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Dynamic Spanning in the Consumption-Based Capital Asset Pricing Model *

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Abstract. Under the assumptions of the Consumption-based Capital Asset Pricing Model (CCAPM), Pareto optimal consumption allocations are characterized by each agent's consumption process being adapted to the filtration generated by the aggregate consumption process of the economy. The wealth processes of the agents, however, are adapted to the finer filtration generated by aggregate consumption and the conditional distribution of future aggregate consumption. Therefore, in order to achieve Pareto optimal consumption allocations, a sufficiently varied set of assets must exist such that any wealth process adapted to this finer filtration can be implemented by dynamically trading in that set of assets. We provide sufficient conditions for the existence of such a set of assets based on dynamically trading contingent claims on aggregate consumption. In addition, we give sufficient conditions for the existence of equilibria in a dynamically effectively complete market in which agents are only able to trade in contingent claims on aggregate consumption, the market portfolio of firms, and a (numeraire) zero-coupon bond. We demonstrate the role of short- and long-term contingent claims on aggregate consumption for the implementation of Pareto optimal allocations in the presence of short- and long-term risks. In addition, in the presence of personal risks, we demonstrate the role of insurance contracts.

Key words: Consumption-based capital asset pricing model, CCAPM, dynamic trading.

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1. Introduction

The prime equilibrium model of asset pricing is the Consumption-based Capital Asset Pricing Model (CCAPM) first formulated in continuous time by Breeden

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(1979). Duffie and Zame (1989) provide conditions on the primitives of a continuous-time economy under which equilibria obeying the CCAPM exist. The basic assumptions are homogeneous beliefs, time-additive preferences, and *dynamically complete markets*. The assumption of dynamically complete markets ensures that consumption allocations are Pareto optimal.[1] A basic characteristic of such allocations is that any agent's equilibrium consumption at any date is measurable with respect to the aggregate consumption at that date. Based on this observation, we define a concept called *dynamically effectively complete markets*. It has the property that any consumption plan, which is measurable with respect to aggregate consumption, can be implemented. This is accomplished by ensuring that portfolio strategies exist which 'span' information sets distinguished by either differences in current aggregate consumption or differences in conditional probability distributions for future aggregate consumption.[2] Given standard assumptions, we show that this type of market is sufficient to ensure the existence of an equilibrium with Pareto optimal consumption allocations, despite the fact that agents may not be able to implement any financially feasible consumption plan as they are in a dynamically complete market. In addition, we identify relationships between the types of securities that constitute a dynamically effectively complete market and the characteristics of the process for the conditional probability distributions for future aggregate consumption and the agents' consumption endowment processes. In particular, we investigate the role of short- and long-term contingent contracts on aggregate consumption as well as insurance contracts that facilitate an efficient sharing of personal risks.

As the CAPM and the APT the CCAPM is based on the gains of diversification. That is, the impact of events that influence the returns on individual assets but not the level of aggregate consumption is optimally eliminated by agents through diversification. A dynamically complete market requires securities that enable agents to implement any state-contingent consumption plan, both in terms of diversifiable risk and non-diversifiable risk. Recognizing that agents eliminate diversifiable risk, the securities on the market only have to provide insurance against non-diversifiable risk as depicted by changes in current aggregate consumption and beliefs about future aggregate consumption. Hence, if diversifiable risks exist, the number of securities required to ensure Pareto optimality may be dramatically reduced compared to the number of securities required to ensure a (fully) dynamically complete market.

These points are illustrated in the following using the simple event tree information structure shown in Figure 1. There are two consumption dates, $t = 1$ and $t = 2$, and eight states, $w \in \Omega$, revealed at date $t = 2$ and four signals, $\sigma \in \Sigma$, revealed at date $t = 1$.

[1] There is also a version of the CCAPM for incomplete markets, but the issue of existence of equilibria is unresolved in this setting (cf. Duffie (1996)).

[2] This presumes that the individual agents' consumption endowments are measurable with respect to aggregate consumption. We consider the case of general consumption endowments in Section 7.

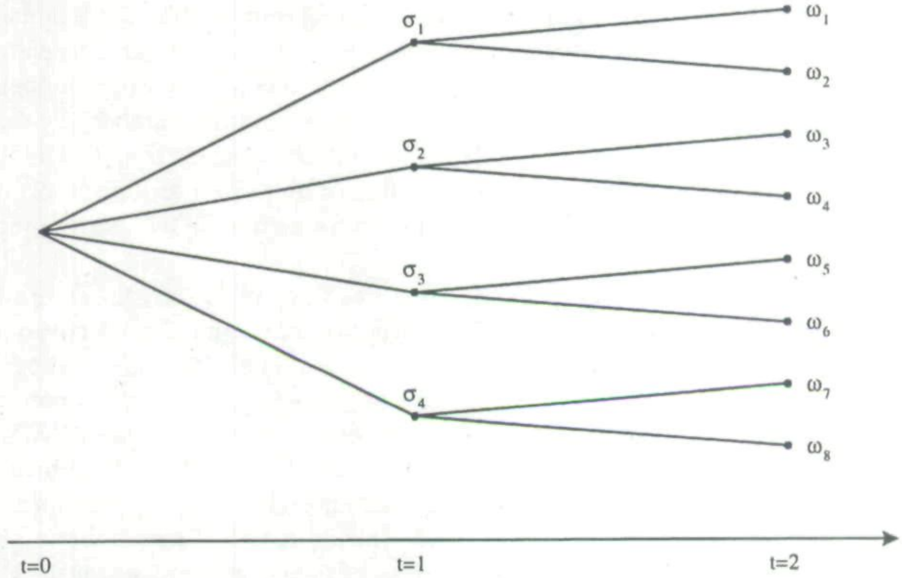

Figure 1. Event tree information structure.

A complete market requires twelve primitive securities, one for each signal at date $t = 1$ and one for each state at date $t = 2$. In that market, all trading occurs at date $t = 0$ with no subsequent trading occurring at date $t = 1$, cf. Rubinstein (1975). However, with dynamic trading only four sufficiently distinct long-lived securities are required to ensure Pareto optimal consumption allocations. This is due to the fact that if, at each date, agents can span wealth vectors with the same dimension as the cardinality of the information set for the following date, i.e. the market is dynamically complete, then any wealth process can be implemented by dynamic trading. Moreover, the minimum number of securities to constitute a dynamically complete market is equal to the maximum cardinality of information sets between any two dates, cf. e.g. Kreps (1979) for discrete-time and finite state-space settings and Duffie and Huang (1985) for the extension to continuous-time models.

The basic assumptions of the consumption based capital asset pricing model are time-additive and state-independent preferences and homogeneous beliefs. With these additional assumptions it is well-known that Pareto optimal consumption plans are measurable with respect to aggregate consumption (AC-measurable), cf. e.g. Breeden and Litzenberger (1978). Therefore, only AC-measurable consumption plans have to be implementable in order to achieve Pareto optimality. As a consequence, much fewer securities may be required to ensure Pareto optimality than to ensure a (dynamically) complete market. Breeden and Litzenberger (1978) demonstrate in a discrete-time model with a finite state space and no endowment risks that Pareto optimality can be achieved if agents can trade in a complete set of contingent claims on aggregate consumption, i.e. if there exists a primitive claim on aggregate consumption for each aggregate consumption level at each date. Consequently, all trading occurs at the initial trading date. As it is the case for the standard Arrow-Debreu model, the Breeden-Litzenberger concept of an effectively complete market does not generalize to a continuous-time model with continuous state space, since this would require an infinite number of primitive claims on aggregate consumption. However, in the same spirit as in Duffie and Huang (1985) we show that continuous trading of long-lived contingent claims on aggregate consumption can substitute the need for an infinite number of primitive securities.

In our discrete-time and finite state-space setting, as depicted in Figure 1, we illustrate in the following example the concept of a dynamically effectively complete market in which any consumption plan, measurable with respect to aggregate consumption, can be achieved through dynamic trading. In the main part of the paper we formalize this concept to the continuous-time setting of the CCAPM.

Agents' consumption possibilities are determined by the return from their portfolio strategies and their consumption endowments. However, suppose for simplicity that agents have no consumption endowments such that agents' consumption processes are equal to the dividend processes of their portfolio strategies.[3] Hence, in a dynamically effectively complete market, it is possible for agents to choose a portfolio strategy with any financially feasible AC-measurable dividend process. However, in general, the value process of any such portfolio strategy and, therefore, the agent's wealth process, will not be AC-measurable.

This can be illustrated by considering, in Figure 2, the same event tree as in Figure 1 augmented with the level of aggregate consumption, γ_t, for each signal at each date. Note, in particular, that a number of signals have identical aggregate consumption levels, i.e. the risk in asset returns associated with these signals is diversifiable. Consider a given portfolio strategy with payoffs that are measurable with respect to aggregate consumption, γ_t, at each date. Hence, the dividends at date $t = 1$ of this portfolio strategy depend only on whether signal σ_1 is obtained or not, whereas the date $t = 2$ dividends depend only on whether an even or an odd state occurs. Hence, no matter which signal occurs at date $t = 1$, the portfolio chosen at date $t = 1$ has the same payoff vectors at date $t = 2$. However, the ex-dividend values at date $t = 1$ of those portfolios will be different for the four signals due to differences in the state-contingent prices. These prices are the products of conditional probabilities for the future states and the marginal rates of substitution between future state-contingent consumption and current consumption.

