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Optimal Portfolio Choice under Heterogeneous Beliefs

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Abstract. This paper analyzes how an investor who is convinced that he can "beat the market" should behave when the equilibrium price process is endogenous. The investor's optimal portfolio is shown to consist of three components: (1) a tangency portfolio, (2) a hedge portfolio against changes in the market's valuation of securities, and (3) a hedging position against changes in the divergence between the investor's and the market's beliefs. The sign and magnitude of this third component will depend on investor preferences and on the divergence in the investor's and the market's quality of information. A numerical example illustrates that the effect of heterogeneous beliefs on optimal portfolio allocations can be significant.

Key words: hedging, heterogeneous beliefs, portfolio choice.

JEL classification codes: G11, G14.

1. Introduction

A number of papers in the literature analyze the question of how agents' diverse information is incorporated into prices. Grossman (1976, 1978) argues that the equilibrium price aggregates the available information perfectly. If prices reveal all relevant information, then rational investors ignore their own piece of information and only consider the market price when making decisions. As a result, investors have homogeneous equilibrium beliefs about the joint distribution of asset returns. While homogeneous beliefs are analytically convenient and theoretically appealing, they are counterfactual. Extending Grossman's model, Hellwig (1980) shows that when the supply of the risky asset is random, the equilibrium price will not be fully revealing. Rather, heterogeneity in beliefs persists in equilibrium, with agents

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drawing information from their own private sources as well as from the market price.

Taking heterogeneous beliefs as given, this paper analyzes how an investor who is convinced that he can "beat the market" should behave. Besides providing an optimal behavior rule, the results presented below show explicitly *which factors* investors must take into account when forming their portfolios under heterogeneous beliefs. The analysis distinguishes two types of heterogeneous beliefs: relative optimism and confidence. Relative optimism characterizes a situation in which the individual investor's and the market's assessment of future dividend growth differ. Relative confidence arises when the investor's and the market's uncertainty about their respective estimates of future dividend growth differ. It is shown that each type of heterogeneity in beliefs will have different implications for the composition of agents' optimal portfolios. Whereas agents' relative optimism influences the tangency component of their optimal asset demand, their relative confidence influences their hedging demand.

The analysis below is related to a number of papers in the literature. Williams (1977) analyzes continuous-time portfolio choice and equilibrium asset pricing with heterogeneous beliefs. He assumes that asset prices follow a geometric Brownian motion with an unknown, constant drift. He shows that investors choosing an optimal portfolio proceed in two steps. First, they replace the unknown expected returns with their current conditional expectations, i.e. their best estimate of the unknown expected returns based on the history of asset prices. Second, they take a "hedging" position similar to that derived by Merton (1971) to protect themselves against unfavorable changes in their estimate of expected returns. In equilibrium, an intertemporal CAPM-like relationship arises.

This paper takes a somewhat different perspective. First, it considers an *individual* investor whose expectations are not identical with those of the market. Second, instead of assuming an unknown *asset price* drift, the analysis below considers the case in which the drift of the *dividend* process is unknown. Assuming that the individual investor has zero weight in setting market prices, the asset price is completely determined by the market's assessment of future dividend growth. With this alternative specification, the investor whose expectations differ from those of the market must take an additional factor into account when forming his optimal portfolio: he must recognize that market prices and their dynamics are driven by the market and *anticipate* the market's reaction to dividend news.

It is important to note that in the model presented below, although beliefs are heterogeneous, there is no private information. The individual investor and the market "agree to disagree" and make no attempt to extract private information by observing each other's actions. This distinguishes our analysis from that in papers concerned with the optimal exploitation of private information such as Kyle (1985) and Holden and Subrahmanyam (1992). In Kyle (1985), an informed trader acts as an intertemporal monopolist in the asset market, taking into account explicitly the effect his trading has on the current price and on future trading opportunities. As a result, his information is incorporated gradually into prices. Holden and Subrahmanyam (1992) develop a model in which multiple privately informed agents strategically exploit their information. They show that these traders compete aggressively, causing much of their private information to be revealed very rapidly.

In contrast to these papers, in the setting considered below, the individual investor takes the market's beliefs, their evolution and the market's pricing function as *given* when forming his portfolio. The investor's optimal portfolio demand can be shown to consist of three components:

- The first is the *tangency* portfolio. When forming it, the investor uses the *market's* valuation function together with his *own* assessment of the dynamics of the dividend process to determine the expected return on the risky asset.
- The second is a hedging position against unfavorable random changes in the *market's* valuation of securities. Its magnitude depends on the market's degree of confidence in its estimate of the unknown expected growth in dividends.
- The third is a hedging position against unfavorable random shifts in the *divergence* between the individual investor's and the market's beliefs. This hedging component is driven by the difference between the individual investor's and the market's degree of confidence in their estimate of the unknown expected growth in dividends.

The analysis of this paper is closely related to the work of Detemple and Murthy (1994). These authors analyze portfolio choice and asset pricing under heterogeneous beliefs when agents have logarithmic utility. They consider two types of agents with different prior beliefs about the future production growth rate and demonstrate that relatively optimistic agents take more aggressive positions in the risky asset than relatively pessimistic agents. Because they consider the case of logarithmic utility, however, Detemple and Murthy do not analyze the consequences of heterogeneous beliefs for hedging demand. By considering an agent with non-logarithmic utility, our analysis shows that a diverse *degree of confidence (quality of information)* among agents influences their hedging demand. An example illustrates that the sign and magnitude of this influence is driven *both* by investor preferences and by the extent of the difference in confidence between them. Numerical computations demonstrate that heterogeneity in beliefs leads to significant differences in optimal portfolio allocations.

