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Decreasing Yield Curves in a Model with an Unknown Constant Growth Rate

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Abstract. The effect of incomplete information on the term structure of interest rates is examined in the framework of a pure exchange economy under uncertainty where aggregate output grows at a constant rate. If the growth rate is known, the term structure is flat. In contrast, the term structure is a decreasing curve when agents do not know the growth rate. Long term yields are less than the short rate and the yield of long term bonds is determined by the worst possible realizations of future short rates.

Key words: incomplete information, term structure of interest rates.

JEL classification codes: D5, D9, E4, G1.

1. Introduction

The effect of incomplete information on the term structure of interest rates is examined in the framework of a Lucas (1978) – type pure exchange economy in which aggregate consumption is modeled as a geometric Brownian motion whose drift coefficient (the growth rate) is an unknown constant. Agents use the output data to estimate the growth rate. The following results are obtained. Under complete information, the short rate is a linear function of the growth rate and the term structure is flat. Under incomplete information, the short rate is also a linear function of the (now estimated) growth rate. However, the term structure is *decreasing* under incomplete information. Moreover, the long rate[1] is equal to the lowest possible value the short rate can ex ante assume.

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[1] The 'long rate' is understood here as the asymptotic value of the yield curve as time to maturity tends to infinity.

This result may appear quite unexpected. Although consumers know that, long enough into the future, the growth rate will be practically known, the shape of the yield curve diverges from the complete information one. In particular, yields of long term bonds are close to the value they have in the worst case.[2] To give an example. Assume that the lowest possible short rate, a priori, is zero and the long yield is also zero. After a while, the short rate may be at 5%, the amount of uncertainty may have decreased in the sense that consumers have more information and thus a better estimator with a smaller variance of the growth rate, but the long yield is still equal to its initial value, 0%! Isn't this a violation of rational expectations? Should we not expect the long yield to approach the value of the short rate as the short rate converges to its 'true' value and varies less and less? The aim of the present paper is to show that, contrary to one's naive intuition, this is not the case.

The intuition for our result is as follows. Bond prices are given by the expected marginal utility of one unit of the consumption good at maturity. In our framework, this present value is an exponential function of the growth rate. Under incomplete information, the bond's value is therefore the expectation of an exponential of the unknown growth rate, given the available information. Convexity of the exponential function and Jensen's inequality imply that short term yields must be lower than the short rate. As far as the long rate is concerned, note that the asymptotic behavior of an average of exponential functions is determined by the function which decays at the lowest rate.[3] Thus, it is the lowest value the growth rate can assume with positive probability which determines the asymptotic behavior of bond prices and the long rate, independent of the shape of the beliefs which become more and more concentrated around the 'true' value of the short rate. The worst case, so to speak, determines the long rate. Even though agents learn more and more about the growth rate, the worst case stays at the same level, and so does, a fortiori, the long rate.

Our result relates to an important theorem of Dybvig et al. (1996) who establish that the absence of arbitrage implies that the long rate can never fall. Translated to our model, this means that the long rate must be equal to the lowest possible value the short rate can assume. For, if the long rate rose above this level, the 'true' value of the short rate, and hence also the 'true' value of the long yield, could be lower with positive probability forcing the long yield to fall back to this 'true' level in the long run. Since the theorem of Dybvig et al. (1996) precludes this, the long rate must stay forever equal to the essential infimum of the ex-ante distribution of the growth rate. Stated differently, the long rate could rise only if one could exclude with absolute certainty some values for the future short rates. Although

[2] The worst case occurs if the growth rate assumes the lowest possible value it can ex ante assume.

[3] To illustrate this remark, consider $a > b$. Then the average $\frac{1}{2}\left(e^{-a\tau} + e^{-b\tau}\right)$ behaves asymptotically as $\frac{1}{2}e^{-b\tau}$ as τ approaches infinity. Thus, only the lower discount factor plays a role in the long run.

the agents learn more and more about the unknown growth rate, they never obtain this absolute certainty.

Incomplete information models of the asset market have also been investigated by Detemple (1986), Detemple (1991), Dothan and Feldman (1986), Feldman (1989), and Gennotte (1986) in a series of papers. Gennotte (1986) studies the problem of portfolio choice. He shows that estimation of unknown parameters and optimal portfolio choice can be separated. This makes it possible to apply the Bellman methodology to derive conditions for optimal portfolios. Although we do not use the Bellman approach here, the separation principle holds also true in our framework of a pure exchange economy. In this context, it is more convenient to use marginal utility of the representative agent as a state-price deflator in order to calculate equilibrium asset prices. Dothan and Feldman (1986) and Detemple (1986) use also techniques from nonlinear filtering theory to derive partial differential equations for asset prices and investment strategies. Moreover, they extend the analysis to characterize bond prices and the term structure in equilibrium.

