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Pricing and Hedging Discount Bond Options in the Presence of Model Risk *

F. S. LHABITANT
Union Bancaire Privee (Switzerland) and Thunderbird, The American Graduate School of International Management. E-mail: lhabitant@attglobal.net. Author for correspondence

C. MARTINI and A. REGHAI
INRIA Sophia - Antipolis, France

Abstract. This paper focuses on pricing and hedging options on a zero-coupon bond in a Heath–Jarrow–Morton (1992) framework when the value and/or functional form of forward interest rates volatility is unknown, but is assumed to lie between two fixed values. Due to the link existing between the drift and the diffusion coefficients of the forward rates in the Heath, Jarrow and Morton framework, this is equivalent to hedging and pricing the option when the underlying interest rate model is unknown. We show that a continuous range of option prices consistent with no arbitrage exist. This range is bounded by the smallest upper-hedging strategy and the largest lower-hedging strategy prices, which are characterized as the solutions of two non-linear partial differential equations. We also discuss several pricing and hedging illustrations.

1. Introduction

Fixed income and interest rate contingent claims markets have grown tremendously in the last two decades, both in their volume of trading and in the breadth of coverage of underlying markets. Several sophisticated products have been created to satisfy the various needs of investors. When a financial institution buys or sells such contracts, two questions arise: which price to charge or pay, and how to offset the risk?

Since the early works of Merton (1973) and Black and Scholes (1973), option pricing and hedging are inextricably linked. The essence of contingent claim pricing resides in the ability to replicate exactly a complex payoff by a dynamic self-financing strategy using more basic securities. If such a strategy exists, then,

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it will hedge the option, and its initial cost should be equal to the option price to prevent arbitrage opportunities.

In practice, however, things are far less simple. Both the option price and its replicating strategy depend on the assumption of a parametric model to describe the dynamics of the underlying asset. This includes identifying relevant state variables, specifying their transition probabilities, determining their associated risk premiums, and estimating the values of some unknown – and sometimes unobservable – parameters. The fairness of the price and/or the success of the hedge will depend crucially on the validity of the underlying model and on the efficiency of the estimation procedure.

In the interest rate derivatives area, a plethora of continuous-time models are simultaneously used among academics and practitioners, with little agreement on which one is preferable.[1] Most of them are chosen for their analytical tractability – they provide a simple formula for the option price and explicit option hedging ratios – but they represent reality imperfectly or rely on economically unrealistic assumptions. As a consequence, the path of the underlying asset to be realized will almost never be drawn from the possible paths of the selected model. This model risk yields to a structural incompleteness of the market that precludes the exact pricing, hedging or replication of contingent claims. Therefore, it seems natural to look for alternative pricing and hedging strategies that would work regardless of the realized model among a set of "possible" ones (which is not reduced as usual to a singleton).

In this paper, we will demonstrate that in the general framework of Heath, Jarrow, and Morton (1992), the ability to bound the forward interest rate curve's volatility in an interval allows us to determine a range of discount bond option prices consistent with the no-arbitrage condition. This is possible even without knowing the exact stochastic process followed by the forward interest rates. Due to the essential role played by volatility in the Heath, Jarrow and Morton framework and also in derivative pricing, the model we will elaborate provides an answer to both model risk and estimation risk. Since discount bond options serve as building blocks for other instruments such as options on yields or caps, floors and collars, the extension to these instruments is straightforward.

Our approach relies on the change of numeraire technique and on stochastic control theory. Rather than replicating the payoff of an option, we propose a model of super-replication (or super-hedging), and find the lowest price for an upper-hedging strategy and the highest price for a lower-hedging strategy. These non trivial super-hedging costs are characterized by a terminal value problem involving a non-linear partial differential equation, and correspond to worst-case scenarios of future volatility for hedging a long or a short position in the option. Our model is an extension of the results of Avellaneda, Levy and Paras (1995) and Lyons (1995) on stock derivatives, and sheds a new light on the particular case considered in

[1] See Gibson et al. (1998) for a survey.

Avellaneda and Lewicki (1996), in which the volatility function for forward rates is not allowed to depend on maturity.

This paper is structured as follows: Section 2 introduces the Heath, Jarrow and Morton (1992) framework and develops the concepts of super-hedging and of uncertain volatility models. Section 3 presents our model and discusses its major implications. Section 4 illustrates our methodology with examples of pricing and hedging several discount bond contingent claims. Finally, section 5 outlines future research and possible extensions.

2. Framework and Tools

2.1. THE FRAMEWORK

For ease of exposition, this section summarizes the single stochastic factor Heath, Jarrow and Morton (1992) framework (hereafter HJM). The fundamental building block of the HJM approach is the instantaneous forward rate curve. Let $f_t(\theta)$ be the instantaneous forward interest rate at date t (i.e. the rate set at date t for instantaneous risk-free borrowing or lending at date $\theta \geqslant t$). At each trading date, the forward rate curve $\{f_t(\theta)\}_t$ satisfies the following stochastic differential equation

$$df_t(\theta) = \alpha(\theta)_t\, dt + \sigma(\theta)_t\, dW_t \tag{1}$$

where W_t is a one-dimensional Brownian motion, and $\alpha(\theta)_t$ and $\sigma(\theta)_t$ are the drift and diffusion coefficients, which can depend on time (t), maturity (θ), or the level of the forward rates curve ($f_t(\theta)$). A boundary value, $f_0(\theta)$ is set equal to the observed initial forward curve, so that the model perfectly fits the observed term structure.