The conditional probabilities are identical for the first three signals, but the marginal rates of substitutions are higher for σ_1 than for signals σ_2 and σ_3 since Pareto optimal consumption plans, at date $t = 1$, are strictly increasing in aggregate consumption, cf. e.g. Breeden and Litzenberger (1978). Thus, the value of the

[3] Dividends are here interpreted as the net-amount withdrawn from the portfolio at any date, i.e. the current cum-dividend value of the portfolio acquired at the previous date minus the ex-dividend value of the portfolio acquired at the current date.

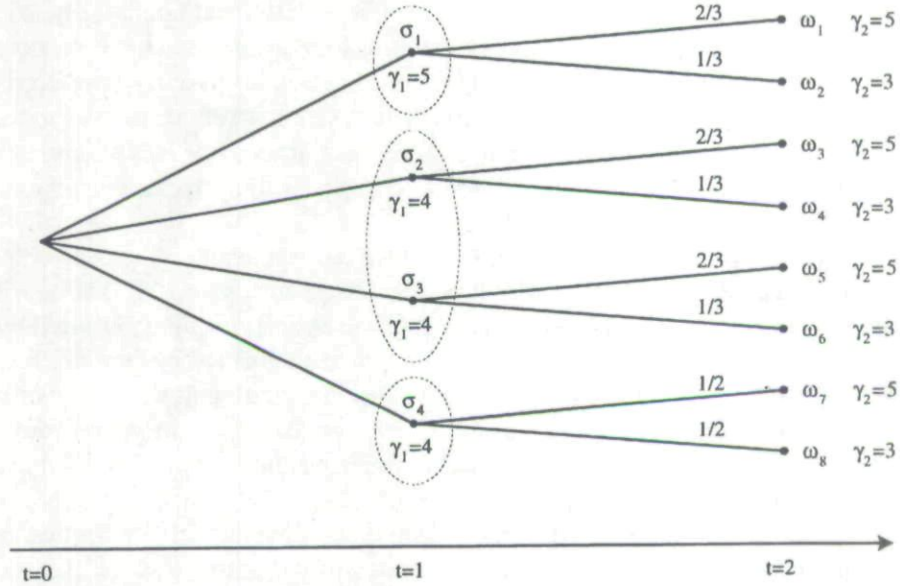

Figure 2. Event tree information structure.

portfolio acquired at date $t = 1$ is higher for σ_1 than the value of the portfolios acquired for signals σ_2 and σ_3.

For signal σ_4 the conditional probabilities have shifted such that the probability of the low level of future aggregate consumption has increased. The marginal rates of substitutions, however, are identical for σ_2, σ_3, and σ_4. Therefore, the state-contingent price for the low state increases whereas the state-contingent price for the high state decreases. Since the portfolios acquired at date $t = 1$ have identical state-contingent payoff vectors, the value of the portfolio acquired for σ_4 is, in general, different from the value of the portfolios acquired for σ_2 and σ_3.[4]

The dividends at date $t = 1$ of the portfolio acquired at date $t = 0$ is higher for σ_1 than for the other three signals since Pareto optimal individual consumption plans are strictly increasing in aggregate consumption. Since the ex-dividend value of the portfolio acquired at $t = 1$ is also higher for σ_1 than for σ_2 and σ_3, the cum-dividend value of the portfolio at date $t = 1$ must be higher for σ_1 than for σ_2 and σ_3, which again, in general, is different from the portfolio value for σ_4. However, the cum-dividend value as well as the ex-dividend value must be identical for σ_2 and σ_3. These two signals are precisely characterized by the fact that current (aggregate) consumption as well as the probability distribution for future aggregate (and individual) consumption are identical. Therefore, for the financial market to be dynamically effectively complete it only has to span the three information sets at date $t = 1$ shown by the dashed circles and, thus, only three sufficiently distinct

[4] In the special case of logarithmic utility, there will be no difference in the values of the portfolio acquired at $t = 1$ due to the myopic behavior of this utility function.

long-lived securities are required at date $t = 0$ to ensure Pareto optimality, whereas a dynamically complete market would require four securities. However, note that even though there are only two aggregate consumption levels at $t = 1$ we need three securities to have a dynamically effectively complete market due to the variation in conditional probabilities for future aggregate consumption levels. This corresponds to what Merton (1973) abstractly terms "changes in the investment opportunity set."

In the following we formalize the concept of a dynamically effectively complete market in a continuous-time and continuous state-space economy. That is, without endowment risks, a Pareto optimal consumption allocation can be ensured even if the financial market only spans information sets distinguished by either differences in current aggregate consumption or differences in conditional probability distributions for future aggregate consumption. If investors have consumption endowment risks, then additional insurance contracts or financial claims may be needed in order to hedge these personal risks.

In Section 2 we outline the basic continuous-time model. In Section 3 and Section 4 we characterize the wealth processes of the agents and we characterize which and how many securities the agents need in order to implement the Pareto optimal consumption plans. Moreover, we expand the existence results in Section 5, and in Section 6 we show that short-term uncertainty is insured by short-term contingent claims and long-term uncertainty is insured by long-term contingent claims. Finally, we extend the characterization of the wealth processes of the agents to include personal risk in Section 7. All proofs are deferred to the appendix.

2. The Continuous-Time Economic Model

The continuous-time and continuous state-space setting of the CCAPM follows that of Duffie and Huang (1985), Duffie (1986), and Duffie and Zame (1989). The primitives are a complete probability space $(\Omega, \mathcal{F}, P)$, a finite time horizon $\mathbb{T} := [0, T]$, and a filtration $\mathbf{F} := \{\mathcal{F}_t\}_{t \in \mathbb{T}}$ generated by a K-dimensional Wiener process, W, completed in the usual way. That is, $\mathbf{F}$ fulfills the usual conditions. The consumption space, L,[5] is the vector space of square-integrable previsible real-valued stochastic processes representing the agents' consumption at each date of the single commodity. Each of the agents $i \in \mathcal{I}$ in the finite set $\mathcal{I}$ is represented by $(U^i, \hat{c}^i, \hat{\theta}^i)$, where U^i is a von Neumann-Morgenstern utility function on L_+ representing agent i's preferences, $\hat{c}^i \in L_+$ is agent i's consumption endowment process, and $\hat{\theta}^i$ is an $\mathcal{F}_0$-measurable stochastic variable representing agent i's endowed portfolio of real assets. Agents can invest in a finite set of assets, $\mathcal{A}$, containing both the set of real assets, $\mathcal{J}$, and the set of financial assets, $\mathcal{N}$. That is,

[5] Formally, L is defined as in Duffie and Zame (1989, Footnote 4, p. 1283). That is $L = L^2(\Omega \times [0, T], \mathcal{P}, \nu)$, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0, T]$, (that is, the σ-field generated by the left-continuous adapted processes) and ν is the product measure of P and the Lebesgue measure on $[0, T]$ restricted to $\mathcal{P}$.

$\mathcal{A} = \mathcal{J} \cup \mathcal{N}$ and, moreover, by construction $\mathcal{J} \cap \mathcal{N} = \emptyset$. Hence, there is a finite set, $\mathcal{J}$, of firms each represented by a real dividend process, $\delta^j \in L$, $j \in \mathcal{J}$.[6] For later use, define the cumulative dividend processes of the real assets as

$$D_t^j := \int_0^t \delta_s^j ds, \qquad j \in \mathcal{J}.$$

In addition, there is a finite set, $\mathcal{N}$, of financial assets in zero net supply with cumulative dividend processes D^n, $n \in \mathcal{N}$, also measured in units of consumption, with ΔD_t denoting the lump-sum dividends paid at date t. To simplify the notation, we stack the cumulative dividend processes of real and financial assets in the same vector, D, of dimension $|\mathcal{A}|$. To separate asset dividends for a set of assets we use projections. E.g., to separate out financial asset dividends, we define the projection of D, denoted $D^{\mathcal{N}}$, also of dimension $|\mathcal{A}|$ such that coordinate a is

$$(D^{\mathcal{N}})^a := \begin{cases} D^n, & \text{if } a \in \mathcal{N}, \\ 0, & \text{if } a \notin \mathcal{N}. \end{cases}$$

Hence, $D = D^{\mathcal{J}} + D^{\mathcal{N}}$. We will use this notation for asset prices, portfolios, gain processes, etc., as well.