The paper is organized as follows. Section 2 presents the model. Section 3 computes the asset's equilibrium price process. Section 4 characterizes the agent's optimal portfolio under heterogeneous beliefs. Section 5 provides an illustrative example. Section 6 concludes. Technical details are provided in the Appendix.

2. The Model

Consider a continuous-time economy with a single firm and a large number of consumers, none of which have influence on prices. The firm produces a single good. It is completely financed by equity and has one share outstanding. The firm pays a dividend to its shareholders at a rate x_t at time t. Suppose that the dividend process follows

$$dx_t = \mu x_t dt + \sigma x_t dB_t, \tag{1}$$

where μ is the constant instantaneous increase in expected dividends, σ denotes the dividend process' constant instantaneous volatility, and B denotes a standard Brownian motion.

Suppose, however, that the agents do not know the true mean μ and must therefore estimate it from past data. We assume that at initial time $t = 0$, all but one agent (call him i) have the following prior information on μ: they view μ as normally distributed with mean $\overline{m}_0$ and variance $\overline{V}_0 = E((\overline{m}_0 - \mu)^2)$. These investors have no additional prior information on μ. They only have the filtration $\mathcal{F}_t^x = \sigma(x_s, s \leq t)$.

Agent i, the deviant investor, does not know the true mean μ either, and must also estimate it from past data. We assume that at initial time $t = 0$, i views μ as normally distributed with mean m_0 and variance $V_0 = E((m_0 - \mu)^2)$. Although he knows the other agents' priors $(\overline{m}_0, \overline{V}_0)$, agent i only uses his own prior information and the history of dividends to infer the true value of μ. That is, he can be considered as only having the filtration $\mathcal{F}_t^x = \sigma(x_s, s \leq t)$.

From filtering theory, as new dividend information arrives, all agents (except i) update their estimate $\overline{m}_t$ of the mean growth in dividends using the relationship[1]

$$d\overline{m}_t = \frac{\overline{V}_t}{\sigma} d\overline{B}_t, \tag{2}$$

where $d\overline{B}_t = dB_t + \frac{\mu - \overline{m}_t}{\sigma} dt$. Using the fact that $d\overline{B}_t = \frac{1}{\sigma}\left(\frac{dx_t}{x_t} - \overline{m}_t dt\right)$, one can rewrite (2) as

$$d\overline{m}_t = \frac{\overline{V}_t}{\sigma^2}\left(\frac{dx_t}{x_t} - \overline{m}_t dt\right). \tag{3}$$

Thus, all investors except i update their estimate $\overline{m}_t$ by multiplying the "surprise" component of the change in dividend, $\left(\frac{dx_t}{x_t} - \overline{m}_t dt\right)$, with the term $\frac{\overline{V}_t}{\sigma^2}$, which is a measure of their relative uncertainty about μ.

In expressions (2) and (3), $\overline{V}_t = E_t((\overline{m}_t - \mu)^2)$ denotes the mean square error of $\overline{m}_t$ and has dynamics

$$d\overline{V}_t = -\frac{\overline{V}_t^2}{\sigma^2} dt. \tag{4}$$

[1] See Theorem 11.2 of Liptser and Shiryayev (1978). Gennotte (1986), Detemple (1986), Dothan and Feldman (1986), Feldman (1989, 1992) and Brennan (1998) provide applications of filtering theory to portfolio choice under incomplete information. Honda (1997) uses filtering theory to price an asset with unobservable regime-switching mean earnings growth.

This latter equation, together with the initial condition $\overline{V}_0$, implies

$$\overline{V}_t = \frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t}. \tag{5}$$

The mean square error $\overline{V}_t$ is a measure of the agents' uncertainty about the true value of μ. It can also be thought of as a measure of the (inverse) quality of their information.

Investor i performs a similar inference. Starting with his prior beliefs m_0 and V_0, he updates his estimate m_t of the mean growth in dividends using the relationship

$$dm_t = \frac{V_t}{\sigma} d\tilde{B}_t, \tag{6}$$

where $d\tilde{B}_t = dB_t + \frac{\mu - m_t}{\sigma} dt$. The mean square error of m_t, $V_t = E_t((m_t - \mu)^2)$, has dynamics $dV_t = -\frac{V_t^2}{\sigma^2} dt$, implying

$$V_t = \frac{1}{\frac{1}{V_0} + \frac{1}{\sigma^2}t}. \tag{7}$$

Note that in this simple model, all the heterogeneity in beliefs stems from the fact that the market and agent i have different priors. Because of these different prior beliefs, agent i and the market interpret the same dividend news differently. Following this initial disagreement, their posterior beliefs differ beyond time 0 as well.[2]

3. The Equilibrium Price

Before considering the deviant investor's optimal portfolio choice problem, we must compute the equilibrium price process and short rate in this economy. The equilibrium price of the share is given by

$$S_t = \frac{1}{\pi_t} E\left(\int_t^T \pi_s x_s ds \middle| \mathcal{F}_t^x \right), \tag{8}$$

where π_t denotes the state-price deflator.[3] Since agent i's influence on asset prices is assumed to be negligible, the equilibrium price of the share is determined by the other agents' preferences and beliefs. Suppose that the other agents have power utility,

$$U(c) = E\left(\int_t^T \frac{c_s^\alpha}{\alpha} e^{-\rho s} ds \right), \tag{9}$$

[2] See Detemple and Murthy (1994) for an extensive description of this issue.

[3] See Honda (1997) for an example of equilibrium asset pricing under incomplete information using state-price deflator techniques.

where ρ is nonnegative and c_s denotes current consumption.