As shown by HJM, the model (1) is entirely specified by the volatility structure[2] $\sigma(\theta)_t$, because the no-arbitrage condition links the drift to the volatility:

$$\alpha(\theta)_t = \sigma(\theta)_t \int_t^\theta \sigma(s)_t ds \tag{2}$$

The HJM specification of the forward rates is equivalent to specifying the entire family of discount bond prices, since

$$P_t(\theta) = \exp\left[-\int_t^\theta f_t(s)ds\right]$$

[2] Note that a particular forward rate is the spot interest rate r_t, which is defined as

$$r_t \equiv \lim_{\theta \to t} f_t(\theta)$$

Therefore, one can show that all popular short-term interest rate models correspond to an HJM model with a particular specification of the volatility. For instance, a constant volatility results in the continuous time limit of the Ho and Lee (1986) and Jamshidian (1990) model, while an exponential form gives the Vasicek (1977) model. For other specifications, see Gibson, Lhabitant and Talay (1998).

where $P_t(\theta)$, $t \leq \theta$ is the price at time t of a discount bond maturing and paying one currency unit at time θ. Equivalently, the bond price dynamics can be written as

$$\frac{dP_t(\theta)}{P_t(\theta)} = r_t\,dt + S(\theta)_t\,dW_t \tag{3}$$

where r_t is the risk-free instantaneous rate, and $S(\theta)_t$ is linked to $\sigma(\theta)_t$ by

$$S(\theta)_t = -\int_t^\theta \sigma(s)_t\,ds \tag{4}$$

2.2. OPTION PRICING

In this framework, we are interested in pricing a European path-independent contingent claim with maturity T on a zero coupon bond with maturity $\theta \geqslant T$. This contingent claim settles at time T at a price $\varphi(P_T(\theta))$, which is known contingent upon $P_T(\theta)$. For instance, a call option with strike K would correspond to $\varphi(P_T(\theta)) = Max[P_T(\theta) - K, 0]$.

Under the assumption of a particular volatility function $\sigma(\theta)_t$, the prevailing HJM model is completely specified, we are in a complete market in the Harrison and Pliska (1981) sense, and there exists a self-financing dynamic trading strategy V that exactly replicates the contingent claim, e.g. $V_T = \varphi(P_T(\theta))$. At time t, the contingent claim price will be equal to V_t, the value of the replicating portfolio. Depending on the particular volatility function specification, the contingent claim price and the replicating strategy parameters are obtained either analytically, or in a numerical form (see Goldstein (1997)).

In practice, however, it is difficult to identify the true prevailing HJM model, or equivalently, its volatility structure. Since volatility is non-observable, the most one has is an estimate of its value, dependent upon a model and upon a particular estimation method. However, several models of the HJM family have been tested in the literature (see for instance Amin and Morton (1994) or Buhler et al. (1999)) and they all evinced strong patterns of mispricing.

Since the replicating strategy V is crucially dependent on the specified HJM model and on the exact value of its parameters, there is a considerable risk of model misspecification, with potentially dramatic consequences for pricing and hedging.[3] Therefore, the exact replication argument used by traditional option pricing models is no longer valid. This is where arbitrage bounds and upper and lower hedging come into action.

[3] See for instance Bossy et al. (1998).

2.3. UPPER AND LOWER HEDGING

Rather than looking for a unique price in a particular model, upper and lower hedging attempt to bound the contingent claim price in a limited range, which is valid for a series of models.

Following Delbaen (1992) and El Karoui and Quenez (1995), we define the upper bound W_t^{up} as the minimum initial investment to implement a self-financing investment strategy V that is enough to replicate a long position in the claim (i.e., to upper-hedge a short position in a claim)

$$W_t^{\text{up}} = \inf\{V_t \,|\exists V \text{ self financing:} V_T \geqslant \varphi(P_T(\theta))\}$$

By a similar argument (the buyer of a contingent claim is a seller of minus the same claim), we define the lower bound W_t^{low} as the minimum initial investment to implement a self-financing investment strategy V that is enough to replicate a short position in the claim (i.e., to lower-hedge a long position in a claim).

$$W_t^{\text{low}} = -\inf\{V_t \,|\exists V \text{ self financing:} V_T \geqslant -\varphi(P_T(\theta))\}$$

How can we interpret these two bounds? If a trader is infinitely risk averse, it is clear that W_t^{up} will be his minimum selling price and W_t^{low} his maximum buying price. If there are several similar traders competing on the same contingent claim, by arbitrage, these two bounds will be the bid and ask prices. Between these two bounds, trading options is risky.

Upper-hedging and lower-hedging appear to be a very attractive concept. However, in our case, one has to remember that the arbitrage bounds will be determined by a hedging strategy implemented in the worst case scenario for the forward rate volatility path. As a consequence, the cost of upper-hedging a short European call position will often be equal to the underlying price itself, which implies that the cheapest upper-replicating strategy consists trivially in holding the underlying asset, and the cost of lower-hedging will often be zero.

In order to obtain non trivial bounds, we must reduce the number of possible scenarios, for instance by restricting the possible values for the forward rate volatility. This is the goal of uncertain volatility models.

2.4. UNCERTAIN VOLATILITY MODELS

Uncertain volatility models were concurrently and independently developed by Avellaneda, Levy and Paras (1995) and Lyons (1995) as extensions of the Black and Scholes (1973) framework for stocks. They provide arbitrage bounds for stock contingent claims when the volatility of the underlying asset is unspecified, but remains within a known range of values.

More recently, Avellaneda and Lewicki (1996) have applied their approach to interest rates contingent claims. We briefly recall their major findings. In an HJM framework, they do not specify the functional form of the forward rate volatility $\sigma(\theta)_t$, but restrict it with two important assumptions:

1. $\sigma(\theta)_t$ fluctuates between two known finite and constant bounds,

$$0 \leq \sigma_{\min} \leq \sigma(\theta)_t \leq \sigma_{\max}$$

2. $\sigma(\theta)_t$ does not depend on θ, the maturity of the forward rate $f_t(\theta)$.

Under these assumptions, Avellaneda and Lewicki derive the dynamics of the spot rate r_t and show that the bond price is

$$P_t(\theta) = \exp\left(-r_t(\theta - t) - \frac{V_t}{2}(\theta - t)^2 - \int_t^\theta (f(s) - f(t))ds\right) \tag{5}$$

and that the smallest upper-hedging price of an interest rate contingent claim – denoted F – solves the following PDE:

$$\frac{\partial F}{\partial t} + (f'(t) + V_t)\frac{\partial F}{\partial r} + \tau^2\left[\frac{\partial F}{\partial V} + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\right]\left(\frac{\partial F}{\partial V} + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\right) - rF = 0 \tag{6}$$

where $V_t = \int_0^t \sigma^2(s)\,ds$ is the "accumulated variance" up to time t, $f(t) = f_0(t)$ is the forward rate curve observed at time 0, $f'(t)$ is its first order variation, and $\tau[x]$ is defined as follows:

$$\tau[x] = \begin{cases} \sigma_{\max} \text{ if } x \geq 0 \\ \sigma_{\min} \text{ if } x < 0 \end{cases} \tag{7}$$

Equation (6) has two state variables: it is parabolic in r and hyperbolic in V. The authors suggest a methodology to solve it numerically using an adapted version of their trinomial lattice for stocks.