The entire economy is thus described by a collection

$$\mathcal{E} = ((\Omega, \mathcal{F}, \mathbf{F}, P), \{(U^i, \hat{c}^i, \hat{\theta}^i)\}_{i \in \mathcal{I}}, D).$$

Agents take an $|\mathcal{A}|$-dimensional ex-dividend asset price process, S, for granted. In addition to having the dividends and asset prices denominated in the consumption numeraire (denoted real dividends and real prices), we also use nominal dividends and nominal asset prices, denoted $\tilde{D}$ and $\tilde{S}$, respectively. Assume that one of the financial assets, $n_0 \in \mathcal{N}$, is a zero-coupon bond paying one consumption unit at maturity date T. Nominal dividends and asset prices are denominated in units of the price of this (numeraire) asset, S^{n_0}. To ease the notation denote the nominal price of one consumption unit as π, i.e., $\pi_t = \frac{1}{S_t^{n_0}}$ with $\pi_T = 1$. Hence, the nominal asset price process and dividend process are defined by

$$\tilde{S}_t := \pi_t S_t,$$

$$\Delta \tilde{D}_t := \pi_t \Delta D_t,$$

$$d\tilde{D}_t = \pi_t dD_t.$$

We assume that the (nominal) gain process, defined by $\tilde{G} = \tilde{S} + \tilde{D}$, is an Itô process,

$$d\tilde{G}_t = \mu_t^G dt + \sigma_t^G dW_t,$$

[6] By *real* we mean that the dividend process pays out in units of consumption.

where μ^G is an $|\mathcal{A}|$-dimensional previsible stochastic process and σ^G is an $|\mathcal{A}| \times K$-dimensional previsible stochastic process such that

$$E\left[\int_0^T \sigma_t^G \sigma_t^{G\top} ds\right] < \infty,$$

where $\top$ denotes transpose. This allows us to define cumulative gains from trade for an $|\mathcal{A}|$-dimensional previsible portfolio process, θ, as[7]

$$\int_{(0,t]} \theta_s \cdot d\tilde{G}_s,$$

if θ fulfills the following integrability conditions

$$E\left[\int_0^T \theta_t \cdot (\sigma_t^G \sigma_t^{G\top} \theta_t) dt\right] < \infty$$

and

$$E\left[\int_0^T |\theta_t \cdot \mu_t^G| dt\right] < \infty.$$

A budget-feasible plan for agent i is a consumption process, $c \in L_+$, and a portfolio process, θ, such that, for any date $t \in \mathbb{T}$,

$$\theta_t \cdot (\tilde{S}_t + \Delta \tilde{D}_t) = \int_0^t \theta_s \cdot d\tilde{G}_s + \int_0^t \pi_s (\hat{c}_s^i - c_s) ds + \hat{\theta}^i \cdot \tilde{S}_0^{\mathcal{J}},$$

and such that $\theta_T \cdot \Delta \tilde{D}_T = 0$.[8] A budget-feasible plan, (c, θ), for agent i is optimal for agent i if there is no other budget feasible plan, $(\tilde{c}, \tilde{\theta})$, such that $U^i(\tilde{c}) > U^i(c)$. An equilibrium for the economy, $\mathcal{E}$, is a collection $(S, \{(c^i, \theta^i)\}_{i \in \mathcal{I}})$ such that, given the asset price process, S, the budget-feasible plan, (c^i, θ^i), is optimal for each agent $i \in \mathcal{I}$, and such that markets clear:

$$\sum_{i \in \mathcal{I}} c^i = \hat{c}^0,$$

$$\sum_{i \in \mathcal{I}} \theta^{i,a} = 1, \qquad \forall a \in \mathcal{J},$$

and

$$\sum_{i \in \mathcal{I}} \theta^{i,a} = 0, \qquad \forall a \in \mathcal{N},$$

[7] $x \cdot y$ denotes the standard Euclidean inner product between x and y in $\mathbb{R}^{|\mathcal{A}|}$.

[8] Equalities of random variables are to be understood as almost surely identities.

where the *exogenously* given aggregate consumption process, $\hat{c}^0$, is defined as[9]

$$\hat{c}^0 := \sum_{i \in \mathcal{I}} \hat{c}^i + \sum_{j \in \mathcal{J}} \delta^j.$$

We make the following three assumptions similar to Duffie and Zame (1989, (A.1)–(A.3), pp. 1285–1286) with slightly more restrictive regularity conditions.

ASSUMPTION 2.1. Agent i's utility function, U^i, is represented by a smooth continuously differentiable function, $u_i : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$ of the form,

$$U^i(c) = E\left[\int_0^T u_i(c_t, t)dt\right],$$

where, for each $t \in \mathbb{T}$, the function $u_i(\cdot, t) : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, increasing, and three times continuously differentiable on $(0, \infty)$, with first derivative, denoted $u_{ic}(\cdot, t)$, satisfying $\lim_{k \downarrow \downarrow 0} u_{ic}(k, t) = +\infty$.

ASSUMPTION 2.2. The aggregate consumption process, $\hat{c}^0$, is an Itô process, uniformly bounded away from zero, represented as

$$d\hat{c}_t^0 = \mu_t^0 dt + \sigma_t^0 \cdot dW_t, \tag{1}$$

such that $E[\int_0^T |\mu_t^0|^2 dt] < \infty$, $E[\int_0^T ||\sigma_t^0||^2 dt] < \infty$, μ^0 and σ^0 have continuous sample paths. Moreover, we assume that, for all t, a regular conditional distribution of $\{\hat{c}_r^0\}_{r \in (t,T]}$ given $\mathcal{F}_t$ (denoted $P(\hat{c}_{t\to}^0 | \mathcal{F}_t)$) exists such that the conditional marginal distribution for aggregate consumption, $P(\hat{c}_r^0 \in \cdot | \mathcal{F}_t)$, is absolutely continuous for all r and t with $t \leq r$.

ASSUMPTION 2.3. A (numeraire) zero-coupon bond with maturity T exists in addition to a subset of securities, $\mathcal{L} \subseteq \mathcal{A}$, for which the accumulated dividend process, $D^{\mathcal{L}}$, is an Itô process represented as

$$dD^{\mathcal{L}} = \mu_t^D dt + \sigma_t^D dW_t,$$

with $E[\int_0^T \sigma_t^D \sigma_t^{D\top} dt] < \infty$ and $E[\int_0^T |\mu_t^D| dt] < \infty$, such that the vector martingale defined by $M_t^{\mathcal{L}} = \int_0^t \sigma_s^D dW_s$ forms a martingale generator for $\mathbf{F}$ (under the measure P). That is, for a given one-dimensional $\mathbf{F}$-martingale, $\hat{M}$, an $\mathbf{F}$-previsible process ψ exists such that

$$\hat{M}_t = \hat{M}_0 + \int_0^t \psi_s \cdot dM_s^{\mathcal{L}}.$$

[9] Even though we have introduced firms, the economy is an exchange economy since production decisions are exogenously given.

From Duffie and Zame (1989)[10] and Duffie (1988)[11] we collect the following results.

PROPOSITION 2.4. An equilibrium with a representative agent $(u_\lambda, \hat{c}^0)$ exists such that, for any date t, the equilibrium price of a cumulative dividend process, Δ, is

$$S_t^\Delta = \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_t^T u_{\lambda c}(\hat{c}_s^0, s) d\Delta_s \middle| \mathcal{F}_t \right], \qquad t \in [0, T), \tag{2}$$

for Δ such that $E|\Delta|_T^2 < \infty$, where $|\Delta|_T$ denotes the total variation of the process Δ in the time interval $[0, T]$.

Furthermore, the equilibrium consumption of agent i, at date t, c_t^i, is measurable with respect to aggregate consumption at date t. That is, measurable non-negative functions $k^i : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$, for $i \in \mathcal{I}$, exist such that

$$c_t^i = k^i(\hat{c}_t^0, t). \tag{3}$$

After collecting these results on the existence and characterization of equilibria in dynamically complete markets, we turn to the implementation of Pareto optimal consumption plans.

3. Characterization of the Wealth Processes of the Agents

As we have demonstrated in the discrete-time, finite state-space example in the introduction, aggregate consumption affects individual consumption (see also Equation (3) in Proposition 2.4) and conditional probability distributions for future aggregate consumption affect implicit state prices and, thus, the current value of the investors' future consumption stream. Similarly, in order to characterize the relevant information for the agents' implementation of their Pareto optimal consumption plans in the continuous-time and continuous state-space economy, the following σ-fields define this information

$$\tilde{\mathcal{G}}_t := \sigma(\hat{c}_t^0),$$

$$\mathcal{G}_t := \bigvee_{s \le t} \tilde{\mathcal{G}}_s,$$

[10] Duffie and Zame (1989, Theorem 1, p. 1287 and Corollary, p. 1289) still holds even though we have added additional structure to the model. The real dividends from the firms do not change anything except giving the representative agent additional endowments. Moreover, in Assumptions 2.1 and 2.2 we included additional assumptions on the primitives of the model, but not on the equilibrium that we want to find. Finally, we have relaxed the spanning assumption in Assumption 2.3 such that we allow for a subset of the assets to span the uncertainty generated by the filtration **F**.

[11] Under slightly less restrictive regularity conditions, Duffie (1988) proves Equation (3) as Exercise 25.3. The result follows from the first order conditions for the mathematical program defining the utility function for the representative agent. In addition, cf. Huang (1987, Proposition 3.3, p. 125).

$$\tilde{\mathcal{H}}_t := \sigma(\hat{c}_t^0, P(\hat{c}_{t\rightarrow}^0 | \mathcal{F}_t)),$$

$$\mathcal{H}_t := \bigvee_{s \leq t} \tilde{\mathcal{H}}_s$$

where $\hat{c}_{t\rightarrow}^0$ is a shorthand notation for $\{\hat{c}_r^0\}_{r\in(t,T]}$. Assume that the σ-fields are completed by adjoining all P-null-sets. Furthermore, we denote the filtrations $\{\mathcal{G}_t\}_{t\in\mathrm{T}}$ resp. $\{\mathcal{H}_t\}_{t\in\mathrm{T}}$ as **G** resp. **H**. The filtration **G** keeps track of past values of aggregate consumption, whereas **H** keeps track of both past values of aggregate consumption and related conditional probabilities given the investors' information, $\mathcal{F}_t$. Note that, since aggregate consumption is exogenously given, both the filtrations **G** and **H** are exogenously given. The following example illustrates the key differences between the filtrations **F**, **H**, and **G**.