An explicit investigation of the resulting term structure can be found in Feldman (1989). He uses a Cox et al. (1985) type production economy in which the growth rate of the physical asset is unobservable. The growth rate is modeled as an Ornstein-Uhlenbeck-process. With complete information, the resulting short rate is also an Ornstein-Uhlenbeck process like in Vasiček's (1977) model. In particular, interest rates are already stochastic with complete information. Under incomplete information, the short rate exhibits still the mean-reverting property of the Ornstein-Uhlenbeck process. The difference is that, due to the filtering error, the diffusion coefficient of the short rate changes over time. The resulting term structure belongs to the class of Hull-White (1990) models.

In Feldman's model, agents are faced with uncertainty about future values of the growth rate even under complete information because the growth rate is a stochastic process. In contrast to Feldman's approach, we do not impose any exogenous stochastics on the growth rate a priori. Instead, we assume it to be constant – there is absolute no uncertainty about future values of the short rate under complete information. Under incomplete information, the estimator of the growth rate is a stochastic process because agents revise their beliefs as new information becomes available. We therefore compare the state of complete certainty with the state of some uncertainty (which disappears in the long run). Although agents learn the unknown parameter quickly in our model, the impact on the shape of the yield curve is not negligible. Finally, Feldman assumes a normal prior belief about the unobserved state variable whereas we also treat the general case of arbitrary priors.

The paper is organized as follows. The model is set up in Section 2. In Section 3 the estimating procedure used by the Bayesian agents is presented. In Section 4 the equilibrium is derived and the results for the term structure are given. The last section concludes.

2. The Economy

Our aim is to understand the effect of incomplete information on interest rates in the framework of the standard exchange model of financial economics in continuous time. A single, perishable good is traded in the economy whose aggregate output evolves according to the stochastic differential equation

$$\frac{dK_t}{K_t} = \mu dt + dW_t, \qquad K_0 = 1,$$

where W is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Unlike in the standard model, the growth rate μ is not a constant parameter, but a square-integrable random variable independent of the Brownian motion W. Under complete information, agents observe the realization of the growth rate μ at time 0. Their information at time t is therefore given by the filtration

$$\mathcal{F}_t = \sigma\left(\mu, (K_s)_{s \le t}\right).$$

Under incomplete information, μ will not be observed by the agents. Their information is thus given by the filtration

$$\mathcal{G}_t = \sigma\left((K_s)_{s \le t}\right).$$

Based on this information, agents estimate the growth rate as

$$\bar{\mu}_t = E\left[\mu | \mathcal{G}_t\right]. \tag{1}$$

The economy has a finite horizon $\bar{T} < \infty$.

The consumption space L^c resp. L^i is the space of nonnegative, square integrable processes that are progressively measurable with respect to $\mathcal{F}$ in the case of complete information resp. with respect to $\mathcal{G}$ in the case of incomplete information:

$$\begin{aligned} L^c &= \left\{C \in \mathcal{L}^2(\Omega \times [0, \bar{T}], \mathcal{O}(\mathcal{F}), P \otimes dt) : C \ge 0\right\} \\ L^i &= \left\{C \in \mathcal{L}^2(\Omega \times [0, \bar{T}], \mathcal{O}(\mathcal{G}), P \otimes dt) : C \ge 0\right\} \end{aligned}$$

where $\mathcal{O}(\mathcal{F})$ denotes the σ-field of $\mathcal{F}$-progressively measurable events. A representative agent has preferences over consumption streams given by a time-additive von Neumann-Morgenstern expected utility functional U:

$$U(C) = E \int_0^{\bar{T}} e^{-\rho t} u(C_t)\, dt.$$

The felicity function u exhibits constant relative risk aversion δ, that is

$$u(x) = \frac{x^{1-\delta} - 1}{1 - \delta}$$

for some $\delta > 0$. The case $\delta = 1$ is allowed for and corresponds to log-utility.

Agents have two investment possibilities. On the one hand, there is the risky stock yielding the dividend K. It is traded at some price S. The stock can be interpreted as the 'market portfolio' paying off the output of the economy as a dividend. Moreover, there is a market for lending and borrowing at a short interest rate r. $\beta_t = \exp(\int_0^t r_u du)$ is called the money market account. The space of admissible asset prices is the space of observable semimartingales. Under complete information, prices are thus required to be adapted to $\mathcal{F}$, whereas they are adapted to $\mathcal{G}$ under incomplete information. The respective class of semimartingales is large enough[4] to contain possible equilibrium prices S and β, which are determined endogenously.