The major drawback of the Avellaneda and Lewicki model is the maturity independence assumption for the volatility function. This restriction rules out almost all of the popular short term interest rate models such as Vasicek (1977), Cox, Ingersoll and Ross (1985) or Hull and White (1990), with the exception of the continuous-time version of the Ho and Lee (1986) model, which uses a constant volatility. In addition, it is inconsistent with the empirical observation that the volatility level depends on both maturity and interest rates level (see for instance Chan et al (1992) or Bühler et al (1999)).

To avoid this, our model will simply relax this assumption.

3. Bounds on a Discount Bond Option

3.1. OUR MODEL

We adopt the general HJM family of single-factor models. Similarly to Avellaneda and Lewicki, we restrict the (unknown) volatility value to remain between two known finite and constant bounds,[4]

$$0 \leq \sigma_{\min} \leq \sigma(\theta)_t \leq \sigma_{\max}$$

but we do not make any other assumption on the functional form of the volatility.

The "natural" numeraire at any time is the current currency unit. However, a clever change of numeraire can greatly reduce the computational work needed to obtain arbitrage-free prices or hedging parameters. Furthermore, since the work of Geman, El Karoui and Rochet (1995), it is also firmly established that a change of numeraire does not affect the self-financing property of a portfolio and does not change the replicating or hedging portfolios either. Hereafter, we will alternatively work under the original numeraire, where the bond price will be denoted $P_t(\theta)$, under the money-market numeraire, with

$$Z_t(\theta) = \frac{P_t(\theta)}{\exp(\int_0^t r_u du)}$$

or under the forward numeraire with

$$P_t^F(\theta) = \frac{P_t(\theta)}{P_t(T)}$$

Consider first upper-hedging a short contingent claim position. Since we don't know $\sigma(\theta)_t$, we are looking for an alternative pricing function $W_t^{\mathrm{up}} \equiv W^{\mathrm{up}}(P_t^F(\theta), t)$, such that

$$W^{\mathrm{up}}(P_T^F(\theta), T) \geq \varphi(P_T^F(\theta)) \tag{8}$$

For a function $W^{\mathrm{up}}(P_t^F(\theta), t)$ which is regular enough, Ito's lemma yields

$$\begin{aligned} W^{\mathrm{up}}(P_t^F(\theta), t) &= W^{\mathrm{up}}(P_0^F(\theta), 0) + \int_0^t \frac{\partial W^{\mathrm{up}}(x, s)}{\partial s} dP_s^F(\theta) \\ &\quad + \int_0^t \left(\frac{\partial W^{\mathrm{up}}(x, s)}{\partial s} + \frac{1}{2} \frac{d < P^F(\theta) >_s}{ds} \frac{\partial^2 W^{\mathrm{up}}(x, s)}{\partial x^2} \right) ds \end{aligned} \tag{9}$$

Equation (9) has an interesting economic interpretation. The left-end side is the theoretical value of our upper-hedging strategy. The first two terms at the right-hand side represent the value of a standard delta-hedging strategy.[5] The last term

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[4] There exists several pragmatic procedures to obtain these two bounds. For instance, one can take the extreme values of implied volatilities, some lower and upper quantiles of historical or implied volatily distribution, or the confidence interval around a volatility estimator (given an interest rate model and an estimation procedure).

[5] The hedging strategy is based on the W^{up} pricing function. The two instruments used to replicate the claim are the discount bonds with maturity θ and T. Since we are in the forward numeraire, all $P_t^F(T)$ related terms disappear.

can be interpreted as a tracking error, that is, as the difference between the required value $W^{\mathrm{up}}(P_t^F(\theta), t)$ and the results provided by the delta-hedging strategy.

To upper-hedge, we must ensure that this tracking error is never positive. For any t and θ, we must have

$$\int_0^t \left(\frac{\partial W^{\mathrm{up}}(x,s)}{\partial s} + \frac{1}{2} \frac{d < P^F(\theta) >_s}{ds} \frac{\partial^2 W^{\mathrm{up}}(x,s)}{\partial x^2} \right) ds \leqslant 0 \tag{10}$$

Using (3) in the forward numeraire yields

$$d < P^F(\theta) >_t = (S(\theta)_t - S(T)_t)^2 (P_t^F(\theta))^2 \, dt$$

For finite fixed and known T and θ, since $\sigma(\theta)_t$ is bounded, it is easy to show that $(S(\theta)_t - S(T)_t)$ is also bounded:

$$0 \leq (\theta - T)^2 \sigma_{\min}^2 \leq (S(\theta)_t - S(T)_t)^2 \leq (\theta - T)^2 \sigma_{\max}^2$$

Therefore, a sufficient condition for (10) to hold is that

$$\frac{\partial W^{\mathrm{up}}(x,s)}{\partial s} + \frac{1}{2}(\theta - T)^2 \sup_{\sigma \in [\sigma_{\min}, \sigma_{\max}]} \sigma^2 x^2 \frac{\partial^2 W^{\mathrm{up}}(x,s)}{\partial x^2} \leqslant 0 \tag{11}$$

Any trader will be ready to sell the contingent claim for a price $W^{\mathrm{up}}(Z_0^F(\theta), 0)$ that solves (8) and (11). Since we are looking for the smallest upper-hedging strategy (i.e., the minimum initial investment), Equations (8) and (11) can be rewritten as equalities.