EXAMPLE 3.1. Let Y be a state process such that $(\hat{c}^0, Y)$ is a Markov process governed by the following stochastic differential equation,

$$\begin{pmatrix} d\hat{c}_t^0 \\ dY_t \end{pmatrix} = \mu(\hat{c}_t^0, Y_t)dt + \sigma(\hat{c}_t^0, Y_t)\begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix},$$

where σ is a non-singular matrix function suitably chosen to meet the conditions of Assumption 2.2. Clearly, $\mathcal{G}_t = \sigma(\hat{c}_s^0 : s \leq t)$ and $\mathcal{H}_t \subseteq \sigma(\hat{c}_s^0, Y_s : s \leq t)$ (up to P-null-sets). The σ-fields $\mathcal{G}_t$ and $\mathcal{H}_t$ will, therefore, in general, not coincide if $\hat{c}^0$ is not Markov by itself. Hence, the filtration **G** is a proper sub-filtration of **H**, in general.

Moreover, the filtration **H** can easily be a proper sub-filtration of **F**. To give an example of this related to the economy of this paper consider the case where agents have no consumption endowments and the firms' market shares of aggregate dividends are affected by the Wiener process, $W = (W^3, \dots, W^K)$ with $K > 2$. Assume that the market share of firm j, a^j, defined from the relation $\delta_t^j = a_t^j \hat{c}_t^0$, is an Itô process on the form

$$da_t^j = \mu^a dt + \sigma^a dW_t.$$

Although the process for aggregate consumption is only affected by the first two coordinates of the Wiener process, individual firm dividends (and also prices of the individual assets) are affected by all the coordinates of the Wiener process.

The agents, however, would diversify their portfolio holdings of the real assets such that they are not gambling on the outcome of which of the individual firms pay high and which pay low dividends. This is so because the uncertainty related to which of the firms pay high and which pay low dividends is independent of the uncertainty related to the level of aggregate consumption. Therefore, the agents would not be worse off if they could only invest in a mutual fund holding all the real assets, which has a **G**-adapted dividend process.

However, in order to implement a Pareto-optimal allocation the investors may in addition to the fund of real assets need a second asset to dynamically hedge the uncertainty associated with the two state variables. That is, in general, they need three assets, two assets with gain processes that span the filtration $\mathbf{H}$ and one numeraire security.

As indicated in the introduction, current aggregate consumption and the conditional probability distribution for future aggregate consumption and, thus, the filtration $\mathbf{H}$, play a key role in describing the state prices in the economy. We start by showing that the price density process pertaining to aggregate consumption, $p_r(c)$, is adapted to the filtration $\mathbf{H}$.

LEMMA 3.2. The implicit price density process, $\{p_r^t(c)\}_{t\in[0,r]}$, pertaining to aggregate consumption of level c, at a given future date, r, is adapted to the filtration $\mathbf{H}$.

Moreover, in the following lemma we derive the continuous-time version of the basic discrete-time and finite state-space valuation equation in Breeden and Litzenberger (1978, Theorem 2).

LEMMA 3.3. The equilibrium price of an absolutely continuous cumulative dividend process,

$$\Delta := \left\{ \int_0^t \delta_r dr \right\}_{t\in\mathbf{T}},$$

satisfying the integrability condition of Proposition 2.4, is

$$S_t^{\Delta} = \int_t^T \int_{\mathbb{R}} p_r^t(c) E[\delta_r | \hat{c}_r^0 = c, \mathcal{F}_t] dc\, dr. \tag{4}$$

Using Lemma 3.2 and Lemma 3.3 the price process of a security will not be $\mathbf{H}$-adapted unless its dividend process is $\mathbf{G}$-adapted, in general. Moreover, even if the dividend process is $\mathbf{G}$-adapted, the price process will *not* be $\mathbf{G}$-adapted, because the implicit price density processs is only $\mathbf{H}$-adapted.

The key characteristic of valuation Equation (4) is that risk adjustments only pertain to variations in aggregate consumption whereas there is no risk adjustment for diversifiable risk not affecting aggregate consumption. That is, the agents' preferences with respect to intertemporal consumption and risk are captured by the implicit price densities pertaining to aggregate consumption. This can also be seen from the measure transformation to the risk adjusted probability measure where the gain processes are martingales. The following lemma demonstrates that this transformation is **H**-adapted.

LEMMA 3.4. Let $q_t = u_{\lambda c}(\hat{c}_t^0, t)$ and let Δ be an absolutely continuous cumulative dividend process,

$$\Delta := \left\{ \int_0^t \delta_r dr \right\}_{t \in \mathrm{T}},$$

satisfying the integrability condition of Proposition 2.4. An equivalent measure, Q, exists such that the equilibrium price of the cumulative dividend process, Δ, is

$$S_t^\Delta = S_t^{n_0} E^Q \left[\int_t^T \frac{1}{S_s^{n_0}} d\Delta_s \middle| \mathcal{F}_t \right].$$

The Radon-Nikodym derivative, $\frac{dQ}{dP}|_{\mathcal{F}_s}$, can be expressed as

$$N_s := \frac{1}{E[q_T]} S_s^{n_0} q_s. \tag{5}$$

The real price of the numeraire security, S^{n_0}, and the Radon-Nikodym derivative, N, are both **H**-adapted.

In nominal terms, the equilibrium price of the nominal cumulative dividend process, $\tilde{\Delta}$, can be determined as

$$\tilde{S}_t^\Delta = E^Q[\tilde{\Delta}_T | \mathcal{F}_t] - \tilde{\Delta}_t,$$

and the corresponding gain process, $\tilde{G}^\Delta$, is an $(\mathbf{F}, Q)$-martingale.

Using the valuation results in the previous lemmas, the equilibrium wealth processes of the agents can be characterized by interpreting the equilibrium wealth of an agent as the current value of the agent's future equilibrium consumption process.[12] It follows from Proposition 2.4 that the agent's future equilibrium consumption process is **G**-adapted. Hence, using Lemma 3.2 and Lemma 3.3 the current value of this consumption process is **H**-adapted.

PROPOSITION 3.5. The equilibrium wealth process of agent i, w^i, is adapted to the filtration **H**, for all $i \in \mathcal{I}$.

Observe that, in general, the wealth process will *not* be adapted to the filtration **G** generated by the aggregate consumption process. That is, even though current

[12] Note that the filtration **H** is precisely constructed as the minimal filtration that makes the equilibrium wealth process for all the agents adapted to that filtration. In a no-arbitrage, two-date consumption framework, Chamberlain (1988) constructs a minimal Wiener filtration that makes the agents' final-date consumption measurable with respect to the final-date σ-algebra of that filtration. This implies that the wealth processes are adapted to the Wiener filtration. In general, this Wiener filtration is a finer filtration than **H**.

and past aggregate consumption are identical for two events, the equilibrium wealth may differ for those events since the implicit price density may differ. This point is also illustrated in our simple discrete-time and finite state-space example presented in the introduction of the paper. However, in single-period models the wealth process is adapted to the filtration **G**, since in those models wealth equals consumption. This result can be re-established in a multi-period setting if the aggregate consumption process is Markov as the following result shows.

COROLLARY 3.6. Suppose $\hat{c}^0$ is Markov. Then the optimal wealth process of agent i, w^i, is adapted to the filtration **G**, for all $i \in \mathcal{I}$.

Having demonstrated that the agents' equilibrium wealth processes are **H**-adapted, we turn to the question of characterizing the type of assets that the agents need in order to implement Pareto optimal consumption plans.

4. Implementation of Pareto Optimal Consumption Plans

In this section we show, given a spanning condition to be made precise, that the agents are able to implement their Pareto optimal consumption plans by trading contingent claims on aggregate consumption. However, even though the equilibrium wealth processes are **H**-adapted, the trading strategies that implement these wealth processes are not necessarily **H**-adapted. We begin with a definition of martingale generators for sub-filtrations of **F** that recognizes this.[13]

DEFINITION 4.1. Let $\mathbf{L} := \{\mathcal{L}_t\}_{t\in\mathbb{T}}$ be a filtration such that $\mathcal{L}_t \subseteq \mathcal{F}_t$, for all $t \in \mathbb{T}$, and let M be an n-dimensional $(\mathbf{F}, Q)$-martingale. M is an $(\mathbf{F}, Q)$-martingale generator for **L** if, for any given one-dimensional **L**-adapted $(\mathbf{F}, Q)$-martingale N, an n-dimensional **F**-previsible process, ψ, exists such that

$$N_t = N_0 + \int_{(0,t]} \psi_s \cdot dM_s,$$

Let $\mathcal{M} \subseteq \mathcal{N}$, be a subset of the financial assets consisting of contingent claims on aggregate consumption. That is, the corresponding cumulative dividend processes, $D^{\mathcal{M}}$, are **H**-adapted. Note by Lemma 3.4 that $\tilde{G}^{\mathcal{M}}$ is an $(\mathbf{F}, Q)$-martingale

[13] Note that for any **F**-previsible process, ψ, it will not, in general, be the case that the martingale generated as

$$\int_{(0,\cdot]} \psi_s \cdot dM_s$$

is **L**-adapted.

denoting the gain process of these contingent claims (augmented with zeros on the coordinates outside the set $\mathcal{M}$).

PROPOSITION 4.2. Let an equilibrium, $(S, \{(c^i, \theta^i)\}_{i\in\mathcal{I}})$, be given. If $\tilde{G}^{\mathcal{M}}$ forms an $(\mathbf{F}, Q)$-martingale generator for $\mathbf{H}$ and the agents' consumption endowment processes, $\hat{c}^i$, are adapted to the filtration $\mathbf{G}$, for all $i \in \mathcal{I}$, then the equilibrium consumption plan can be implemented by trading only in the set of financial assets $\mathcal{M}$, the market portfolio of real assets, and the numeraire zero-coupon bond.