Since the only source of consumption is the market dividend, $c_t = x_t$ and the state-price deflator π_t is given by $\pi_t = e^{-\rho t} x_t^{\alpha-1}$.[4] Then, as shown in Appendix A, the equilibrium share price S_t equals

$$S_t = x_t F(\overline{m}_t, t), \tag{10}$$

where $F(\overline{m}_t, t) \equiv \int_t^T H(\overline{m}_t, s) ds > 0$ and

$$H(\overline{m}_t, s) \equiv \exp\left(\alpha\left(\overline{m}_t - \frac{\rho}{\alpha} - \frac{1}{2}(1-\alpha)\sigma^2\right)(s-t) + \frac{\alpha^2}{2}\overline{V}_t (s-t)^2\right) \tag{11}$$

Note that $F(\overline{m}_t, t)$ represents the present value of future dividends, scaled by the current level of dividends. Using Ito's formula, the price process' dynamics can be computed as

$$\begin{aligned} dS_t &= \left((1-\alpha)\overline{m}_t + \rho + \frac{1}{2}\alpha(1-\alpha)\sigma^2 - \frac{1}{F(\overline{m}_t, t)}\right. \\ &\qquad \left. + \alpha(1-\alpha)\overline{V}_t \frac{G(\overline{m}_t, t)}{F(\overline{m}_t, t)}\right) S_t dt + \sigma\left(1 + \alpha \frac{\overline{V}_t}{\sigma^2} \frac{G(\overline{m}_t, t)}{F(\overline{m}_t, t)}\right) S_t d\overline{B}_t \\ &\equiv \mu_S(\overline{m}_t, t) S_t dt + \sigma_S(\overline{m}_t, t) S_t d\overline{B}_t, \end{aligned} \tag{12}$$

where $G(\overline{m}_t, t) \equiv \int_t^T (s-t) H(\overline{m}_t, s) ds > 0$ is a measure of the "duration" of the share. From equation (12), it is easy to see that $\sigma_S > \sigma$ whenever $\alpha > 0$, indicating that the price overreacts to changes in dividends when the agents are less risk-averse than the log-utility investor. The opposite occurs when $\alpha < 0$.

As shown in Chapter 10 of Duffie (1996), the equilibrium short rate r_t can be computed as $r_t = -\frac{\mu_\pi}{\pi_t}$, where μ_π denotes the drift of the state-price deflator. Applying Ito's formula to $\pi_t = e^{-\rho t} x_t^{\alpha-1}$ and using the dynamics of x_t yields

$$r(\overline{m}_t) = \rho + (1-\alpha)\left(\overline{m}_t - \frac{1}{2}(2-\alpha)\sigma^2\right). \tag{13}$$

4. The Deviant Agent's Problem

Suppose that agent i is endowed with initial wealth W_0 and has utility

$$U(c) = E\left(\int_t^T u(c_s, s) ds + B(W_T)\right), \tag{14}$$

[4] Under the assumptions made here that all agents except i have identical beliefs and utility functions and that agent i's influence on the market price is negligible, Rubinstein's (1974) aggregation results apply and a representative agent can be defined for pricing purposes.

where u is increasing and concave in current consumption c_s and B in terminal wealth W_T. His problem is to choose consumption c and an optimal portfolio w so as to maximize his expected lifetime utility of consumption conditional on his information at time t, $\mathcal{F}_t^x$.

Note that the investor's investment opportunity set is determined completely by three factors: (1) the market's assessment of the future growth in dividends, $\overline{m}_t$, which drives the expected return on the stock $\mu_S(\overline{m}_t, t)$, its volatility $\sigma_S(\overline{m}_t, t)$ and the interest rate $r(\overline{m}_t)$, (2) the investor's own assessment of dividend growth, m_t, and (3) time, t.[4] It would therefore be natural to write the investor's indirect utility function at time t as a function $\overline{m}_t$, m_t, t and of wealth W_t. An equivalent formulation is to write the deviant investor's indirect utility function as

$$J(W_t, \overline{m}_t, \zeta_t, t) = \max_{c,w} E\left(\int_t^T u(c_s, s)ds + B(W_T)\Big|\mathcal{F}_t^x\right), \tag{15}$$

where $\zeta_t = m_t - \overline{m}_t$ denotes the *divergence in beliefs* between him and the market. Thus, ζ_t measures the extent to which agent i is more optimistic than the market about future dividend growth. Although less intuitive, this formulation will allow a more precise characterization of the investor's optimal portfolio policy below.

In order to solve for the optimal portfolio policy, the dynamics of the state variables W_t, $\overline{m}_t$ and ζ_t under the deviant investor's beliefs have to be determined. Given a portfolio w and consumption c, the budget constraint is given by

$$dW_t = W_t\left(w\left((\mu_S^C(\overline{m}_t, t) - r(\overline{m}_t))dt + \sigma_S(\overline{m}_t, t)d\overline{B}_t\right) + r(\overline{m}_t)dt\right) - c_t dt, \tag{16}$$

where $\mu_S^C = \mu_S + \frac{x_t}{S_t} = \mu_S + \frac{1}{F(\overline{m}_t,t)}$ denotes cum dividend instantaneous expected return *as conceived by the market*. Using the fact that $d\overline{B}_t = d\tilde{B}_t + \frac{m_t - \overline{m}_t}{\sigma} = d\tilde{B}_t + \frac{\zeta_t}{\sigma}$, this expression can be rewritten under the deviant agent's beliefs as

$$\begin{aligned} dW_t = W_t\Bigg(w\Bigg(\Bigg(\mu_S^C(\overline{m}_t, t) - r(\overline{m}_t) + \frac{\sigma_S(\overline{m}_t, t)}{\sigma}\zeta_t\Bigg)dt \\ +\sigma_S(\overline{m}_t, t)d\tilde{B}_t\Bigg) + r(\overline{m}_t)dt\Bigg) - c_t dt. \end{aligned} \tag{17}$$

Similarly, one can write $d\overline{m}_t = \frac{\overline{V}_t}{\sigma}d\overline{B}_t$ as

$$d\overline{m}_t = \frac{\overline{V}_t}{\sigma}d\tilde{B}_t + \frac{\overline{V}_t}{\sigma^2}\zeta_t dt. \tag{18}$$

Finally, the dynamics of the divergence in beliefs, ζ_t, are given by

$$d\zeta_t = dm_t - d\overline{m}_t = \frac{V_t}{\sigma}d\tilde{B}_t - \frac{\overline{V}_t}{\sigma}d\overline{B}_t = \frac{V_t - \overline{V}_t}{\sigma}d\tilde{B}_t - \frac{\overline{V}_t}{\sigma^2}\zeta_t dt. \tag{19}$$

The dynamics of ζ_t are thus driven by two effects:

[4] I am grateful to the referee for pointing this out to me.