A similar procedure can be applied for lower-hedging a long contingent claim position. This leads us to the following proposition, which is the major result of our paper:

PROPOSITION 1. If there exists a regular solution $W^{\mathrm{up}}(x, t)$ which is $C^{2,1}$ to the following system

$$\begin{cases} W^{\mathrm{up}}(x, T) = \varphi(x) \\ \dfrac{\partial W^{\mathrm{up}}}{\partial t} + \frac{1}{2}(\theta - T)^2 \sup\limits_{\sigma \in [\sigma_{\min}, \sigma_{\max}]} \sigma^2 x^2 \dfrac{\partial^2 W^{\mathrm{up}}}{\partial x^2} = 0 \end{cases} \tag{12}$$

then, it is the *smallest upper-hedging strategy* in the model considered so far. As a corollary there is at most one such function. If there exists a regular solution $W^{\mathrm{low}}(x, t)$ which is $C^{2,1}$ to the following system

$$\begin{cases} W^{\mathrm{low}}(x, T) = \varphi(x) \\ \dfrac{\partial W^{\mathrm{low}}}{\partial t} + \frac{1}{2}(\theta - T)^2 \inf\limits_{\sigma \in [\sigma_{\min}, \sigma_{\max}]} \sigma^2 x^2 \dfrac{\partial^2 W^{\mathrm{low}}}{\partial x^2} = 0 \end{cases} \tag{13}$$

then, it is the *highest lower-hedging strategy* in the model considered so far. As a corollary there is at most one such function.

Proof. See appendix. ■

In general, Equations (12) and (13) do not have closed-form solutions and must be solved numerically (for instance, through an explicit finite difference solver). Note that when $\sigma_{\min} = \sigma_{\max} = \sigma(\theta)_t$ (i.e., when there is no volatility uncertainty), Equations (12) and (13) shrink into one simple linear partial differential equation. Its solution is the price of the contingent claim in the continuous-time version of the Ho and Lee (1986) model.

3.2. INTERPRETATION AND DISCUSSION

The control problems (12) and (13) have a nice economic interpretation. At each time-period, the upper and lower hedging strategies simply use the worst case volatility specification, $\sigma_{\min}$ or $\sigma_{\max}$, depending on the local convexity of the contingent claim. This is equivalent to locally price/hedge with a Ho and Lee (1986) model, with the volatility value switching between $\sigma_{\min}$ (locally concave claim) or $\sigma_{\max}$ (locally convex claim). As a consequence, our model will only be useful for contingent claims whose payoff function is not uniquely convex or concave,[6], typically spreads, butterflies, or more complex positions. For such claims, as the underlying asset price changes, the position's gamma may change sign, resulting in switches from $\sigma_{\min}$ to $\sigma_{\max}$. In a sense, the model uses locally $\sigma_{\max}$ if we are "locally short gamma" and $\sigma_{\min}$ if we are "locally long gamma". Note that if the realized volatility is different from the worst case time-series of $\sigma_{\min}$ and $\sigma_{\max}$, the hedger receives a positive (random) income stream. Similar results were obtained by Avellaneda and Lewicki (1996).

What is the difference between our upper-strategy risk-averse price (W^{up}) and Avellaneda and Lewicki's upper-strategy risk-averse price (the F function)? Although the two prices are derived under different conditions, solve very different PDEs, and are expressed in different numeraires, it is possible to show that they are equal (after an adjustment due to the change of numeraire).

PROPOSITION 2. We have the following equality:

$$F(r_t, V_t, t) = P_t(T)\, W^{\text{up}}\Big(\frac{P_t(\theta)}{P_t(T)}, t\Big) \tag{14}$$

Proof. See appendix. ■

However, in addition to a much more general volatility specification, our model offers several advantages over Avellaneda and Lewicki. Equation (12) has only one

[6] See Bergman et al. (1996) and Bergman (1998) for the necessary conditions for the price of an option with a convex payoff profile to be an increasing function of the volatility parameter of the underlying asset.

state variable while (6) has two, including the accumulated variance to date, which is unobservable and difficult to estimate. In addition, in (12), the first order space derivative is deleted and the second order term is neater.

3.3. A NOTE ON HEDGING

In addition to pricing, our model can also provide hedging parameters. Since equations (12) and (13) do not have closed-form solutions, the corresponding upper and lower delta-hedging strategies can only be derived numerically.

These hedging strategies offer two advantages over traditional (parametric) ones. First, they are universal, in the sense that they depend only on the bounds we impose on the volatility and not on a particular volatility functional form. The "true" model which rules the forward rates curve is simply irrelevant, as long as forward rates volatility remains within its pre-specified bounds. Second, our delta hedging strategies protect against any movement of the term structure, whereas specific interest rate models typically delta-hedge against one particular movement. For instance, in a Ho and Lee (1986) model, $\sigma(\theta)_t = \sigma_0$, delta-hedging protects the portfolio against parallel shifts in the forward rates, while in a linear absolute model, $\sigma(\theta)_t = \sigma_0 + \sigma_1(\theta - t)$, delta-hedging means protecting against a twist of the term structure. In our model, we are hedged against any volatility path, and therefore, any movement of the forward rate curve.

3.4. PORTFOLIO CONSIDERATIONS

Another interesting feature of (12) and (13) is the model risk diversification within a portfolio. The bounds obtained at the portfolio level will generally be tighter than the sum of bounds of the individual components. That is,

$$W^{\text{up}}(A+B) \leqslant W^{\text{up}}(A) + W^{\text{up}}(B)$$

and

$$W^{\text{low}}(A+B) \geqslant W^{\text{low}}(A) + W^{\text{low}}(B)$$

In a sense, the worst case for a portfolio can be better, but never worse, than the sum of the worst cases for the individual options. This is due to the cancellation of volatility risk that occurs when the overall portfolio has mixed convexity. Therefore, the capital required to upper-hedge a portfolio of options will be less than the capital required to upper-hedge each of its constituents separately. As a consequence, calculations should be performed at the portfolio level whenever possible. Here again, a similar result was obtained for stocks by Avellaneda, Levy, and Paras.

4. Some Applications

In this section, we examine some empirical applications of our model. We first present a numerical application to the pricing of a zero-coupon bond contingent claim. Next, we show how the methodology can also be applied to options on yields and to caps, floors and collars. Then, we provide an example of the model risk diversification at the portfolio level. Finally, we illustrate the advantage of our model for hedging.

4.1. OPTIONS ON ZERO-COUPON-BONDS

As an illustration, let us consider a call-spread option on a zero-coupon bond maturing at time θ. A call-spread option consists of a long call with exercise price K_1 and of a short call with exercise price $K_2 > K_1$. The terminal payoff of the call-spread is equal to

$$\varphi(P_T(\theta)) = \mathrm{Min}\,[\mathrm{Max}\,[P_T(\theta) - K_1, 0]\,,\; K_2 - K_1] \tag{15}$$

This payoff is both concave and convex with respect to the underlying asset price. By convention, we assume $P_\theta(\theta) = 1$.