The proposition demonstrates that financial assets contingent on aggregate consumption can play a key role in implementing Pareto optimal consumption plans when the agents' consumption endowment processes, $\hat{c}^i$, are adapted to the filtration $\mathbf{G}$.[14] When there are 'personal risks' in the agents' consumption endowments, more securities (or insurance contracts) may be needed to facilitate an efficient insurance against those risks. We return to that issue in Section 7.

We focus on the role of contingent claims on aggregate consumption for spanning purposes because a martingale generator for $\mathbf{H}$ must be adapted to $\mathbf{H}$.[15] Note by Proposition 2.4, that the gain process of a contingent claim with cumulative dividend process adapted to $\mathbf{G}$, which we term a *simple contingent claim on aggregate consumption*, is $\mathbf{H}$-adapted. This result is formalized in the following proposition.

PROPOSITION 4.3. Let an equilibrium, $(S, \{(c^i, \theta^i)\}_{i\in\mathcal{I}})$ be given. The gain process of a simple contingent claim on aggregate consumption is $\mathbf{H}$-adapted.

Hence, these simple contingent claims on aggregate consumption are natural candidates for forming a martingale generator. Non-simple contingent claims on aggregate consumption may or may not have $\mathbf{H}$-adapted gain processes. In continuation of Example 3.1, the following example illustrates that if aggregate consumption and the conditional probability distribution for future aggregate

14 In an N-dimensional Markov setting, Merton (1973) demonstrates a similar mutual fund separation result in which any budget-feasible consumption process can be obtained by continuous trading in $N + 2$ portfolios including one hedge portfolio for each of the N state variables. In a similar vein as in Example 3.1 suppose aggregate consumption is a Markov process in itself and suppose it is one of Merton's state variables. Then, Proposition 4.2 shows that only three mutual funds are needed in order to implement a Pareto optimal allocation whereas Merton uses $N + 2$. However, asset prices depend on all state variables.

15 Note that the proposition gives a separation result and is not a statement on the minimal number of financial asset needed in order to implement a Pareto optimal consumption allocation. For example, real assets might also be used as part of the martingale generator. In particular, the market portfolio of real assets can be used, since its gain process is $\mathbf{H}$-adapted given the assumption on the consumption endowments.

consumption is a Markov process as in Example 3.1, then non-simple contingent claims on aggregate consumption will also have **H**-adapted gain processes.

EXAMPLE 4.4 Let the situation be as in Example 3.1. Let δ be the rate of dividends on a contingent claim with an **H**-adapted absolutely continuous cumulative dividend process. Hence, it follows from Proposition 2.4 that the real price of the contingent claim is given by

$$\begin{aligned} S_t^{\Delta} &= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_t^T u_{\lambda c}(\hat{c}_s^0, s)\delta_s ds \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_t^T u_{\lambda c}(\hat{c}_s^0, s)\delta_s ds \,\middle|\, \mathcal{H}_t\right], \end{aligned}$$

where the last equality follows from $(\hat{c}^0, Y)$ being a Markov process. Since all terms on the right hand side are **H**-adapted, the price of the claim is **H**-adapted. It then follows from Lemma 3.4 that the nominal price and the gain process are both **H**-adapted.

However, claims with dividend processes not adapted to **H** will rarely have gain processes adapted to **H**. Examples of simple contingent claims on aggregate consumption are forward contracts and European options written directly on the level of aggregate consumption, whereas examples of non-simple contingent claims are futures, American options, and options written on a forward contract on the level of aggregate consumption.

5. Existence of Equilibria in Dynamically Effectively Complete Markets

In obtaining the result of Proposition 4.2 it has been assumed that the market is dynamically complete, i.e., securities forming a martingale generator for **F** exist, cf. Assumption 2.3. However, it is a direct consequence of Proposition 4.2 that the equilibrium can be sustained in an otherwise identical (dynamically incomplete) economy in which the contingent claims on aggregate consumption forming a martingale generator for **H** and the numeraire zero-coupon bond are the *only* financial assets traded. We term this economy *dynamically effectively complete*. In fact, this statement can be extended to prove the existence of equilibria for dynamically effectively complete markets as opposed to the existence of equilibria for dynamically complete markets shown in Duffie and Zame (1989).

PROPOSITION 5.1. Let an economy $\mathcal{E} = ((\Omega, \mathcal{F}, \mathbf{F}, P), \{(U^i, \hat{c}^i, \hat{\theta}^i)\}_{i\in\mathcal{I}}, D)$ satisfying Assumptions 2.1–2.2 be given. Suppose **H** is generated by some linear function of the Wiener process, i.e., $\mathcal{H}_t = \sigma(\{AW_s\}_{s\in[0,t]})$, where A is a constant matrix, and that the agents' consumption endowment processes, $\hat{c}^i$, are adapted to the filtration **G**, for all $i \in \mathcal{I}$. Financial assets contingent on aggregate consumption can be constructed such that an equilibrium exists in the economy augmented with these assets and the numeraire zero-coupon bond with maturity date T.

The proposition demonstrates that any economy satisfying Assumptions 2.1–2.2 with $\mathbf{H}$ generated by some linear function of Wiener processes can be augmented with financial assets contingent on aggregate consumption such that an equilibrium exists for the augmented economy. However, the choice of dividend processes for the financial assets that ensure that their gain processes form a martingale generator for $\mathbf{H}$ depends on the equilibrium prices in the extended economy. If, instead, we assume that the cumulative dividend processes for the financial assets as opposed to the gain processes are Itô processes and form a martingale generator for $\mathbf{H}$, then the necessary financial assets can be specified exogenously independently of equilibrium prices as in Assumption 2.3. However, this would exclude the spanning role of option-like contingent claims on aggregate consumption which have non-continuous dividend processes. The corresponding gain processes of these claims, on the other hand, are typically assumed to be Itô processes and, therefore, it is natural to state the spanning condition in terms of the gain processes forming a martingale generator for $\mathbf{H}$.

6. Short- and Long-Term Contingent Claims

Proposition 5.1 provides sufficient conditions for the existence of equilibria in dynamically effectively complete markets in terms of existence of a set of abstract contingent claims on aggregate consumption. The following propositions provide relationships between the process of conditional probability distributions for future aggregate consumption and characteristics of the simple contingent claims on aggregate consumption which are useful to implement Pareto optimal consumption plans.

PROPOSITION 6.1. Suppose the conditional probability distribution for aggregate consumption after date t_1, $P(\hat{c}^0_{t_1\rightarrow}|\mathcal{F}_t)$, is $\mathcal{H}_{t_0}$-measurable for all $t \in [t_0, t_1]$ and there is a set, $\mathcal{L}$, of long-term simple contingent claims on aggregate consumption such that

$$D^{\mathcal{L}}_{t'} = D^{\mathcal{L}}_{t''}, \qquad \forall t', t'' \in [t_0, t_1]$$

and that $\sigma(D^{\mathcal{L}}_s - D^{\mathcal{L}}_{t_1} : s > t_1) \subseteq \sigma(\hat{c}^0_{t_1\rightarrow})$. Then the set of gain processes of the assets in the set $\mathcal{L}$, $\{\{\tilde{G}^l_t\}_{t\in[t_0,t_1]}\}_{l\in\mathcal{L}}$, has a Q-martingale multiplicity of one in the time interval $[t_0, t_1]$.

The key assumption in the proposition is that no new information is revealed in the time period from t_0 to t_1 about aggregate consumption possibilities after t_1.[16]

[16] Note that this implies, e.g., that the term structure of forward rates is deterministic for maturities later than t_1.

In that case, long-term simple contingent claims on aggregate consumption are not useful in the implementation of short-term Pareto optimal consumption plans. In order to implement short-term Pareto optimal consumption plans, the agents must be able to trade in a sufficiently varied set of short-term simple contingent claims on aggregate consumption (or non-simple contingent claims on aggregate consumption).

PROPOSITION 6.2. Suppose the process of aggregate consumption before date t_1, $\hat{c}^0_{\to t_1}$, is $\mathcal{H}_{t_0}$-measurable, for a given $t_0 < t_1$, and there is a set, $\mathcal{L}$, of short-term simple contingent claims on aggregate consumption such that

$$D^{\mathcal{L}}_{t'} = D^{\mathcal{L}}_{t''}, \qquad \forall t', t'' > t_1.$$

Then the set of gain processes of the assets in the set $\mathcal{L}$, $\{\{\tilde{G}^l_t\}_{t\in[t_0,t_1]}\}_{l\in\mathcal{L}}$, has a Q-martingale multiplicity of one in the time interval $[t_0, t_1]$.

The key assumption in this proposition is that no new information is revealed about short-term aggregate consumption possibilities. The information revealed pertains only to long-term aggregate consumption possibilities. Consequently, only long-term contingent claims on aggregate consumption are useful in order to implement Pareto optimal consumption plans.

The last two propositions demonstrate that there is a close connection between the maturity structure of useful simple contingent claims on aggregate consumption and the information that is revealed about future aggregate consumption. An efficient sharing of short-term risk in aggregate consumption is facilitated by trading short-term simple contingent claims, whereas an efficient sharing of long-term risk in aggregate consumption requires trading in long-term simple contingent claims (or short-term non-simple contingent claims on aggregate consumption).