- The first is the drift $-\frac{\overline{V}_t}{\sigma^2}\zeta_t$, which induces the divergence in beliefs to decrease through time at a rate proportional to the product of the market's uncertainty $\overline{V}_t$ and of the *current divergence in beliefs*. When agent i is more optimistic than the market ($m_t > \overline{m}_t$, i.e. $\zeta_t > 0$), the drift of ζ_t is negative. The reverse holds when agent i is less optimistic than the market ($\zeta_t < 0$). As a result, ζ_t tends towards its long-run mean of zero.
- The second factor driving the dynamics of ζ_t, $\frac{V_t-\overline{V}_t}{\sigma}d\tilde{B}_t$, results from the different *degree of confidence* between agent i and the market. When $V_t > \overline{V}_t$, so that agent i is less confident about his estimate of μ than the market, good news ($d\tilde{B}_t > 0$) increase ζ_t because agent i revises his estimate m_t upward by more than the market revises $\overline{m}_t$. The reverse holds when $d\tilde{B}_t < 0$. When $V_t < \overline{V}_t$, so that agent i is more confident about his estimate of μ than the market, good news lead to a reduction in ζ_t because agent i revises his estimate m_t upward by less than the market revises $\overline{m}_t$. The reverse holds when $d\tilde{B}_t < 0$. When $V_t = \overline{V}_t$, ζ_t is a deterministic function of time because the magnitude of revisions in agents' beliefs in response to a given $d\tilde{B}_t$ is the same.[6]

Using the above dynamics of the state variables, the necessary optimality condition for (15) is

$$0 = \max_{c,w}\left(u(c,t) + \mathcal{D}J\right), \tag{20}$$

where

$$\begin{aligned}\mathcal{D}J &= J_W\left(W_t\left(w\left(\mu_S^C - r + \frac{\sigma_S}{\sigma}\zeta_t\right) + r\right) - c_t\right) + J_t + J_{\overline{m}}\frac{\overline{V}_t}{\sigma^2}\zeta_t \\ &\quad - J_\zeta\frac{\overline{V}_t}{\sigma^2}\zeta_t + \frac{1}{2}J_{WW}W_t^2 w^2\sigma_S^2 + \frac{1}{2}J_{\overline{m}\overline{m}}\frac{\overline{V}_t^2}{\sigma^2} + \frac{1}{2}J_{\zeta\zeta}\left(\frac{V_t-\overline{V}_t}{\sigma}\right)^2 \\ &\quad + J_{W\overline{m}}W_t w\frac{\sigma_S}{\sigma}\overline{V}_t + J_{W\zeta}W_t w\frac{\sigma_S}{\sigma}(V_t - \overline{V}_t) + J_{\overline{m}\zeta}\frac{\overline{V}_t(V_t-\overline{V}_t)}{\sigma^2},\end{aligned} \tag{21}$$

with the boundary condition $J(W, \overline{m}, \zeta, T) = B(W)$, where the dependence of μ_S^C, σ_S and r on $\overline{m}_t$ and t has been kept implicit for notational convenience. Differentiating (20) partially with respect to the decision variables yields the following first-order conditions:

$$0 = u_c(c,t) - J_W, \tag{22}$$

$$\begin{aligned}0 &= J_W W_t\left(\mu_S^C - r + \frac{\sigma_S}{\sigma}\zeta_t\right) + J_{WW}W_t^2 w\sigma_S^2 \\ &\quad + J_{W\overline{m}}W_t\frac{\sigma_S}{\sigma}\overline{V}_t + J_{W\zeta}W_t\frac{\sigma_S}{\sigma}(V_t - \overline{V}_t).\end{aligned} \tag{23}$$

[6] This is not to say that $dm_t = d\overline{m}_t$. Unless $m_t = \overline{m}_t$, $d\tilde{B}_t \neq d\overline{B}_t$ and, therefore, $dm_t \neq d\overline{m}_t$.

Equation (22) is the usual consumption optimality condition derived by Merton (1971). Equation (23), which describes the optimal portfolio decision, can be rewritten as

$$w = \frac{J_W}{-J_{WW}W_t}\frac{\mu_S^C - r + \frac{\sigma_S}{\sigma}\zeta_t}{\sigma_S^2} + \frac{J_{W\overline{m}}}{-J_{WW}W_t}\frac{\overline{V}_t}{\sigma\sigma_S} + \frac{J_{W\zeta}}{-J_{WW}W_t}\frac{V_t - \overline{V}_t}{\sigma\sigma_S}. \tag{24}$$

In essence, (24) is similar to the expression derived by Merton (1971). Because of heterogeneous beliefs, however, the investor's optimal portfolio now consists of *three* components, one tangency portfolio and *two* hedging portfolios. Relative *optimism* ($\zeta_t \neq 0$) induce a change in the composition of the tangency portfolio. When *confidence* differs between the investor and the market ($V_t \neq \overline{V}_t$), the investor's hedging against unfavorable changes in his investment opportunity set consists of two distinct components, one against changes in the *market's* valuation of securities ($\overline{m}_t$), and one against changes in the *divergence* in beliefs between him and the market (ζ_t). More specifically, the three components of the investor's optimal portfolio can be characterized as follows:

- When forming the *tangency* portfolio

$$\frac{J_W}{-J_{WW}W_t}\frac{\mu_S^C - r + \frac{\sigma_S}{\sigma}\zeta_t}{\sigma_S^2}, \tag{25}$$

  investor i uses the *market's* valuation function (summarized in μ_S^C, σ_S and r) together with his *own* assessment of the dynamics of the dividend process, summarized in ζ_t, to compute the expected return on the risky asset. When he is more optimistic than the market ($m_t > \overline{m}_t$, i.e. $\zeta_t > 0$), he increases his holdings of the risky asset. When he is more pessimistic than the market ($m_t < \overline{m}_t$, i.e. $\zeta_t < 0$), he reduces his holdings.

  Note that agent i just adds the term $\frac{\sigma_S}{\sigma}\zeta_t$ to the expected total return as conceived by the market when forming his portfolio. The reason for the correction factor $\frac{\sigma_S}{\sigma} = \left(1 + \alpha\frac{\overline{V}_t}{\sigma^2}\frac{G(\overline{m}_t,t)}{F(\overline{m}_t,t)}\right)$ has to do with the stock price's reaction to changes in dividends. Recall from the analysis in Section 3 that when the market is less risk-averse than the log-utility investor ($\alpha > 0$), the stock price overreacts to changes in dividends. When forming his tangency portfolio, agent i takes this fact into account and adjusts the expected return as viewed by the market, μ_S^C, by *more* than the difference between his and the market's estimate of the growth in dividends, ζ_t. The reverse is true when the market is more risk-averse than the log-utility investor ($\alpha < 0$). The expression $\mu_S^C + \frac{\sigma_S}{\sigma}\zeta_t$ actually represents the total expected return on the risky asset from agent i's perspective.

- The second component of optimal portfolio demand,

$$\frac{J_{W\overline{m}}}{-J_{WW}W_t}\frac{\overline{V}_t}{\sigma\sigma_S}, \tag{26}$$

results from agent i's attempt to hedge against unfavorable random changes in the *market's* valuation of securities i.e. in the variables μ_S^C, σ_S and r, which are all completely determined by $\overline{m}_t$ and t. Note that the magnitude of this hedging demand is driven by the *market's* degree of uncertainty about $\overline{m}_t$, $\overline{V}_t$. This is so because $\overline{V}_t$ determines the magnitude of the market's future revisions in $\overline{m}_t$ in response to new information.
- The third component,

$$\frac{J_{W\zeta}}{-J_{WW}W_t}\frac{V_t-\overline{V}_t}{\sigma\sigma_S}, \tag{27}$$

results from agent i's attempt to hedge against unfavorable random shifts in the *divergence* between his and the market's beliefs, $\zeta_t = m_t - \overline{m}_t$. The greater the divergence in confidence between the market and agent i, $V_t - \overline{V}_t$, the greater this component of hedging demand. From equation (19), note that when the market's and agent i's confidence levels are the same, $d\zeta_t = -\frac{\overline{V}_t}{\sigma^2}\zeta_t dt$, a deterministic function of time, and this component of hedging demand disappears.

Thus, the effects of heterogeneous beliefs on portfolio demand depend on the concrete nature of heterogeneity at hand. Whereas relative optimism is reflected in the agent's tangency portfolio, differences in confidence influence his hedging behavior. It goes without saying that when both relative optimism or pessimism and relative confidence are present, the agent's portfolio behavior is affected both through the tangency portfolio and through the hedging portfolio. Note that in the case of homogeneous beliefs, $m_t = \overline{m}_t$ and $V_t = \overline{V}_t$, both the tangency portfolio effect and the hedging effect of heterogeneity disappear and portfolio demand reduces to

$$w = \frac{J_W}{-J_{WW}W_t}\frac{\mu_S^C - r}{\sigma_S^2} + \frac{J_{W\overline{m}}}{-J_{WW}W_t}\frac{\overline{V}_t}{\sigma\sigma_S}. \tag{28}$$

5. An Example

In order to analyze the effects of heterogeneous beliefs on investor i's portfolio demand in more detail, let us consider an example.[7] Suppose for simplicity that investor i derives utility exclusively from final consumption,

$$U = E(B(W_T)), \tag{29}$$

[7] Brennan (1998) uses a similar example to analyze the role of learning in dynamic portfolio decisions.

where W_T denotes his terminal wealth. Suppose further that his terminal utility of wealth is of the isoelastic class,

$$B(W_T) = \frac{W_T^{\alpha_i}}{\alpha_i}, \qquad \alpha_i < 1. \tag{30}$$

The investor chooses the share of his wealth invested in the risky asset, w, so as to maximize his expected utility from terminal wealth, conditional on his information at time t,

$$\max_w E\left(B(W_T)|\mathcal{F}_t^x\right) = \max_w E\left(\frac{W_T^{\alpha_i}}{\alpha_i}\Big|\mathcal{F}_t^x\right), \tag{31}$$

subject to the budget constraint (16). Under the assumed preferences, the value function $J(W, \overline{m}_t, \zeta_t, t) \equiv \max E\left(B(W_T)\right) = \max E\left(\frac{W_T^{\alpha_i}}{\alpha_i}\right)$ can be rewritten as the product of two functions,[8]

$$J(W_t, \overline{m}_t, \zeta_t, t) = \frac{W^{\alpha_i}}{\alpha_i} I(\overline{m}_t, \zeta_t, t). \tag{32}$$