The representative agent initially owns one share of the risky stock. He takes the asset prices as given and chooses a portfolio-consumption strategy $((\theta, \eta), C)$ that is progressively measurable with respect to $\mathcal{F}$- resp. $\mathcal{G}$ and satisfies the integrability conditions

$$C \in L^c \text{ resp.} L^i \tag{2}$$

$$E \int_0^{\bar{T}} \theta_u^2 d[S]_t < \infty \tag{3}$$

$$E \int_0^{\bar{T}} \eta_u d\beta_u < \infty. \tag{4}$$

The portfolio strategy (θ, η) in the stock and the money account finances the consumption stream C, that is, the value of the portfolio $V_t = \theta_t S_t + \eta_t \beta_t$ satisfies

$$V_0 = S_0 \tag{5}$$

$$dV_t = \theta_t (dS_t + K_t dt) + \eta_t d\beta_t - C_t dt \tag{6}$$

$$\text{for all } t \leq \bar{T} \quad V_t \geq 0 \quad a.e. \tag{7}$$

A strategy $((\theta, \eta), C)$ is called admissible if it meets conditions (2) through (7). If (θ^1, θ^2, c) is an admissible strategy, then c is said to be affordable. Denote by $\mathcal{A}(S, \beta)$ the set of all affordable consumption streams if prices are (S, β).

DEFINITION 2.1. A representative agent equilibrium consists of a pair of $\mathcal{F}$ resp. $\mathcal{G}$-adapted asset prices (S, β) such that it is rational for the representative agent to consume aggregate output:

$$K = \arg\max_{C \in \mathcal{A}(S,\beta)} U(C).$$

[4] The fact that asset prices are semimartingales when agents have smooth utility functionals and the endowment is a diffusion has first been shown by Huang (1987).

3. Estimating the Growth Rate

At date t, agents estimate the growth rate μ using the observed realizations of the output $(K_s)_{s\leq t}$. The general methodology of Bayesian inference has been extensively described in the previous literature. For our purposes, it suffices to know that the Brownian motion with drift $\epsilon_t = \mu t + W_t$ carries the same information as K does, since $\epsilon_t = \log K_t + \frac{1}{2}t$. Therefore, the agents have to estimate the unknown drift of a Brownian motion with drift μ. The estimator is $\bar{\mu}_t = E\left[\mu | \mathcal{G}_t\right]$, compare (1). The process

$$\bar{W}_t = \epsilon_t - \int_0^t \bar{\mu}_u du$$

is the residual which is not explained by the cumulative estimates. It is called the innovation process in filtering theory and is a Brownian motion with respect to the information filtration $\mathcal{G}$.

THEOREM 3.1. The process

$$\bar{W}_t = \epsilon_t - \int_0^t \bar{\mu}_u du$$

is a $\mathcal{G}$-adapted Brownian motion. In terms of $\bar{W}$ the dynamics of aggregate consumption are

$$\frac{dK_t}{K_t} = \bar{\mu}_t dt + d\bar{W}_t .$$

Proof. This is Theorem 7.12, p. 258 in Liptser and Shiryayev (1978).

If the prior belief about the growth rate is a normal distribution, it is particularly easy to obtain the estimate $\bar{\mu}$ and the conditional distribution of μ in closed form.

THEOREM 3.2. If the prior belief is normal, $\mu \sim N(m, 1)$, the estimate of the growth rate is

$$\begin{aligned} \bar{\mu}_t &= \frac{m + \epsilon_t}{1+t} \\ &= m + \int_0^t \frac{1}{1+u} d\bar{W}_u . \end{aligned} \tag{8}$$

The conditional distribution of μ given the information $\mathcal{G}_t$ is

$$\mathcal{L}\left(\mu | \mathcal{G}_t\right) = N\left(\frac{m+\epsilon_t}{1+t}, \frac{1}{1+t}\right) . \tag{9}$$

Proof. This is a corollary to the Main Theorem of Filtering Theory, confer (Liptser and Shiryayev 1978, Theorem 8.1, also Theorem 12.2.). Since everything is very simple here, we sketch the proof.