7. Personal Risk

In Proposition 4.2 we showed that agents can implement Pareto optimal consumption plans by restricting their trading strategies to the market portfolio of real assets, contingent claims on aggregate consumption, and the numeraire zero-coupon bond. This result was shown under the assumption that the agents' consumption endowment processes contain no personal risks, i.e. that the agents' consumption endowment processes are adapted to the filtration generated by aggregate consumption, **G**. When this is not the case, more securities (or insurance contracts) are needed to facilitate an efficient insurance against personal risks.

One simple way of implementing the agents' optimal consumption plan in the case where the agents are exposed to personal risks is to introduce an insurance fund. This insurance fund offers contracts with the following dividend processes

$$\epsilon^i = \{E[\hat{c}^i_t|\mathcal{G}_t] - \hat{c}^i_t\}_{t\in\mathrm{T}},$$

for all $i \in \mathcal{I}$. Clearly, contract i is explicitly tailored to agent i in the sense that contract i is designed to insure precisely against agent i's personal risk.

The value of contract i at date t is

$$\int_t^T \int_{\mathbb{R}} p_r^t(c) E[E[\hat{c}_r^i | \mathcal{G}_r] - \hat{c}_r^i \mid \hat{c}_r^0 = c, \mathcal{F}_t] dc\, dr,$$

for $i \in \mathcal{I}$. Note that the value of the contract is zero at date zero, such that it has no budget consequences at date zero for the agent to buy this insurance contract. Moreover, note that if agent i buys insurance contract i at date zero, the agent's insured endowment process is $\{E[\hat{c}_t^i | \mathcal{G}_t]\}_{t \in \mathrm{T}}$, which is **G**-adapted. Hence, after having introduced the insurance fund, we are back in the case of Proposition 4.2, in which the individual agents' personal endowment processes are **G**-adapted.

The insurance fund is funded by issuing a real asset with the dividend process, $-\sum_{i \in \mathcal{I}} \epsilon^i$. The portfolio consisting of the real assets including the asset issued by the insurance fund forms the market portfolio. The dividend process of the market portfolio is

$$\left\{ \sum_{i \in \mathcal{I}} \hat{c}_t^i + \sum_{j \in \mathcal{J}} \delta_t^j - \sum_{i \in \mathcal{I}} E[\hat{c}_t^i | \mathcal{G}_t] \right\}_{t \in \mathrm{T}} = \left\{ \hat{c}_t^0 - \sum_{i \in \mathcal{I}} E[\hat{c}_t^i | \mathcal{G}_t] \right\}_{t \in \mathrm{T}}.$$

Hence, the dividend process of the market portfolio is **G**-adapted. The following result now follows almost immediately from Proposition 4.2.

PROPOSITION 7.1. Let a subset of financial assets, $\mathcal{M} \subseteq \mathcal{N}$, and an equilibrium for a dynamically complete market, $(S, \{(c^i, \theta^i)\}_{i \in \mathcal{I}})$, be given. Denote the corresponding gain processes for $\mathcal{M}$ by $\tilde{G}^{\mathcal{M}}$. If $\tilde{G}^{\mathcal{M}}$ forms an $(\mathbf{F}, Q)$-martingale generator for **H** and the agents have access to the insurance contracts with dividend processes $\{\epsilon^i\}_{i \in \mathcal{I}}$, then the equilibrium consumption plan can be implemented by buying the individually tailored insurance contracts and dynamically trading only in the set of financial assets, $\mathcal{M}$, the market portfolio of real assets including the asset issued by the insurance fund, and the numeraire zero-coupon bond.

Note that the agents buy-and-hold the individually tailored insurance contracts, since the insurance contracts insure the consumption endowments for the whole lifetime of the agents. If these individually tailored long-term insurance contracts are not available, the Pareto optimal consumption allocations are implementable if the set of financial assets and short-term insurance contracts is rich enough for the agents to hedge their personal risks through short-term insurance contracts and dynamically trading of financial assets. In order to investigate this issue, define the following σ-fields,

$$\tilde{\mathcal{P}}_t := \sigma(\hat{c}_t^0, P(\hat{c}_{t\to}^0 | \mathcal{F}_t), \hat{c}_t^i, \{\{E[\hat{c}_r^i | \hat{c}_r^0 = c, \mathcal{F}_t]\}_{c \in \mathfrak{C}_t^r}\}_{r \in (t,T]} : i \in \mathcal{I}, \hat{c}_{t\to} \in \mathfrak{C}_t),$$

$$\mathcal{P}_t := \bigvee_{s \leq t} \tilde{\mathcal{P}}_s,$$

and the filtration, $\mathbf{P} := \{\mathcal{P}_t\}_{t\in\mathbf{T}}$. The equivalent proposition to Proposition 4.2 now takes the following form.

PROPOSITION 7.2. Let a subset of financial assets, $\mathcal{M} \subseteq \mathcal{N}$, and an equilibrium for a dynamically complete market, $(S, \{(c^i, \theta^i)\}_{i\in\mathcal{I}})$, be given. Denote the corresponding gain processes for $\mathcal{M}$ by $\tilde{G}^{\mathcal{M}}$. If $\tilde{G}^{\mathcal{M}}$ forms an $(\mathbf{F}, Q)$-martingale generator for $\mathbf{P}$, then the equilibrium consumption plan can be implemented by trading only in the set of financial assets $\mathcal{M}$, the market portfolio of real assets, and the numeraire zero-coupon bond.

The key issue in implementing Pareto optimal consumption plans with personal risks is that agents through insurance contracts and dynamic trading are able to insure themselves against not only variations in current consumption endowments but also in variations of the value process of their future consumption endowment process. However, note that all new information about the agents' future consumption endowments does not have to be captured by the martingale generator – only information relevant for the value process of future consumption endowments. Irrespective of equilibrium prices, this is assured by the assumption that the information generated by each individual agent's future expected consumption endowments conditional on levels of aggregate consumption is captured by the martingale generator. Hence, for example, new information about conditional variances of the agents' future consumption endowments has no impact on the value process.

Appendix A. Proofs

The following equivalent characterization of the filtration $\mathbf{H}$ is useful in several of the following proofs:

LEMMA A.1. The σ-field, $\mathcal{H}_t$, can, up to P-null-sets, be written as

$$\sigma(\hat{c}^0_s, E[g(\hat{c}^0_{s\to})|\mathcal{F}_s] \;:\; g \in C_b(\mathfrak{C}_s, \mathbb{R}), s \leq t),$$

where $\mathfrak{C}_s$ denotes all future possible aggregate consumption paths, given the information at date s, and $C_b(\mathfrak{C}_s, \mathbb{R})$ denotes the set of bounded continuous functions mapping from $\mathfrak{C}_s$ to $\mathbb{R}$.

PROOF OF LEMMA 3.2. Let $F^t_r(c)$ denote the value, at date t, of a claim paying out one unit of consumption if, and only if, the aggregate consumption level is less than or equal to c at date r. By Proposition 2.4

$$F^t_r(c) = \frac{E[u_{\lambda c}(\hat{c}^0_r, r)1_{\{\hat{c}^0_r \leq c\}}|\mathcal{F}_t]}{u_{\lambda c}(\hat{c}^0_t, t)}.$$

$F_r^t(c)$ is $\mathcal{H}_t$-measurable by Lemma A.1. Moreover, $F_r^t(\cdot)$ is non-decreasing and right continuous.

By the Radon-Nikodym Theorem and the assumption (cf. Assumption 2.2) that the conditional marginal distribution for aggregate consumption, $P(\hat{c}_r^0 \in \cdot|\mathcal{F}_t)$, is absolutely continuous, a non-negative, $\mathcal{H}_t$-measurable function, $p_r^t(\cdot)$, exists such that

$$F_r^t(c) = \int_{-\infty}^{c} p_r^t(u)du. \tag{6}$$

□

PROOF OF LEMMA 3.3. By Proposition 2.4

$$\begin{aligned} S_t^\Delta &= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_t^T u_{\lambda c}(\hat{c}_r^0, r) d\Delta_r \middle| \mathcal{F}_t\right] \\ &= \int_t^T E\left[\frac{u_{\lambda c}(\hat{c}_r^0, r)\delta_r}{u_{\lambda c}(\hat{c}_t^0, t)} \middle| \mathcal{F}_t\right] dr \\ &= \int_t^T E\left[\frac{u_{\lambda c}(\hat{c}_r^0, r) E[\delta_r|\mathcal{F}_t \vee \tilde{\mathcal{G}}_r]}{u_{\lambda c}(\hat{c}_t^0, t)} \middle| \mathcal{F}_t\right] dr. \end{aligned}$$

Using (i) the conditional density derived in Equation (6) and (ii) the relation from Hoffmann-Jørgensen (1994, pp. 448–449) between conditional probabilities and conditional probabilities *given* a fixed aggregate consumption level, c, yields Equation (4). □

PROOF OF LEMMA 3.4. In order to show that N, defined in Equation (5), is a Radon-Nikodym derivative we must show that N is a non-negative $(\mathbf{F}, P)$-martingale with $E[N_T] = 1$. Note that from Proposition 2.4 the real price of the numeraire security is given by

$$S_t^{n_0} = \frac{1}{q_t} E[q_T|\mathcal{F}_t]. \tag{7}$$

$E[q_T]$ is well-defined since q_T is bounded. This follows from the assumption that $\hat{c}^0$ is uniformly bounded away from zero and that $u_{\lambda c}(\cdot, T)$ is decreasing and strictly positive. Multiplying both sides of Equation (7) with q_t yields that N is a non-negative $(\mathbf{F}, P)$-martingale. Moreover, it follows from Equation (7), Lemma A.1, and the definition of N that both S^{n_0} and N are $\mathbf{H}$-adapted.