Using Equation (12) with $\varphi(x)$ defined as in (15) allows us to compute the upper-strategy risk-averse prices for various sets of parameters. For instance, Figure 1 shows the upper-strategy risk-averse prices we obtain in the case of a six month call-spread on a zero-coupon bond with a maturity varying from one to ten years. The volatility uncertainty is represented by varying $\sigma_{\min}$ from 1% to 30%, while $\sigma_{\max}$ is held constant at 30%. The initial rate curve is upward sloping between 3% (short term rate) and 5% (30-years rate). Clearly, we do not obtain explosive prices, despite a very large volatility uncertainty.

To demonstrate the practical relevance of our approach, Table I presents the results of pricing a 6-month call-spread and put-spread on a 5-year discount bond using (12) and (13). The exercise prices are K_1= 95% and K_2=105% (all prices are expressed as a percentage of the underlying asset initial price). We represent the volatility uncertainty by a symmetric range around a volatility estimate (called the "mid" volatility and denoted $\overline{\sigma}$). The price bounds obtained with $\sigma_{\min} = \sigma_{\max} = \overline{\sigma}$ (i.e., no uncertainty) are the Ho and Lee price, where an explicit dynamic strategy for exact replication is available. When the volatility uncertainty increases, that is, when the range widens, we obtain a bid and an ask price corresponding to the minimum selling price and the maximum buying price for which hedging strategies exist. As one could expect, the price range widens around the Ho and Lee price as the volatility uncertainty becomes larger.

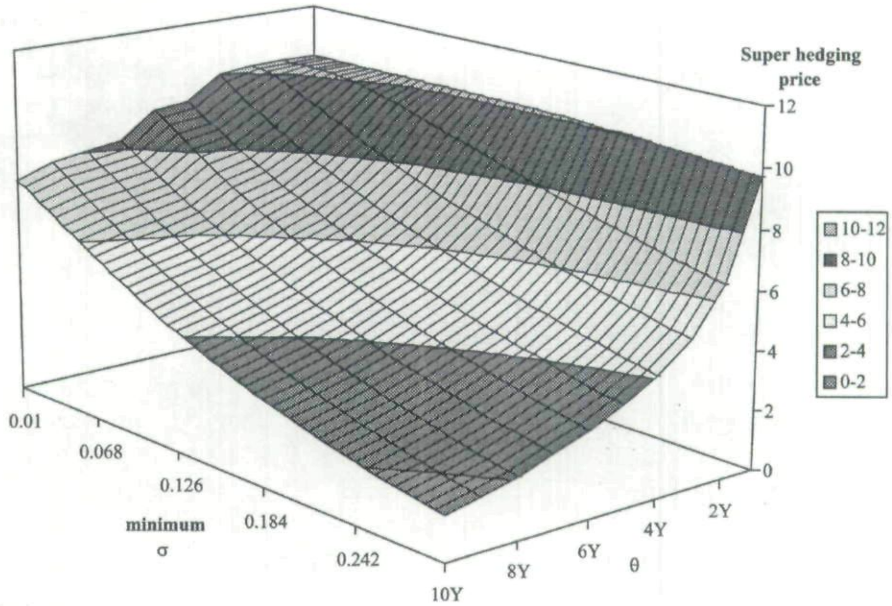

Figure 1. Upper-hedging price for a six-month call-spread on a zero-coupon bond. The maturity of the underlying bond (θ) varies from one to ten years. The volatility uncertainty is represented by an interval $[\sigma_{\min}, \sigma_{\max}]$. For illustrative purposes, we vary $\sigma_{\min}$ from 1% to 30%, while $\sigma_{\max}$ is held constant at 30%. When $\sigma_{\min} = \sigma_{\max} = 30\%$, we obtain the Ho & Lee prices. The initial rate curve is upward sloping between 3% (short term rate) and 5% (30-years rate).

4.2. OPTIONS ON YIELDS AND OTHER ASSETS

Of course, options on zero-coupon bonds are not quoted on the markets. But they serve as building blocks for other instruments such as options on yields or caps, floors and collars.

Pricing formulas for European put and call options on yields have been derived by Longstaff (1990) in a Cox, Ingersoll and Ross (1985) framework. Options on yields differ fundamentally from options on zero-coupon bonds as their underlying asset is not traded and does not need to follow a martingale under an equivalent martingale measure. This particularity results in a set of very unusual properties, such as being unbounded above by the underlying asset value and unbounded below by their intrinsic value. In addition, their price is not necessarily an increasing function of the time to maturity or of the yield variance.

However, as the yield is a function of a zero-coupon price, we can consider a yield option as being a particular option on a zero-coupon. For instance, the payoff of a European call on a yield can be rewritten as a non-linear, but smooth function of a zero-coupon bond price.

$$\varphi(P_T(\theta)) = \mathrm{Max}\left[-\frac{\ln P_T(\theta)}{\theta - T} - K, 0\right]$$

Table I. "Bid-ask" prices induced by the volatility uncertainty. The volatility uncertainty is represented here by a symmetric range around a volatility estimate (called the "mid" volatility and denoted as $\overline{\sigma}$).

Call spread, $T = 0.5, \theta = 5, K_1 = 95\%, K_2 = 105\%$					
Mid volat.	Volatility bounds				
$\overline{\sigma}$	$[\overline{\sigma}, \overline{\sigma}]$	$[0.9\overline{\sigma}, 1.1\overline{\sigma}]$	$[0.8\overline{\sigma}, 1.2\overline{\sigma}]$	$[0.7\overline{\sigma}, 1.3\overline{\sigma}]$	$[0.6\overline{\sigma}, 1.4\overline{\sigma}]$
5.0%	4.70	4.31–5.08	3.95–5.46	3.58–5.84	3.20–6.23
7.5%	4.53	4.12–4.96	3.71–5.39	3.30–5.81	2.91–6.24
10.0%	4.38	3.94–4.82	3.52–5.27	3.10–5.74	2.68–6.20
12.5%	4.22	3.77–4.68	3.33–5.14	2.92–5.62	2.51–6.11
Put spread, $T = 0.5, \theta = 5, K_1 = 95\%, K_2 = 105\%$					
Mid volat.	Volatility bounds				
$\overline{\sigma}$	$[\overline{\sigma}, \overline{\sigma}]$	$[0.9\overline{\sigma}, 1.1\overline{\sigma}]$	$[0.8\overline{\sigma}, 1.2\overline{\sigma}]$	$[0.7\overline{\sigma}, 1.3\overline{\sigma}]$	$[0.6\overline{\sigma}, 1.4\overline{\sigma}]$
5.0%	5.30	4.92–5.68	4.54–6.05	4.16–6.42	3.77–6.80
7.5%	5.47	5.04–5.88	4.61–6.29	4.19–6.70	3.76–7.09
10.0%	5.62	5.18–6.06	4.73–4.48	4.26–6.90	3.80–7.32
12.5%	5.78	5.32–6.23	4.86–6.67	4.38–7.08	3.89–7.49

Therefore, we can use (12) and (13) to price or hedge yield options in a general HJM framework, and we can do this without making assumptions on the underlying interest rate model.