The current value ϵ_t of the growth rate is a sufficient statistic[5] for estimating the unknown drift μ which implies $\bar{\mu}_t = E[\mu|\epsilon_t]$. Calculating $\bar{\mu}_t$ is then a standard exercise in Bayesian inference. Note that the conditional density of μ given ϵ_t is by Bayes' rule

$$\begin{aligned} P(\mu \in dz|\epsilon_t = x) &= \frac{P(\epsilon_t \in dx|\mu = z)P(\mu \in dz)}{P(\epsilon_t \in dx)} \\ &= \frac{\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-zt)^2}{2t}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z-m)^2}{2}\right)}{P(\epsilon_t \in dx)} . \end{aligned} \tag{10}$$

Being the sum of two independent normal random variables, ϵ is normally distributed. Thus, the total probability in the denominator is

$$P(\epsilon_t \in dx) = \frac{1}{\sqrt{2\pi t(1+t)}}\exp\left(-\frac{(x-mt)^2}{2t(1+t)}\right).$$

Plugging this value into (10), one obtains the conditional distribution of μ given $\epsilon_t = x$ as

$$\mathcal{L}(\mu|\epsilon_t = x) = N\left(\frac{m+x}{1+t}, \frac{1}{1+t}\right)$$

and $\bar{\mu}_t = \frac{m+\epsilon_t}{1+t}$ follows.

Apply Itô's formula (and neglect the drift term because $\bar{\mu}$ is a martingale) to obtain the dynamics

$$d\bar{\mu}_t = \frac{1}{1+t}d\bar{W}_t .$$

An alternative way to derive the preceding theorem is to use the differential equation for the optimal filter derived in Liptser and Shirayev (1978) and used in the previous literature on incomplete information, compare Feldman (1989, p. 794). For example, in our framework, this would give

$$\begin{aligned} d\bar{\mu}_t &= \gamma_t d\bar{W}_t , \\ d\gamma_t &= -\gamma_t^2 dt , \end{aligned}$$

[5] This fact is a special case of the general separation principle. See Detemple (1991) for a general version of this principle in a finance context.

where γ is the variance of the conditional distribution of the estimator $\bar{\mu}$. With the use of (8), we see that $\gamma_t = \frac{1}{1+t}$ and $\bar{\mu}_t = \frac{m+\epsilon_t}{1+t}$ solve these equations.

4. Equilibrium and the Term Structure

4.1. COMPLETE INFORMATION

Under complete information, it is well known how to derive the representative agent equilibrium. A spot consumption price process is given by the marginal utility of the representative agent. Set

$$\psi_t = e^{-\rho t} u'(K_t)\,.$$

We use ψ as a state-price deflator for the asset prices. The stock price S is chosen such that the discounted value is equal to the discounted expected value of the future dividend stream:

$$S_t \psi_t = E\left[\int_t^{\bar{T}} K_u \psi_u \, du | \mathcal{F}_t\right].$$

The short rate r is determined to make a martingale of the discounted money account,

$$\beta_t \psi_t = E\left[\beta_{\bar{T}} \psi_{\bar{T}} | \mathcal{F}_t\right],$$

or, stated differently,

$$r_t = -E\left[\left.\frac{d\psi_t}{\psi_t}\right| \mathcal{F}_t\right].$$

We obtain the following theorem:

THEOREM 4.1. Under complete information, the representative agent equilibrium is given by

$$\begin{aligned} S_t &= K_t \frac{1 - \exp(-\gamma(\bar{T} - t))}{\gamma} \\ r_t &= \rho + \delta\mu - \frac{(1+\delta)\delta}{2} \end{aligned}$$

where $\gamma = \rho - (1-\delta)(\mu - \frac{\delta}{2})$.

Proof. See the appendix. □

4.2. INCOMPLETE INFORMATION

Under incomplete information, agents revise their beliefs about the unknown parameter μ according to the incoming information generated by the observed output process K. With the aid of the innovation process derived in the preceding section, they endogenously identify a σ – algebra equivalent economy with complete information[6] which is given by the probability space $(\Omega, \mathcal{F}, P)$, the filtration $\mathcal{G}$, the Brownian motion $\bar{W}$, and the aggregate output K of the economy which in the derived model evolves according to

$$\frac{dK_t}{K_t} = \bar{\mu}_t dt + d\bar{W}_t .$$

Agents therefore use the innovation process $\bar{W}$ and the estimated growth rate $\bar{\mu}$ instead of the unknown W and μ in the derived model. With this derived complete information economy at hand, we are able to derive the equilibrium of the market and the corresponding interest rates along the same lines as for the complete information case. Again, the spot consumption price process is given by the marginal utility of the representative agent,

$$\psi_t = e^{-\rho t} u'(K_t) .$$

Using ψ as a state-price deflator and the filtration $\mathcal{G}$ instead of $\mathcal{F}$, the candidate for the equilibrium stock price is

$$\overline{S_t} = \frac{1}{\psi_t} E\left[\int_t^{\bar{T}} K_u \psi_u \, du | \mathcal{G}_t\right] .$$