From the expression of the Radon-Nikodym derivative it follows that

$$S_t^\Delta = \frac{1}{q_t} E\left[\int_t^T q_s d\Delta_s \middle| \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= E\left[\int_t^T \frac{q_s}{q_t} d\Delta_s \middle| \mathcal{F}_t\right] \\
&= E\left[\int_t^T \frac{N_s}{N_t} \frac{S_t^{n_0}}{S_s^{n_0}} d\Delta_s \middle| \mathcal{F}_t\right] \\
&= E^Q\left[\int_t^T \frac{S_t^{n_0}}{S_s^{n_0}} d\Delta_s \middle| \mathcal{F}_t\right] \\
&= S_t^{n_0} E^Q\left[\int_t^T \frac{1}{S_s^{n_0}} d\Delta_s \middle| \mathcal{F}_t\right]
\end{aligned}
$$

The nominal price formula follows directly from substitution of the definitions of nominal terms, i.e.

$$
\begin{aligned}
\tilde{S}_t^{\Delta} &= E^Q\left[\int_t^T d\tilde{\Delta}_s \middle| \mathcal{F}_t\right] \\
&= E^Q[\tilde{\Delta}_T | \mathcal{F}_t] - \tilde{\Delta}_t.
\end{aligned}
$$

Since $\tilde{G}_t^{\Delta} = \tilde{S}_t^{\Delta} + \tilde{\Delta}_t$, it follows that $\tilde{G}^{\Delta}$ is an $(\mathbf{F}, Q)$-martingale. □

PROOF OF PROPOSITION 3.5. Define the cumulative consumption process, C^i, of agent i by

$$
C_t^i := \int_0^t c_s^i ds.
$$

Using Lemma 3.3 and Proposition 2.4, agent i's wealth at date t can be written as

$$
\begin{aligned}
w_t^i &= S_t^{C^i} = \int_t^T \int_{\mathbb{R}} p_r^t(c) E[c_r^i | \hat{c}_r^0 = c, \mathcal{F}_t] dc\, dr \\
&= \int_t^T \int_{\mathbb{R}} p_r^t(c) k^i(c, r) dc\, dr. \qquad (8)
\end{aligned}
$$

Lemma 3.2 then gives that w^i is adapted to the filtration $\mathbf{H}$. □

PROOF OF COROLLARY 3.6. If $\hat{c}^0$ is Markov, then the process $\{P(\hat{c}_{r\rightarrow}^0 | \mathcal{F}_s)\}_{s\in[0,r)} = \{P(\hat{c}_{r\rightarrow}^0 | \hat{c}_s^0)\}_{s\in[0,r)}$ is adapted to the filtration $\mathbf{G}$, for all $r \in \mathbb{T}$. That is, $\mathcal{H}_t \subseteq \mathcal{G}_t$, for all $t \in \mathbb{T}$. Hence the two filtrations $\mathbf{G}$ and $\mathbf{H}$ are identical. □

PROOF OF PROPOSITION 4.2. The proof goes in two steps:

(i) Allocate trading strategies to each agent which generate that agent's equilibrium consumption process and which clears markets.

(ii) Show that no agent has any incentive to deviate from the allocated trading strategy.

Step i: Choose an agent, $i_0 \in \mathcal{I}$. Define $\mathcal{I}_0 := \mathcal{I} \backslash \{i_0\}$. For each agent, $i \in \mathcal{I}_0$, let $n_t^i = c_t^i - \hat{c}_t^i$ be the net-consumption requirement from the agent's trading strategy. Since c^i is adapted to $\mathbf{G}$ by Proposition 2.4, and $\hat{c}^i$ is adapted to $\mathbf{G}$ by assumption, n^i is adapted to $\mathbf{G}$. Denote agent i's portfolio wealth process as $X^i := \theta^i \cdot (S + \Delta D)$. Clearly, the portfolio wealth is going to finance all future net-consumption requirements. Hence, using Proposition 2.4, X^i is given by

$$\begin{aligned} X_t^i &= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_t^T u_{\lambda c}(\hat{c}_r^0, r) n_r^i dr \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} \int_t^T E[u_{\lambda c}(\hat{c}_r^0, r) n_r^i | \mathcal{F}_t] dr. \end{aligned} \tag{9}$$

Since n^i is adapted to the filtration $\mathbf{G}$ and the σ-field $\mathcal{H}_t$ includes conditional probabilities of $\mathbf{G}$-adapted processes, it follows that $E[u_{\lambda c}(\hat{c}_r^0, r) n_r^i | \mathcal{F}_t]$ is $\mathcal{H}_t$-measurable. Hence, by Equation (9), X is adapted to the filtration $\mathbf{H}$.

Define the cumulative net consumption requirements as

$$N_t^i := \int_0^t n_s^i ds.$$

Since N^i is $\mathbf{H}$-adapted, it then follows from Lemma 3.4 that

$$M_t^i := \tilde{X}_t^i + \tilde{N}_t^i = E^Q[\tilde{N}_T^i | \mathcal{F}_t] \tag{10}$$

is an $\mathbf{H}$-adapted $(\mathbf{F}, Q)$-martingale. Hence, an $\mathbf{F}$-previsible trading strategy, ψ_i, exists such that

$$M_t^i = \tilde{X}_0^i + \int_0^t \psi_s^i \cdot d\tilde{G}_s^{\mathcal{M}}, \tag{11}$$

since by assumption $\tilde{G}^{\mathcal{M}}$ forms an $(\mathbf{F}, Q)$-martingale generator for $\mathbf{H}$. Without loss of generality we can choose $\psi_T^i = 0$. Recall that ψ^i is augmented with zeros outside the set of coordinates $\mathcal{M}$.[17] ψ^i is agent i's new trading strategy in the financial assets contingent on aggregate consumption. The only thing missing is to specify the amount traded in the numeraire security such that

$$\tilde{X}^i = \psi^i \cdot (\tilde{S} + \Delta\tilde{D}) = \psi^i \cdot (\tilde{S}^{\mathcal{M}} + \Delta\tilde{D}^{\mathcal{M}}) + \psi^{in_0}.$$

Consequently, ψ^{in_0} must be determined as

$$\psi_t^{in_0} := \tilde{X}_t - \psi_t^i \cdot (\tilde{S}_t^{\mathcal{M}} + \Delta\tilde{D}_t^{\mathcal{M}}), \qquad t \in \mathbb{T}.$$

[17] In particular, $(\psi^i)^{\mathcal{J}} = 0$, sincc agent i is now only trading in the set of financial assets, $\mathcal{N} \supseteq \mathcal{M}$.

It follows that the new trading strategy, ψ^i, is budget feasible since by Equation (10) and (11)

$$\begin{aligned}\tilde{X}_t^i &= \int_0^t \psi_s^i \cdot d\tilde{G}_s^{\mathcal{M}} - \tilde{N}_t^i + \tilde{X}_0^i \\ &= \int_0^t \psi_s^i \cdot d\tilde{G}_s - \tilde{N}_t^i + \tilde{X}_0^i \\ &= \int_0^t \psi_s^i \cdot d\tilde{G}_s - \int_0^t \pi_s n_s^i ds + \hat{\theta}^i \cdot \tilde{S}_0^{\mathcal{J}}.\end{aligned}$$

That $\int_0^t \psi_s^i \cdot d\tilde{G}_s^{\mathcal{M}} = \int_0^t \psi_s^i \cdot d\tilde{G}_s$ follows since $\tilde{G}^{n_0}$ is identically one by construction of the numeraire security.

Thus, we have shown that the trading strategy, ψ_i, implements the equilibrium consumption process, c^i, of agent $i \in \mathcal{I}_0 = \mathcal{I}\backslash\{i_0\}$. The trading strategies of agent i_0 is such that all asset markets clear:

$$\psi^{\mathcal{J}i_0} = 1 - \sum_{i \in \mathcal{I}_0} \psi^{\mathcal{J}i} = 1,$$

$$\psi^{\mathcal{N}i_0} = -\sum_{i \in \mathcal{I}_0} \psi^{\mathcal{N}_i}.$$

Step ii: By construction, agents $i \in \mathcal{I}_0$ have no incentive to deviate from their allocated trading strategies: they precisely implement their equilibrium consumption process, and that process is weakly preferred to any other budget feasible consumption process. Similarly, we need to show that agent i_0's trading strategy implements her equilibrium consumption process. But this follows from the linearity of stochastic integrals and market clearing of the equilibrium consumption processes both with the old and the new trading strategies. □

PROOF OF PROPOSITION 5.1. First, we show that if the (exogenously given) filtration $\mathbf{H}$ is generated by Wiener processes, then contingent claims on aggregate consumption exist forming a martingale generator for $\mathbf{H}$. Note that in Example 3.1, the filtration $\mathbf{H}$ is generated by the Wiener process (W_1, W_2), such that for this example (two) contingent claims on aggregate consumption exist which form a martingale generator for $\mathbf{H}$.