In fact, any financial asset that can be decomposed into a portfolio of European options on zero-coupon bonds can be handled by our model. For instance, pricing a caplet – i.e., a call on an interest rate – is equivalent to pricing a put on a zero-coupon bond (see Briys et al. (1991)). Therefore, the arbitrage bounds for each caplet can easily be computed using (12) and (13). A similar procedure applies to interest rate floorlets and interest rate collarlets.

However, the volatility risk for more complex assets such as caps, floors and collars can only be estimated by summing the individual upper and lower hedging prices of each of their components, since our model handles any option on one single zero-coupon bond, but not a portfolio of options on distinct zero-coupon bonds.

4.3. POOLED VERSUS INDIVIDUAL PRICES

As mentioned previously, our model accounts for diversification of volatility risk through assets, i.e., the capital required to upper-hedge a portfolio of options will

Figure 2. Examples of upper- and lower-hedging prices for a one-year call-spread on a five year zero-coupon bond. The two exercise prices of the call spread are 95 and 105, and the volatility is unknown, but assumed to lie in the interval [5%, 10%]. The dotted lines represent the hedging prices obtained when the spread is decomposed into a portfolio of calls, and the hedging prices for each of them are computed. The plain lines represent the hedging prices obtained when the spread is considered as one single instrument.

be generally lower than the capital required to upper-hedge each of its constituents separately.

Figure 2 illustrates this phenomenon in the case of a one-year call-spread on a five year zero-coupon bond. The two exercise prices – as a percentage of the underlying asset – of the call spread are 95 and 105, and the volatility is unknown, but assumed to lie in the interval [5%, 10%]. The two dashed lines represent the upper and lower hedging price computed using the upper and lower hedging prices of individual call options and summing the results. For the upper-hedging price of the call-spread, we consider the upper-hedging price of the first call (strike 95) minus the lower hedging price of the second call (strike 105). For the lower-hedging price of the call-spread, we consider the lower-hedging price of the first call (strike 95) minus the upper hedging price of the second call (strike 105). This explains why in some situations, the lower hedging price computed from individual prices may become negative. The two plain lines represent the upper and lower hedging prices computed using (12) and (13) with $\varphi(x)$ defined directly as the terminal payoff of the call spread. As expected, the bounds obtained by hedging each constituent separately will generally be wider.

4.4. HEDGING

To illustrate the relevance of our model for hedging purposes, we use Monte-Carlo simulations to evidence the divergence between the value of the replicating portfolio and the terminal option price when one uses an incorrectly specified model rather than our model.

In all simulations, we assume that the dynamics of the forward rate curve evolves according to an HJM model with a forward rate volatility exponentially decaying with time to maturity. That is:

$$\sigma(\theta)_t = \sigma \exp^{-\kappa(\theta - t)}. \tag{16}$$

This corresponds to the Hull and White (1990) model, also called the extended Vasicek model. All interest rate contingent claims are priced according to this model, which is unknown by market participants. We set $\sigma = 0.03$ and $\kappa = 0.07$, with a forward rate curve sloping upward from 6% (short rate) to 8% (10 years).

In this framework, we consider a six-month call spread on a three-year discount bond, with exercise prices set at 97% and 103% of the original bond price. Seven traders quote prices on the spread, deal at these prices, and delta-hedge their position until maturity, using the underlying asset and a zero-coupon bond maturing with the spread. All positions will be rebalanced twice a day, without any transaction costs.

Each trader i ($i = 1$ to 6) believes a specific HJM model,

$$\sigma(\theta)_t = \widehat{\sigma_i} \exp^{-\widehat{\kappa_i}(\theta - t)}$$

with his own estimates $\widehat{\sigma_i}$ and $\widehat{\kappa_i}$ of the true values σ and κ. Therefore, each trader quotes one single price (no bid-ask, since he believes he is in a complete market and knows the underlying model), and delta-hedges according to his own model. Traders 1, 2 and 3 each use an extended Vasicek model, with a correct estimator for $\widehat{\kappa_i}$ $(= \kappa)$. Trader 1 uses an implicit volatility re-estimated at each rebalancing, so that $\widehat{\sigma_1} = \sigma = 0.03$. Trader 2 overestimates the volatility parameter ($\widehat{\sigma_2} = 0.04 > \sigma$) while trader 3 underestimates it ($\widehat{\sigma_3} = 0.02 < \sigma$). These three traders allow us to assess the replication errors due to the discrete nature of the hedging strategy (trader 1) and also due to estimation risk on one parameter (trader 2 and 3). Traders 4, 5 and 6 believe in a Ho and Lee (1986) model, that is, in a constant volatility with $\widehat{\kappa_i} = 0$. Trader 4 overestimates the volatility parameter ($\widehat{\sigma_4} = 0.04 > \sigma$), trader 5 correctly estimates it ($\widehat{\sigma_5} = 0.03 = \sigma$), and trader 6 underestimates it ($\widehat{\sigma_6} = 0.02 < \sigma$). These traders allow us to estimate the impact of model risk.

Finally, trader 7 does not assume any particular model for $\sigma(\theta)_t$, but believes that $0.02 \leq \sigma(\theta)_t \leq 0.04$. He will quote a bid and an ask price, and will use our upper and lower hedging model, depending on whether the hedged position is long or short.