Then, the control problem (20) can be rewritten as

$$\begin{aligned} 0 \;=\; & \max_w \alpha_i I\left(w\left(\mu_S^C - r + \frac{\sigma_S}{\sigma}\zeta_t\right) + r - \frac{1}{2}(1-\alpha_i)w^2\sigma_S^2\right) \\ & + I_t + I_{\overline{m}}\left(\frac{\overline{V}_t}{\sigma^2}\zeta_t + \alpha_i w \frac{\sigma_S}{\sigma}\overline{V}_t\right) + I_\zeta\left(\alpha_i w \frac{\sigma_S}{\sigma}(V_t - \overline{V}_t) - \frac{\overline{V}_t}{\sigma^2}\zeta_t\right) \\ & + \frac{1}{2} I_{\overline{m}\overline{m}} \frac{\overline{V}_t^2}{\sigma^2} + \frac{1}{2} I_{\zeta\zeta}\left(\frac{V_t - \overline{V}_t}{\sigma}\right)^2 + I_{\overline{m}\zeta}\frac{\overline{V}_t(V_t - \overline{V}_t)}{\sigma^2}, \end{aligned} \tag{33}$$

with the boundary condition $I(\overline{m}, \zeta, T) = 1$. The optimal portfolio (24) becomes

$$w \;=\; \frac{1}{1-\alpha_i}\left(\frac{\mu_S^C - r + \frac{\sigma_S}{\sigma}\zeta_t}{\sigma_S^2}\right) + \frac{\overline{V}_t}{(1-\alpha_i)\sigma\sigma_S}\frac{I_{\overline{m}}}{I} + \frac{V_t - \overline{V}_t}{(1-\alpha_i)\sigma\sigma_S}\frac{I_\zeta}{I}. \tag{34}$$

Observe that the investor's tangency portfolio demand is a linear function of his relative optimism or pessimism, ζ_t. To characterize the investor's hedging demand more precisely, note that by non-satiation we know that expected lifetime utility must rise as investment opportunities improve i.e. $J_{\overline{m}} > 0$ and $J_\zeta > 0$. Since $I_{\overline{m}} = \frac{J_{\overline{m}}}{W^{\alpha_i}/\alpha_i}$, this implies that $I_{\overline{m}} > 0$ for $\alpha_i > 0$ and $I_{\overline{m}} < 0$ for $\alpha_i < 0$.[9] Similarly, $I_\zeta = \frac{J_\zeta}{W^{\alpha_i}/\alpha_i}$, so $I_\zeta > 0$ for $\alpha_i > 0$ and $I_\zeta < 0$ for $\alpha_i < 0$.

[8] See Brennan (1998) and Merton (1971).

[9] See Brennan (1998).

Table I. Effect of heterogeneous beliefs on hedging demand depending on preferences α_i and on the relative confidence of investor i and the market.

Value of α_i	Relative confidence	Hedging demand
$\alpha_i > 0$	$V_t > \overline{V}_t$	$+$
	$V_t < \overline{V}_t$	$-$
$\alpha_i < 0$	$V_t > \overline{V}_t$	$-$
	$V_t < \overline{V}_t$	$+$

Using these results, one can analyze the two components of hedging demand in more detail. Note that in the special case of logarithmic utility ($\alpha_i = 0$), both components of hedging demand are zero. Consider first investor i's attempt to hedge against unfavorable changes in the *market's* valuation function. When he is less risk-averse than the log-utility investor ($\alpha_i > 0$), this component of hedging demand is positive and increases his demand for the risky asset. Conversely, when he is more risk-averse than the log-utility investor ($\alpha_i < 0$), he reduces his demand for the risky asset.

The sign and magnitude of the second component of investor i's hedging demand, which is driven by his attempt to hedge against unfavorable changes in the *divergence* between his and the market's beliefs, follows a somewhat more complicated pattern. As can be seen in Table I, this second component of hedging demand is driven by two effects: investor preferences α_i and the divergence in confidence between him and the market, $V_t - \overline{V}_t$.

When investor i is *less* risk-averse than the log-utility investor ($\alpha_i > 0$), this component of hedging demand will be positive when his degree of confidence is lower than that of the market ($V_t > \overline{V}_t$). Conversely, when investor i is more confident than the market ($V_t < \overline{V}_t$), his hedging demand is negative. When investor i is *more* risk-averse than the log-utility investor ($\alpha_i < 0$), the reverse holds: his hedging demand is negative when his degree of confidence is lower than that of the market ($V_t > \overline{V}_t$), and positive when he is more confident than the market ($V_t < \overline{V}_t$).

To gain some insight into the magnitude of the effect of heterogeneous beliefs on portfolio demand, the optimal portfolio w was determined by solving the control problem (33) using a finite difference scheme. Throughout, a value for α of -2 was used, implying a market relative risk aversion of 3. The other parameter values used are $\overline{m}_0 = 0.08$ per year, $\rho = 0.05$, $\sigma = 0.2$ and $\overline{V}_0 = 0.002$, implying an initial risk-free rate of 5% per year. The value of $\overline{V}_0$ corresponds to the level of confidence of an investor with diffuse initial priors having twenty years of available dividend data to estimate μ. The optimal investment in the risky asset, w, was calculated for different time horizons T and investor preferences α_i.

Table II. Effect of differences in optimism on the investor's portfolio demand. w ($\zeta_0 = 0$) denotes the optimal portfolio allocation under homogeneous beliefs. w ($\zeta_0 = 0.01$) denotes the optimal allocation when investor i is more optimistic than the market. Δ is the difference caused by heterogeneous beliefs.