The candidate for the equilibrium short rate is, as in the complete information case, just replacing the filtration $\mathcal{F}$ with $\mathcal{G}$,

$$\bar{r}_t = -E\left[\frac{d\psi_t}{\psi_t}\middle| \mathcal{G}_t\right] .$$

We obtain the following theorem:

THEOREM 4.2. Under incomplete information, the equilibrium is given by

$$\begin{aligned} \bar{S}_t &= E\left[S_t | \mathcal{G}_t\right] \\ \bar{r}_t &= \rho + \delta\bar{\mu}_t - \frac{(1+\delta)\delta}{2} . \end{aligned}$$

The proof is given in the appendix.

[6] cf. Feldman (1992).

Let us study the term structure. As far as the short rate is concerned, Theorem 4.2 seems to be a very natural result: the short rate is constant under complete information whereas it is the best estimate of the former under incomplete information:

$$\bar{r}_t = E\left[r_t \mid \mathcal{G}_t\right].$$

As a conditional expectation, $\bar{r}$ is a martingale. Being a Bayesian estimate, it converges towards the unobservable value r of the full information world if we let the horizon $\bar{T}$ of the economy tend to infinity.

The puzzling effect appears if we look at the term structure. Under full information, the interest rate is known for all times at time 0 and therefore the term structure is flat,

$$y_t^T = r = \rho + \delta\mu - \frac{(1+\delta)\delta}{2},$$

where y_t^T denotes the yield to maturity T at time t. A natural conjecture for the yields $\overline{y_t^T}$ under incomplete information would be that they are also the expected value of the yields under complete information. In this case, the term structure under incomplete information would be flat, as in the complete information case. However, the yield curve is always *decreasing* with incomplete information. In the case of normally distributed prior beliefs, it is even linearly decreasing under incomplete information as the next theorem shows.

THEOREM 4.3. Under incomplete information and a normally distributed prior belief about $\mu \sim N(m, 1)$, the yields to maturity $\overline{y_t^T}$ are

$$\overline{y_t^T} = \bar{r}_t - \frac{\delta^2}{2}\frac{T-t}{1+t}.$$

Before we give the proof of this theorem, we ask ourselves whether this astonishing result is due to the choice of normal prior beliefs. To decide on this, we consider next the case of a general prior belief whose essential infimum is finite, $b := \operatorname{ess\,inf} \mu > -\infty$. Let us assume that the prior belief is not trivial, $\operatorname{Var}(\mu) > 0$.

Since $\bar{\mu}_t \geq \operatorname{ess\,inf} \mu = b$, the short rate and all yields are bounded below by $y_{\min} = \rho + \delta b - \frac{\delta(1+\delta)}{2}$.

Bond prices are

$$\overline{B_t^{t+\tau}} = E\left[\frac{\psi_{t+\tau}}{\psi_t} \,\middle|\, \mathcal{G}_t\right]. \tag{11}$$

In order to study the asymptotic behavior of the yields, we let the horizon $\bar{T}$ tend to infinity to obtain the family $\left(\overline{B_t^{t+\tau}}\right)_{\tau\geq 0}$ of bond prices for all times to maturity τ.[7] The announced result is

[7] Note that the bond prices in the economies with different horizons $\bar{T}_1 < \bar{T}_2$ are consistent in the sense that for a maturity $T \leq \bar{T}_1$ the corresponding bond price is the same in both economies

THEOREM 4.4. Let $-\infty < b = \operatorname{ess\,inf} \mu$ and $\operatorname{Var}(\mu) > 0$.
Then the yield curve is bounded by the short rate

$$y_{\min} \leq \overline{y_t^{t+\tau}} \leq \bar{r}_t$$

and converges to the lowest possible level, $y_{\min}$:

$$\lim_{\tau \to \infty} \overline{y_\tau^{t+\tau}} = y_{\min} .$$

We give now the proof of both Theorem 4.3 and 4.4:
Proof. The bond prices are equal to a (conditional) Laplace transform:

$$\begin{aligned}
\overline{B_t^{t+\tau}} &= e^{-\rho\tau+\frac{\delta}{2}\tau} E\left[\exp\left(-\delta(\epsilon_{t+\tau}-\epsilon_t)\right) \mid \mathcal{G}_t\right] \\
&= e^{-\rho\tau+\frac{\delta}{2}\tau} E\left[\exp\left(-\delta(W_{t+\tau}-W_t+\mu\tau)\right) \mid \mathcal{G}_t\right] \\
&= e^{-\rho\tau+\frac{\delta}{2}\tau} E\left[E\left[\exp\left(-\delta(W_{t+\tau}-W_t)\right) \mid \mathcal{F}_t\right] e^{\mu\tau} \mid \mathcal{G}_t\right],
\end{aligned}$$

which by independence of the increments of W yields

$$\overline{B_t^{t+\tau}} = e^{-\rho\tau+\frac{\delta}{2}\tau+\frac{\delta^2}{2}\tau} E\left[\exp(-\delta\mu\tau) \mid \mathcal{G}_t\right] . \tag{12}$$

- If μ is normally distributed, we know from Theorem 3.2, (9) that μ is also normally distributed conditional on the information contained in $\mathcal{G}_t$. We therefore obtain in this case

$$\overline{B_t^{t+\tau}} = e^{-\rho\tau+\frac{\delta}{2}\tau+\frac{\delta^2}{2}\tau} e^{-\delta\tau\bar{\mu}_t+\frac{1}{2}\delta^2\tau^2\frac{1}{1+t}} .$$

- In general, we obtain from (12) the yields

$$\begin{aligned}
\overline{y_t^{t+\tau}} &= \rho - \frac{\delta}{2}(1+\delta) - \frac{1}{\tau}\log E\left[\exp(-\delta\mu\tau) \mid \mathcal{G}_t\right] \\
&\leq \rho - \frac{\delta}{2}(1+\delta) + \delta\bar{\mu}_t = \overline{r_t}
\end{aligned}$$

where Jensen's inequality has been used. The yield curve is bounded by its initial value.
For the long-run value of the yields, we have to study the yield curve $\left(y_t^{t+\tau}\right)_{\tau\geq 0}$ as the yield to maturity τ tends to infinity. Denote by

$$L(\lambda) = E\left[\exp(-\lambda\mu) \mid \mathcal{G}_t\right]$$

the conditional Laplace transform of the unknown drift μ.
Lemma C.1 in the appendix shows that the essential infimum of the conditional distribution of μ is equal to the essential infimum of the distribution of μ, namely b. The result, $\overline{y_t^{t+\tau}} \to y_{\min}$ follows then from the lemma on Laplace transforms B.1 in the appendix.

because the price of a bond does not depend on the horizon of the economy, cf. the pricing formula (11).

The preceding theorem shows that decreasing yield curves are not particular to the choice of a normal prior distribution. It is generally true in this model that with complete information the term structure is flat whereas the lack of information about the growth rate leads to a decreasing yield curve. An unexpected result.

From inspection of the proof, we see that bond prices are essentially determined by the conditional Laplace transform of the conditional distribution of the unknown growth rate μ (Equation (12)). Stated differently, in order to price bonds, agents compute an average of possible discount factors according to the possible realizations the growth rate may assume. In the long run, that is for long term bonds, this average is dominated by the lowest values of μ regardless of the probability that such value is really obtained. To illustrate this, consider the sum $p\exp(-r_1\tau) + (1-p)\exp(-r_2\tau)$ with $r_1 < r_2$. For whatever probabilities $p > 0$ this sum behaves in the long run as $p\exp(-r_1\tau)$, that is as if the (worse) case r_1 were true. Only under complete certainty ($p = 0$) the behavior of the sum is determined by r_2.[8] Therefore, long term bonds are priced according to the worst case. This forces the yield curve to be decreasing (at least at the long end) and explains our result.

Another way to look at the result is to relate it to the important Theorem of Dybvig et al. (1996). According to this theorem, the long rate can never fall in an arbitrage free world. Now, in our framework, possible long rates are given by possible values for the short rate under complete information. Dybvig et al. (1996)'s Theorem implies that the long rate under incomplete information must be given by the lowest of all possible values for the short rate. If the long rate under incomplete information was higher, then it could happen with positive probability that the true value for the short rate would be somewhat lower. This in turn would force the long yield to decrease to this level, a contradiction to Dybvig et al. (1996)'s theorem.

Contrary to one's intuition, the fact that agents learn more and more about the unknown parameter of the model and that the economy, loosely speaking, converges to a complete information one, the long rate does not move to its complete information level. Rationality of agents and absence of arbitrage require absolute certainty before the long rate can rise. Although the variance of the estimation error decreases to zero, it remains always positive such that this absolute certainty is never reached.