The effectiveness of the hedging strategy is measured by examining the average value, the volatility and the minimum and maximum values of the relative terminal

Table II. Terminal delta-hedging $P\&L_T$ as a percentage of the initial claim price. The position hedged is a short 6-month call spread position on a three-year discount bond. The true model is an extended Vasicek, with $\sigma = 0.3$ and $\kappa = 0.7$. Traders 1 uses the right model and right parameters, trader 2 and 3 face estimation risk, traders 4 to 6 use an incorrect model, and trader 7 uses our upper and lower-hedging model. Rebalancing the hedge is performed discretely, twice a day. The number of simulations is 5'000

	Extended Vasicek			Ho and Lee			Our model	
Trader no.	1	2	3	4	5	6	7 (upper)	7 (lower)
$\widehat{\sigma}$	0.3	0.4	0.2	0.3	0.4	0.2		
$\widehat{\kappa}$	0.7	0.7	0.7	0	0	0		
$(P\&L_T^F)$	3.6	4.2	3.8	4.0	3.6	4.3	5.7	5.3
$\sigma(P\&L_T^F)$	3.2	23.0	10.0	58.9	77.2	46.2	27.2	31.4
Min $P\&L_T^F$	−3.7	−33.9	−14.1	−79.5	−97.3	−67.0	−3.9	2.7
Max $P\&L_T^F$	10.7	52.8	27.3	80.7	107.7	67.5	117.2	134.1

hedging error (i.e., the trader's $P\&L_T$ relative to the initial spread price). In theory, the optimal hedging strategy should result in $P\&L_T$ always equal to zero, but the discrete nature of the hedge will create a bias. Therefore, trader 1's results will be used as the benchmark, since he uses the right model and the correct parameters estimates and is only affected by the discrete nature of his portfolio revisions.

Our findings are reported in Table II. Trader 1's $P\&L_T$ stays within a range of −3.7 to 10.7 percent of the initial spread value, with an average value of 3.6 percent and a volatility of 3.2 percent. Surprisingly, on average, all the above considered strategies provide surprisingly good and comparable results. The average profit and loss varies between 3.6 and 4.3 percent of the initial spread value, with no model being clearly superior to the others. All average $P\&L_T$ were positive, but it is important here to remember that this is not an advantage for traders 1 to 6, because we have no prior information about whether the spread to be replicated is being purchased or sold. In practice, the client would not reveal if he is a buyer or a seller of the claim until he knows the bid and ask prices. A positive average $P\&L_T$ for a long position would therefore imply a negative average $P\&L_T$ for a short one.

Results deteriorate considerably when we consider the $P\&L_T$ volatility and its extreme values. When a trader makes an estimation error (traders 2 and 3), the volatility of the terminal position value increases considerably (up to 23%), as well as the terminal maximum loss or gain (up to 52.8% of the initial spread value). When the trader makes a model error, the volatility jumps up to 77.2%, and the maximum gain or loss can be larger than the initial spread value.

By contrast, our model (trader 7) has a consistent behavior. Its average $P\&L_T$ is comparable to the other models. Its volatility is large (27.2 or 31.4 percent), but it does not reflect the strong asymmetry in terminal $P\&L_T$. For a short position in the hedged spread, the maximum loss is 3.9% of the initial spread price, while the largest gain was 117.2 percent. For a long position in the hedged spread, the maximum loss is 2.7% of the initial spread price, while the largest gain was 134.1 percent. By using our model, trader 7 has insured his position against downside risk (the small maximum loss only arises because of discrete hedging), while still retaining the upside potential of using a our model.

5. Conclusion

This paper proposes a simple model for pricing and hedging zero-coupon contingent claims in a market where the volatility of the instantaneous forward rates is not precisely known, and consequently, the interest rate process is not uniquely determined. In our model, the price of the smallest upper-hedging strategy and of the largest lower-hedging strategy satisfy a new non-linear partial differential equation. These price bounds are arbitrage based. At equilibrium, they define the set of mutually acceptable prices for a buyer and a seller. Any transaction outside this set will create a free lunch for one party.

The contribution of this paper is twofold: on the theoretical side, it shows that under certain conditions, it is still possible to derive some non-trivial arbitrage price bounds for various discount bond contingent claims without restricting the structure of the interest rates volatility function. This highlights the important role played by the volatility structure in a HJM framework: all the model and estimation risk can be summarized in this single parameter. On the practical side, it provides a consistent way to eliminate the dependance on an unobservable parameter, it allows us to assess the risk of such positions and it can be used to explain the existence of the bid-ask spread in perfect markets in a very natural way, as in the usual case of incomplete markets. It also shows the importance of the diversification of model risk in a portfolio, as the super-hedging price for a global portfolio generally differs from the sum of the super-hedging prices of its individual components.

It should be noted here that our uncertain volatility model does not in itself provide any information about the possible behavior of volatility as a random parameter. Our uncertain volatility model is a stochastic model for interest rates, but it is not stochastic in the Brownian motion sense. It only uses probability theory to assign zero probabilities to particular events. For instance, there is a zero probability that the interest rate volatility will move beyond of a certain range. The zero-probability itself should be understood in the Föllmer and Leukert (1998) sense, that is, the event is not considered by the trader when he establishes his hedging strategy, even if it would lead to a considerable loss.

Several directions for future work are apparent. First, in this paper, model misspecification is confined to errors in volatility estimates. Because a link between the volatility and the drift coefficient in an HJM model exists, one could wonder how to use the information contained in the drift coefficient to tighten the volatility bounds, thus reducing model uncertainty and lowering the hedging price interval. Second, our model could be extended to options on several zero-coupon bonds and to multi-factor HJM framework. Both cases raise several interesting questions, both in terms of (correlation and volatility) matrix bounding, and in terms of computational procedures. Finally, introducing learning in the model – through dynamic variations of $\sigma_{\min}$ and $\sigma_{\max}$ – would allow for optimal model risk management.

A. Appendix

In this part, we present the details of the proofs of our major propositions.

A.1. PROOF OF PROPOSITION 1

Here, we sketch out the demonstration for the upper-hedging strategy; the demonstration for the lower-hedging case follows directly.