Value of α_i	T_i	w ($\zeta_0 = 0$)	w ($\zeta_0 = 0.01$)	Δ
	1	6.337	6.863	0.526
0.5	2	6.700	7.255	0.555
	5	8.008	8.662	0.654
	1	3.155	3.419	0.264
0	2	3.323	3.600	0.277
	5	3.923	4.250	0.327
	1	2.078	2.253	0.175
−0.5	2	2.161	2.345	0.184
	5	2.454	2.672	0.218
	1	1.526	1.657	0.131
−1	2	1.552	1.691	0.139
	5	1.638	1.802	0.164
	1	0.948	1.035	0.087
−2	2	0.888	0.981	0.093
	5	0.673	0.782	0.109
	1	0.632	0.697	0.065
−3	2	0.499	0.569	0.070
	5	0.053	0.141	0.083

Table II reports the optimal portfolio weights w when investor i has expectations similar to those of the market ($\zeta_0 = 0$) and is more optimistic than the market ($\zeta_0 = 0.01$), as well as the difference Δ between the two. The results show that even for the small value of ζ considered here, the difference in the optimal portfolio weights, Δ, is significant. Observe that the higher the value of α_i, the greater the effect of heterogeneous beliefs Δ. This is so because the lower investor i's degree of risk aversion, the more aggressively he reacts to a given difference in expected return between him and the market.

Table III reports the results of similar computations when investor i and the market have identical estimates of the mean growth in dividends ($m_0 = \overline{m}_0 = 0.08$), but different degrees of confidence. The optimal portfolio weight is reported for the case of homogeneous beliefs ($V_0 = \overline{V}_0 = 0.002$) and both when investor i is more ($V_0 = 0.001 < \overline{V}_0$) and less ($V_0 = 0.003 > \overline{V}_0$) confident than the market about his estimate, along with the respective difference Δ compared to the base case of homogeneous beliefs. Observe first that Δ has the sign predicted by

Table III. Effect of differences in confidence on the investor's portfolio demand. w ($V_0 = 0.002$) denotes the optimal portfolio allocation under homogeneous beliefs. w ($V_0 = 0.001$) and w ($V_0 = 0.003$) denote the optimal allocations under heterogeneous beliefs when investor i is more and less confident than the market, respectively. Δ is the difference caused by heterogeneous beliefs.

Value of α_i	T_i	w ($V_0 = 0.002$)	w ($V_0 = 0.001$)	Δ	w ($V_0 = 0.003$)	Δ
	1	6.337	6.188	−0.149	6.498	0.161
0.5	2	6.701	6.401	−0.300	7.029	0.328
	5	8.008	7.211	−0.797	8.990	0.982
	1	3.156	3.156	0	3.156	0
0	2	3.323	3.323	0	3.323	0
	5	3.923	3.923	0	3.923	0
	1	2.078	2.095	0.017	2.061	−0.017
−0.5	2	2.161	2.196	0.035	2.127	−0.034
	5	2.454	2.552	0.098	2.363	−0.091
	1	1.526	1.545	0.019	1.507	−0.019
−1	2	1.552	1.591	0.039	1.515	−0.037
	5	1.638	1.745	0.107	1.545	−0.093
	1	0.948	0.964	0.016	0.932	−0.016
−2	2	0.888	0.921	0.033	0.857	−0.031
	5	0.673	0.750	0.077	0.611	−0.062
	1	0.632	0.645	0.013	0.619	−0.013
−3	2	0.499	0.525	0.026	0.476	−0.023
	5	0.058	0.099	0.041	0.030	−0.028

the theoretical discussion summarized in Table I. That is, when the investor is less risk-averse than the log-utility investor ($\alpha_i > 0$), $\Delta > 0$ when $V_0 > \overline{V}_0$ and $\Delta < 0$ when $V_0 < \overline{V}_0$. The reverse holds when the investor is more risk-averse than the log-utility investor ($\alpha_i < 0$). In the case of logarithmic utility ($\alpha_i = 0$), differences in confidence between investor i and the market have no effect on the optimal portfolio weight w. Secondly, note that the effect of heterogeneous beliefs is greater the longer the time horizon considered. This is so because the influence of unfavorable shifts in the divergence in beliefs ζ_t on expected utility rises with the time horizon, leading to a corresponding increase in the investor's hedging incentive.

Note that although they are in general somewhat smaller in magnitude than in the case of differences in optimism, the values of Δ reported in Table III are significant, even for the short time horizons considered here. For example, an investor with a relative risk aversion of 3 ($\alpha_i = -2$) and an investment time horizon of 5 years would invest 67.3% of his wealth in the risky asset if he ignores differences in confidence. If he takes his higher degree of confidence into account ($V_0 = 0.001$), his optimal allocation rises to 75% ($\Delta = 0.077$), a proportional increase of 11.4%.

It is worth noting that a stronger departure from logarithmic utility need not increase the value of Δ. However, the *proportional* effect of heterogeneous beliefs on the optimal portfolio weight, $\Delta / w(V_0 = 0.002)$, is in general greater the larger the departure from logarithmic utility. For example, in a situation otherwise similar to that just considered, an investor with a risk aversion of 4 ($\alpha_i = -3$) would increase his optimal portfolio weight from 5.8% to 9.9%. Although $\Delta = 0.041$ is somewhat smaller in this case than the previous value of 0.077, the proportional increase in the optimal portfolio weight is now a dramatic 70.7%.

6. Conclusion

This paper analyzes the consequences of heterogeneous beliefs for optimal portfolio choice when the equilibrium price is endogenous. Under heterogeneous beliefs, the deviant investor must anticipate other agents' learning and the resulting price dynamics when forming his optimal portfolio. The results presented above demonstrate that heterogeneity in beliefs influences portfolio demands through two channels, depending on the type of heterogeneity in beliefs at hand.