5. Conclusion

A simple model of an incomplete information economy where only one constant parameter, the growth rate, is unknown is presented in this paper. In such a model, agents use a linear transformation of the growth rate as a discount factor for bond pricing. Under complete information, this leads to a flat term structure since all bonds are priced with the same constant discount factor. Under incomplete information, however, agents price bonds by taking an average over all possible prices

[8] The formal version of this fact is Theorem B.1 in the appendix.

bonds may assume according to the possible realizations of the growth rate. Due to the convexity of bond prices in time to maturity, long term bonds are priced according to the worst case scenario. Higher emphasis is put on the 'bad' realizations of the growth rate than on the 'good' ones.

We relate this result to the important Dybvig et al. (1996) Theorem. According to this result, the long rate can never fall. We show how this theorem can be used to shed light on our result from a different perspective.

Certainly, the yield curves we obtain in our simple model are not the ones we observe in the real world. However, one can learn something about the long yield from this model. Due to the convexity of bond prices, long yields depend much more on 'bad' scenarios of the future than on 'good' ones. In a rational economy, the long rate can never rise above the worst case level because agents would speculate on lower long yields and make arbitrarily large gains since long term bonds react very sensitive to decreases in yields. The worst case, so to speak, determines the long yield.

An interesting avenue of future research may be to find other models which support this finding.

Appendix A: Proof of the Equilibrium Relations

The derivation of the equilibrium is given for the case of incomplete information only. Under complete information, the proof runs along the same lines and the calculations are easier.

Set $\psi_t = e^{-\rho t} K_t^{-\delta}$. By Itô's formula and (8)

$$d\psi_t = -\psi_t \left(\left(\rho + \delta \bar{\mu}_t - \frac{\delta(1+\delta)}{2} \right) dt + \delta d\bar{W}_t \right) .$$

Set

$$\bar{r}_t = \rho + \delta \bar{\mu}_t - \frac{\delta(1+\delta)}{2} \tag{13}$$

and define $\bar{\beta}$ via

$$d\bar{\beta}_t = \bar{r}_t \bar{\beta}_t dt, \qquad \beta_0 = 1 . \tag{14}$$

The candidate for the equilibrium stock price is

$$\bar{S}_t := \frac{1}{\psi_t} E\left[\int_t^{\bar{T}} K_u \psi_u, du \middle| \mathcal{G}_t \right] . \tag{15}$$

Introduce the price functional for consumption streams in L^i: $\Psi(D) = E \int_0^{\bar{T}} D_u \psi_u \, du$. We prove next the plausible result that the "price" $\Psi(D)$ of every affordable consumption stream D is less than the initial endowment S_0.

LEMMA A.1. For every admissible strategy $((\theta, \mu), D)$ we have

$$\Psi(D) \leq S_0 .$$

Proof. This is the standard result that the sum of the deflated value of the portfolio and deflated cumulative consumption $V_t \psi_t + \int_0^t D_u \psi_u \, du$ is a (local) martingale.

To see this, introduce the martingales

$$M_t = E\left[\int_0^{\bar{T}} K_u \psi_u \, du \,\middle|\, \mathcal{G}_t \right]$$
$$N_t = \bar{\beta}_t \psi_t .$$

By the definitions (13) and (15) we have

$$M_t = \bar{S}_t \psi_t + \int_0^t K_u \psi_u \, du .$$

Hence,

$$dM_t = \bar{S}_t d\psi_t + \psi_t dS_t + d[\bar{S}, \psi]_t + \psi_t K_t dt \tag{16}$$
$$dN_t = \bar{\beta}_t d\psi_t + \psi_t d\bar{\beta}_t . \tag{17}$$

Now, using the dynamics of V as defined in (6) we obtain the dynamics of the deflated value of the portfolio

$$\begin{aligned} d(V_t \psi_t) &= \psi_t dV_t + V_t d\psi_t + d[V, \psi]_t \\ &= \psi_t \theta_t \left(d\bar{S}_t + K_t dt \right) + \psi_t \eta_t d\bar{\beta}_t - \psi_t D_t dt \\ &\quad + V_t d\psi_t + \theta d[S, \psi]_t . \end{aligned} \tag{18}$$

On the other hand, (16) and (17) imply

$$\begin{aligned} \theta_t dM_t + \eta_t dN_t &= \theta_t \bar{S}_t d\psi_t + \theta_t \psi_t d\bar{S}_t + \theta_t d[\bar{S}, \psi]_t + \theta_t \psi_t K_t dt \\ &\quad + \eta_t \bar{\beta}_t d\psi_t + \eta_t \psi_t d\bar{\beta}_t \\ &= V_t d\psi_t + \theta_t \psi_t d\bar{S}_t + \theta_t d[\bar{S}, \psi]_t \\ &\quad + \theta_t \psi_t K_t dt + \eta_t \psi_t d\bar{\beta}_t . \end{aligned}$$