By construction, it is clear that (12) defines an upper hedging strategy (the tracking error is non-positive, which is the necessary condition for upper-hedging). To show that it is the smallest one, it is sufficient to build an interest rate model for which (10) is an equality. Define $\psi(x, t)$ as

$$\psi(x,t) = \tau\left[\frac{\partial^2 \overline{W}}{\partial t^2}(x,t)\right]$$

where τ is given by (7). We can build a process X_t satisfying the following stochastic differential equation (under the risk neutral probability):

$$\frac{dX_t}{X_t} = ((T-t)^2 - (\theta-t)^2)\psi^2(X_t,t)dt - (\theta-t)\psi(X_t,t)dW_t$$

where W_t is a standard Brownian motion. This equation admits a unique weak solution (see Stroock and Varadhan (1979)). Consider now the following functions

$$t \longmapsto \xi(t,.)$$

where

$$\begin{aligned} \xi(t,.)\ :\ [T,\theta] &\to \mathbb{R} \\ s &\longmapsto \xi(t,s) \end{aligned}$$

At fixed t, we choose $\xi(t,.) = \psi(X_t, t)$. For this special structure of volatility, we associate an interest rate model given by:

$$\begin{aligned} dZ_t(\theta) &= S(\theta)_t Z_t(\theta) dW_t \\ &= -(\theta - t)Z_t(\theta)\xi(t)dW_t \end{aligned}$$

We easily check that

$$dP_t^F(\theta) = P_t^F(\theta)(\theta - t)^2\xi^2(t)dt - P_t^F(\theta)(\theta - t)\xi(t)dW_t$$

Let us consider the function $\widetilde{W}$ solution of the following partial differential equation:

$$\begin{cases} \widetilde{W}(x, T) = \varphi(x) \\ \dfrac{\partial \widetilde{W}}{\partial t} + \dfrac{1}{2}\left(\displaystyle\int_T^\theta \xi(t, s)ds\right)^2 x^2 \dfrac{\partial^2 \widetilde{W}}{\partial x^2} = 0 \end{cases}$$

Classical hedging theory tells us that $\widetilde{W}(P_0^F(\theta), 0)$ is the smallest self-financing hedging strategy. Besides, from the construction of the function $\widetilde{W}$, we have $\widetilde{W} = W^{\mathrm{up}}$. Therefore, we conclude that W^{up} is the smallest upper-hedging strategy. ■

5.1. A.2. PROOF OF PROPOSITION 2

The proof is purely computational. Write

$$P_t(T)\, W^{\mathrm{up}}\left(\frac{P_t(\theta)}{P_t(T)}, t\right)$$

as a function G depending on r_t, V_t, and t. This may be done using (5).

1. Compute derivatives

$$\frac{\partial G}{\partial t}, \frac{\partial G}{\partial V}, \frac{\partial G}{\partial r}$$

and

$$\frac{\partial^2 G}{\partial r^2}.$$

Let

$$G(r, V, t) = P_t(T)\, W^{\mathrm{up}}\left(\frac{P_t(\theta)}{P_t(T)}, t\right)$$

with

$$P_t(T) = \exp\left[-r(T-t) - \frac{V}{2}(T-t)^2 - \int_t^T (f(s) - f(t))\, ds\right]$$

Using the derivation chain rule we obtain:

$$\frac{\partial G}{\partial t} = \frac{\partial P_t(T)}{\partial t} W^{\mathrm{up}} + P_t(T) \frac{\partial W^{\mathrm{up}}}{\partial x} \frac{\partial}{\partial t}\left(\frac{P_t(\theta)}{P_t(T)}\right) + \frac{\partial W^{\mathrm{up}}}{\partial t}$$

$$\frac{\partial G}{\partial V} = \frac{\partial P_t(T)}{\partial V} W^{\mathrm{up}} + P_t(T) \frac{\partial W^{\mathrm{up}}}{\partial x} \frac{\partial}{\partial V}\left(\frac{P_t(\theta)}{P_t(T)}\right)$$

$$\frac{\partial G}{\partial r} = \frac{\partial P_t(T)}{\partial r} W^{\mathrm{up}} + P_t(T) \frac{\partial W^{\mathrm{up}}}{\partial x} \frac{\partial}{\partial r}\left(\frac{P_t(\theta)}{P_t(T)}\right)$$

But a straightforward computation from (5) gives the following:

$$\frac{\partial P_t(T)}{\partial t} = P_t(T)(r_t + V_t(T-t) + (T-t)f'(t))$$

$$\frac{\partial}{\partial t}\left(\frac{P_t(\theta)}{P_t(T)}\right) = \frac{P_t(\theta)}{P_t(T)}(\theta - T)(V_t + f'(t))$$

$$\frac{\partial P_t(T)}{\partial V} = -P_t(T)\left(\frac{(T-t)^2}{2}\right)$$

$$\frac{\partial}{\partial V}\left(\frac{P_t(\theta)}{P_t(T)}\right) = -\frac{P_t(\theta)}{P_t(T)}\left(\frac{(\theta - T)(\theta + T - 2t)}{2}\right)$$

$$\frac{\partial P_t(T)}{\partial r} = -P_t(T)(T-t)$$

$$\frac{\partial}{\partial r}\left(\frac{P_t(\theta)}{P_t(T)}\right) = -\frac{P_t(\theta)}{P_t(T)}(\theta - T)$$

Replacing, we obtain:

$$\frac{\partial G}{\partial t} = \frac{\partial P_t(T)}{\partial t} W^{\mathrm{up}} + P_t(T) \frac{\partial W^{\mathrm{up}}}{\partial x} \frac{\partial}{\partial t}\left(\frac{P_t(\theta)}{P_t(T)}\right) + \frac{\partial W^{\mathrm{up}}}{\partial t}$$

$$\frac{\partial G}{\partial V} = \frac{\partial P_t(T)}{\partial V} W^{\mathrm{up}} + P_t(T) \frac{\partial W^{\mathrm{up}}}{\partial x} \frac{\partial}{\partial V}\left(\frac{P_t(\theta)}{P_t(T)}\right)$$

$$\frac{\partial G}{\partial r} = \frac{\partial P_t(T)}{\partial r} W^{\mathrm{up}} + P_t(T) \frac{\partial W^{\mathrm{up}}}{\partial x} \frac{\partial}{\partial r}\left(\frac{P_t(\theta)}{P_t(T)}\right)$$

2. Check Avellaneda's partial differential equation.
3. Conclude using unicity of solutions. ■

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