in Table II which is based on conservative estimates of the risk premia, negative cash positions appear only for the longer horizons and are modest in size. In view of the uncertainty that attaches to historical estimates of risk premia, not too much importance should be attached to the precise magnitude of the portfolio allocations reported in the tables. What is important to recognize is the effect of the horizon on the demand for bonds; in all cases the demand for bonds is increasing in the horizon and this hedging effect is most pronounced for the more risk averse investors;[10] it is this hedging effect that creates the positive relation between risk aversion and the bond stock ratio.

As we have noted, when there is no bond with maturity equal to the investment horizon, the size of the bond allocations in Tables II and III depend on the investment horizon. It is noteworthy that the recommended allocations reported in Table I do not depend on the investor's horizon and do not contain specific recommendations about the maturity of the bonds that are to be held. It seems therefore that the usefulness of the recommendations could be enhanced by more specific advice about the maturity of the bonds to be included in the portfolio and the horizon of the investor.

A striking feature of the investment advisors' recommendations is that the recommended cash holdings are all increasing in the investor's risk aversion. Although this is also a feature of the examples reported in Tables II and III, this is a parameter specific result. The cash allocation may increase or decrease with risk aversion depending on assumptions. For example, if we assume that $\pi_S = 2\%$, $\lambda_u = \lambda_r = 0$ and that the investor can only invest in the ten and twenty year bonds, then we find that the cash holdings for an investor with a horizon of twenty years are 8%, 5%, 4%, and 3% for $\gamma = 2, 3, 4, 6$ respectively, which are decreasing with γ.

[10] As risk aversion becomes infinite, the investor places all of his wealth in the bond with maturity equal to the investment horizon.

The foregoing discussion is based on the assumption that the investor can take positions in two (or more) bonds without short-sales constraints, so that the market is complete. If there are binding short-sales constraints, the investor will not be able to hedge perfectly against shifts in the investment opportunity. A similar situation obtains if the investor can invest in only a single bond. For a *given* set of bonds the constrained portfolio optimization problem can be solved numerically. However, in general the introduction of constraints on the size of positions makes the choice of the maturity of the bonds to be included in the portfolio a critical element of the portfolio decision, and this makes the portfolio optimization problem considerably more difficult. For this reason we do not pursue the constrained optimization problem in this paper. Nevertheless, the portfolio allocations will still be horizon-dependent, and we conjecture that the bond-stock allocation will still be increasing in risk aversion for long lived investors.

3. Conclusion

Canner et al. (1997) have pointed out an apparently puzzling aspect of the asset allocation recommendations of popular financial advisors: the recommended ratio of bonds to stocks increases as the investor's risk aversion increases which appears to conflict with the Tobin (1958) Separation Theorem. In this paper we have shown that this phenomenon can be rationalized by supposing that the advice is directed at investors with long horizons who face the risk that the investment opportunity set as measured by the interest rate will move adversely. Under such conditions, Merton (1973) has shown that the investor's asset demands have (at least) two components: the first is a myopic mean-variance efficient portfolio which does satisfy the Separation Theorem, and the second is a portfolio which is designed to hedge against shifts in the future investment opportunity set. In the model of the investment opportunity set that is presented here, the investor must hedge against not only changes in the instantaneous riskless rate, r, but also against changes in the central tendency of the rate, which we denoted by u. We showed that in this setting the optimal bond-stock mix of a long lived investor is an increasing function of his risk aversion which is consistent with the recommendations of the popular advisors. Thus this model "solves" the "asset allocation puzzle". However, we note that the optimal portfolio mix depends on the maturity of the bonds that are chosen for inclusion in the portfolio as well as on the investment horizon. Therefore it appears that the popular recommendations could be improved by making them horizon specific and by specifying the characteristics of the bonds to be included. The model does not provide general predictions about the relation between risk aversion and cash holdings; however, for the parameter values used in the examples reported here cash holdings are also increasing in risk aversion which is also true of the advisors' recommended cash holdings.

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