Comparing terms in (19) and (18) yields

$$d(V_t \psi_t) = \theta_t dM_t + \eta_t dN_t - D_t \psi_t dt .$$

Therefore, $X_t := V_t \psi_t + \int_0^t D_u \psi_u du$ is a local martingale. Because of the no ruin condition (7) X is nonnegative, hence a supermartingale. Hence,

$$\Psi(D) = E \int_0^{\bar{T}} D_u \psi_u du \leq E X_{\bar{T}} \leq X_0 = V_0 \psi_0 = S_0 .$$

A standard Lagrange argument yields the optimality of the strategy $((1,0), C)$. Let $((\theta,\eta), D)$ be an admissible strategy. Then by the above Lemma A.1

$$U(D) \leq U(D) + S_0 - \Psi(D) = S_0 + E \int_0^{\bar{T}} \left(e^{-\rho t} u(D_t) - D_t \psi_t\right) dt\,.$$

The expression $e^{-\rho t} u(D_t) - D_t \psi_t$ is maximized over D_t when the first order condition

$$e^{-\rho t} u'(D_t) = \psi_t$$

holds. By definition of ψ_t, K_t solves the first order condition. Hence, K is optimal among all affordable consumption streams for the representative agent and $(\bar{S}, \bar{\beta})$ a representative agent equilibrium.

Appendix B. A Lemma on Laplace Transforms

In this part of the appendix, we present a lemma on Laplace transforms which is mathematically the key to Theorem 4.4. Apart from its economic application, it has an interest on its own.

LEMMA B.1. Let Y be a random variable and $b > -\infty$ its essential infimum. Then the Laplace transform

$$L_Y(\lambda) = E \exp(-\lambda Y)$$

exists for all nonnegative λ. The asymptotic behaviour of L_Y is given by

$$\lim_{\lambda\to\infty} -\frac{1}{\lambda} \log L_Y(\lambda) = b\,.$$

Proof. Without loss of generality let $b = 0$. Because of $Y \geq 0\,a.e.$, one has $e^{-\lambda Y} \leq 1$ $a.e.$ and by monotonicity of the integral

$$-\frac{1}{\lambda} \log L_Y(\lambda) \geq 0\,.$$

Assume that

$$\limsup -\frac{1}{\lambda} \log L_Y(\lambda) = \epsilon > 0\,.$$

Then there exists a sequence (λ_n) with $\lambda_n \to \infty$ and

$$-\frac{1}{\lambda_n} \log L_Y(\lambda_n) \geq \frac{\epsilon}{2} \quad \text{for all } n\,.$$

This is equivalent to

$$Ee^{-\lambda_n Y} \leq e^{-\lambda_n \frac{\epsilon}{2}}$$

or, multiplying both sides with $e^{\lambda_n \frac{\epsilon}{4}}$,

$$Ee^{-\lambda_n (Y - \frac{\epsilon}{4})} \leq e^{-\lambda_n \frac{\epsilon}{4}}.$$

Fatou's lemma gives

$$E \liminf e^{-\lambda_n (Y - \frac{\epsilon}{4})} \leq \lim Ee^{-\lambda_n (Y - \frac{\epsilon}{4})} = 0$$

what implies

$$e^{-\lambda_n (Y - \frac{\epsilon}{4})} \to 0 \quad a.e.$$

which is only possible if

$$Y \geq \frac{\epsilon}{4} \quad a.e.,$$

a contradiction.

C. The Essential Infimum of the Conditional Distribution of the Growth Rate

LEMMA C.1. The essential infimum of the conditional distribution of μ given ϵ_t is b,

$$ess \inf \mathcal{L}\,(\mu|\epsilon_t) = ess \inf \mu = b\,.$$

Proof. The result is due to the fact that a Brownian motion can assume all possible values. Therefore, on the basis of ϵ_t, it is impossible to exclude a value for μ with certainty if this value is initially possible.

By Bayes' rule,

$$P\,(\mu \leq \alpha|\epsilon_t = x) = \frac{P\,(\epsilon_t \in dx|\mu \leq \alpha)\,P\,(\mu \leq \alpha)}{P\,(\epsilon_t \in dx)}\,.$$

It follows that $P\,(\mu \leq \alpha|\epsilon_t = x) \geq 0$ if and only if $P\,(\mu \leq \alpha) \geq 0$.

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