LEMMA A2. Let an equilibrium, $(S, \{(c^i, \theta^i)\}_{i\in\mathcal{I}})$, for a dynamically complete market be given. Suppose $\mathbf{H}$ is generated by some linear function of the Wiener process, i.e. $\mathcal{H}_t = \sigma(\{AW_s\}_{s\in[0,t]})$, where A is a constant matrix. Then financial assets contingent on aggregate consumption can be constructed such that their gain processes form a martingale generator for $\mathbf{H}$.

PROOF OF LEMMA A.2. Define $\hat{N}$ such that

$$N_t = e^{\hat{N}_t - 1/2\langle \hat{N} \rangle_t},$$

where N is the Radon-Nikodym derivative given in Lemma 3.4. Then $M := AW - \langle \hat{N}, AW \rangle$ is an $(\mathbf{F}, Q)$-martingale generator for $\mathbf{H}$ by Itô's representation theorem. We show that a set of contingent claims on aggregate consumption can be constructed such that the gain process of this set forms the martingale generator M.

Define a set of contingent claims on aggregate consumption, $\mathcal{Q}$, by the cumulative nominal dividend process, $\tilde{D}^{\mathcal{Q}}$, such that

$$\tilde{D}^{\mathcal{Q}}_T = M_T,$$

with $\tilde{D}^{\mathcal{Q}}_t = 0$, for $t < T$. By Lemma 3.4 the equilibrium nominal date t price is

$$\tilde{S}^{\mathcal{Q}}_t = E^Q[\tilde{D}^{\mathcal{Q}}_T | \mathcal{F}_t].$$

Since the contingent claims only pay dividends at date T, the gain process of this set of contingent claims is equal to the price process. Hence,

$$\begin{aligned} \tilde{G}^{\mathcal{Q}}_t &= E^Q[\tilde{D}^{\mathcal{Q}}_T | \mathcal{F}_t] \\ &= E^Q[M_T | \mathcal{F}_t] \\ &= M_t. \end{aligned}$$

Now consider an otherwise identical economy, $\mathcal{E}_1$, in which the original economy, $\mathcal{E}$, is augmented with financial assets such that Assumption 2.3 is fulfilled. As shown in Duffie and Zame (1989) an equilibrium for the dynamically complete economy $\mathcal{E}_1$ exists. It then follows from Lemma A.2 that financial assets contingent on aggregate consumption can be constructed such that their gain processes form a martingale generator for $\mathbf{H}$. Add these financial assets to the economy $\mathcal{E}_1$ and term this economy $\mathcal{E}_2$. Note that the allocations and the price processes of the securities in $\mathcal{E}_1$ are the same as in $\mathcal{E}_2$. Proposition 4.2 then implies that the equilibrium allocation in $\mathcal{E}_1$ can be implemented in $\mathcal{E}_2$ by trading only in the market portfolio of firms, the contingent claims on aggregate consumption added in $\mathcal{E}_2$, and in the numeraire zero-coupon bond. Hence, an equilibrium for the economy $\mathcal{E}$ exists augmented with *only* the financial assets contingent on aggregate consumption and the numeraire zero-coupon bond. □

PROOF OF PROPOSITION 6.1. Choose $l \in \mathcal{L}$. By Proposition 2.4

$$S^l_t = \frac{1}{u_{\lambda c}(\hat{c}^0_t, t)} E\left[\int_t^T u_{\lambda c}(\hat{c}^0_s, s) dD^l_s \middle| \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_{t_1}^{T} u_{\lambda c}(\hat{c}_s^0, s)\, dD_s^l \middle| \mathcal{F}_t\right] \\
&= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_{t_1}^{T} u_{\lambda c}(\hat{c}_s^0, s)\, dD_s^l \middle| \mathcal{H}_{t_0}\right] \\
&= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} K_{t_0}^l, \qquad t \in [t_0, t_1],
\end{aligned}
$$

where $K_{t_0}^l$ is an $\mathcal{H}_{t_0}$-measurable stochastic variable. Since the long-term assets pay no dividends in the time interval $[t_0, t_1]$, the gain process of these assets are given by

$$
\begin{aligned}
\tilde{G}_t^l &= \frac{\pi_t}{u_{\lambda c}(\hat{c}_t^0, t)} K_{t_0}^l + \tilde{D}_{t_0}^l \\
&= K_{t_0}^l M_t + \tilde{D}_{t_0}^l, \qquad t \in [t_0, t_1],
\end{aligned}
$$

where M is a one-dimensional $(\mathbf{F}, Q)$-martingale according to Lemma 3.4:

$$
\begin{aligned}
E^Q[M_s|\mathcal{F}_t] &= E\left[\frac{\pi_s}{u_{\lambda c}(\hat{c}_s^0, s)} \frac{N_s}{N_t} \middle| \mathcal{F}_t\right] \\
&= E\left[\frac{\pi_s}{q_s} \frac{q_s}{q_t} \frac{\pi_t}{\pi_s} \middle| \mathcal{F}_t\right] \\
&= \frac{\pi_t}{q_t} E[1|\mathcal{F}_t] \\
&= M_t, \qquad t < s.
\end{aligned}
$$

Hence, the set of gain processes of the assets in the set $\mathcal{L}$, $\{\{\tilde{G}_t^l\}_{t\in[t_0,t_1]}\}_{l\in\mathcal{L}}$, has a Q-martingale multiplicity of one in the time interval $[t_0, t_1]$. □

PROOF OF PROPISITION 6.2. Choose $l \in \mathcal{L}$. By Proposition 2.4

$$
\begin{aligned}
S_t^l &= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_{t}^{T} u_{\lambda c}(\hat{c}_s^0, s)\, dD_s^l \middle| \mathcal{F}_t\right] \\
&= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_{t}^{t_1} u_{\lambda c}(\hat{c}_s^0, s)\, dD_s^l \middle| \mathcal{F}_t\right] \\
&= \frac{1}{u_{\lambda c}(\hat{c}_t^0, t)} E\left[\int_{t}^{t_1} u_{\lambda c}(\hat{c}_s^0, s)\, dD_s^l \middle| \mathcal{H}_{t_0}\right], \qquad t \in [t_0, t_1],
\end{aligned}
$$

since aggregate consumption follows a deterministic path in the time interval $[t_0, t_1]$. Hence, converting into nominal terms the price process of asset l is

$$
\tilde{S}_t^l = \frac{q_t}{E[q_T|\mathcal{F}_t]} S_t^l = \frac{K_t^l}{E[q_T|\mathcal{F}_t]},
$$

where K_t^l is an $\mathcal{H}_{t_0}$-measurable stochastic variable for all t in $[t_0, t_1]$. Since

$$K_t^l = \tilde{S}_t^l E[q_T | \mathcal{F}_t],$$

then K^l is a continuous $\mathbf{F}$-semi-martingale in the time interval $[t_0, t_1]$. Decompose K^l in a $\mathbf{F}$-local martingale and a bounded variation process. Both elements of the decomposition is adapted to the trivial filtration $\{\mathcal{H}_{t_0}\}_{t\in[t_0,t_1]}$. By a result of Dellacherie and Meyer (1982) the local martingale of the decomposition is then a $\{\mathcal{H}_{t_0}\}_{t\in[t_0,t_1]}$-local martingale and hence almost surely constant. Hence K^l is of bounded variation. The gain process of the asset l is

$$\tilde{G}_t^l = \frac{K_t^l}{E[q_T | \mathcal{F}_t]} + \tilde{D}_t^l \qquad t \in [t_0, t_1].$$

Since $\tilde{G}^l$ is an $(\mathbf{F}, Q)$-martingale according to Lemma 3.4 and both K^l and $\tilde{D}^l$ are of bounded variation, the gain process can also be written

$$\tilde{G}_t^l = A_{t_0}^l M_t + B_{t_0}^l, \qquad t \in [t_0, t_1],$$

where $A_{t_0}^l$ and $B_{t_0}^l$ are $\mathcal{H}_{t_0}$-measurable stochastic variables and M is an $(\mathbf{F}, Q)$-martingale. M is the martingale part of $\{\frac{1}{E[q_T|\mathcal{F}_t]}\}_{t\in[t_0,t_1]}$ and, therefore, independent of l. Hence, the set of gain processes of the assets in the set $\mathcal{L}$, $\{\{\tilde{G}_t^l\}_{t\in[t_0,t_1]}\}_{l\in\mathcal{L}}$, has a Q-martingale multiplicity of one in the time interval $[t_0, t_1]$. □

PROOF OF PROPOSITION 7.2. The proof is similar to the proof of Proposition 4.2. The only change is in Equation (9).

$$\begin{aligned} X_t^i &= \int_t^T \int_{c\in\mathfrak{C}_t^r} p_r^t(c) E[n_r^i | \hat{c}_r^0 = c, \mathcal{F}_t] dc\, dr \\ &= \int_t^T \int_{c\in\mathfrak{C}_t^r} p_r^t(c) E[c_r^i | \mathcal{F}_t] dc\, dr \\ &\quad - \int_t^T \int_{c\in\mathfrak{C}_t^r} p_r^t(c) E[\hat{c}_r^i | \hat{c}_r^0 = c, \mathcal{F}_t] dc\, dr. \end{aligned} \tag{12}$$

Since $\mathcal{H}_t \subseteq \mathcal{P}_t$ and $p_r^t(c) E[\hat{c}_r^i | \hat{c}_r^0 = c, \mathcal{F}_t]$ is $\mathcal{P}_t$-measurable, Equation (12) shows that X^i is adapted to the filtration $\mathbf{P}$. The proof then proceeds as in the proof of Proposition 4.2. □

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