- When the heterogeneity consists of relative optimism or pessimism, asset demand is influenced through the tangency portfolio. The agent forms the tangency portfolio using the *market's* valuation function together with his *own* assessment of the dynamics of the dividend process. When computing the expected return on the risky asset from his perspective, the agent takes the reaction of the asset price to changes in dividends into account. When the market is less (more) risk-averse than the log-utility investor, agent i revises expected total return by more (less) than the divergence in the estimate of the growth in dividends between himself and the market. This result is due to the fact that the asset price overreacts (underreacts) to changes in dividends.
- When the heterogeneity consists of a different level of confidence (quality of information) across agents, heterogeneity in beliefs influences asset demand through the hedging portfolio. In this case, agent i chooses his portfolio so as to hedge against unfavorable random shifts in the divergence between his and the market's beliefs. When the market's and agent i's levels of confidence are the same, the divergence in beliefs is a deterministic function of time and this hedging behavior disappears.

  Using a simple example, the sign of this hedging demand was shown to depend on the investor's degree of risk aversion. Under logarithmic utility, there is no hedging demand. When the investor is *less* risk-averse than the log-utility investor ($\alpha_i > 0$), hedging demand will be positive when his degree of confidence is lower than that of the market ($V_t > \overline{V}_t$). Conversely, when the investor is more confident than the market ($V_t < \overline{V}_t$), his hedging demand is negative. When the investor is *more* risk-averse than the log-utility investor ($\alpha_i < 0$), the reverse holds: his hedging demand is negative when his degree of confidence is lower than that of the market ($V_t > \overline{V}_t$) and positive when he is more confident than the market ($V_t < \overline{V}_t$).

For reasonable parameter values, the effect of both types of heterogeneity in beliefs on the investor's optimal portfolio is significant, even for investors with short time horizons.

The model presented in this paper is a very simple one. Nevertheless, it provides key insights into the factors affecting agents' portfolio behavior under heterogeneous beliefs. The analysis has demonstrated that when agents do not have logarithmic utility, heterogeneous beliefs will have an effect on their intertemporal hedging demand, leading to an optimal portfolio rule that is more complex than that derived by Detemple and Murthy (1994) for the case of logarithmic utility. The next stage is to investigate the consequences of heterogeneous beliefs for agents' optimal consumption patterns and equilibrium asset pricing when agents do not have logarithmic utility.

Appendix A. Computing the Equilibrium Share Price

Using the fact that the state-price deflator is given by $\pi_t = e^{-\rho t} x_t^{\alpha-1}$, the equilibrium pricing relationship (8) becomes

$$S_t = \frac{1}{\pi_t} E\left(\int_t^T \pi_s x_s ds \middle| \mathcal{F}_t^x\right) = x_t^{1-\alpha} E\left(\int_t^T e^{-\rho(s-t)} x_s^\alpha ds \middle| \mathcal{F}_t^x\right). \tag{35}$$

Applying Fubini's theorem, this expression can be rewritten as

$$S_t = x_t^{1-\alpha} \int_t^T e^{-\rho(s-t)} E\left(x_s^\alpha | \mathcal{F}_t^x\right) ds. \tag{36}$$

In order to compute this expectation explicitly, one can make use of the fact that $x^\alpha = \exp(\alpha \ln(x))$. Moreover, as shown in Appendix B,

$$E_t\left(\ln(x_s)\right) = \ln(x_t) + \left(\overline{m}_t - \frac{1}{2}\sigma^2\right)(s-t) \tag{37}$$

and

$$\mathrm{Var}_t\left(\ln(x_s)\right) = \sigma^2(s-t) + \overline{V}_t(s-t)^2. \tag{38}$$

Remembering that for a normally distributed random variable $y = \ln(x)$, $E(x^\alpha) = E(\exp(\alpha y)) = \exp(\alpha E(y) + \alpha^2 \mathrm{Var}(y)/2)$, we have

$$\begin{aligned} E\left(x_s^\alpha | \mathcal{F}_t^x\right) &= \exp\Bigg(\alpha\left(\ln(x_t) + \left(\overline{m}_t - \frac{1}{2}\sigma^2\right)(s-t)\right) \\ &\quad + \frac{\alpha^2}{2}\left(\overline{V}_t(s-t)^2 + \sigma^2(s-t)\right)\Bigg). \end{aligned} \tag{39}$$

Inserting this result in equation (36) yields the following expression for the equilibrium share price S_t:

$$S_t = x_t F(\overline{m}_t, t), \tag{40}$$

where $F(\overline{m}_t, t) \equiv \int_t^T H(\overline{m}_t, s)ds > 0$ and

$$H(\overline{m}_t, s) \equiv \exp\left(\alpha\left(\overline{m}_t - \frac{\rho}{\alpha} - \frac{1}{2}(1-\alpha)\sigma^2\right)(s-t) + \frac{\alpha^2}{2}\overline{V}_t(s-t)^2\right), \tag{41}$$

which is the result used in the text.

Appendix B. Computing the Conditional Mean and Variance of $\ln(x_s)$

This appendix shows how to compute the conditional mean and variance of $\ln(x_s)$ used in Appendix A. Recall that x has dynamics

$$dx_s = \overline{m}_s x_s ds + \sigma x_s d\overline{B}_s, \tag{42}$$

and that

$$d\overline{m}_s = \frac{\overline{V}_s}{\sigma} d\overline{B}_s. \tag{43}$$

Applying Ito's formula yields

$$d\ln(x_s) = \left(\overline{m}_s - \frac{1}{2}\sigma^2\right)ds + \sigma d\overline{B}_s. \tag{44}$$

Since $\overline{m}$ is a martingale, using Fubini's theorem yields

$$\begin{aligned} E_t(\ln(x_s)) &= E_t\left(\ln(x_t) + \int_t^s \left(\overline{m}_u - \frac{1}{2}\sigma^2\right)du\right) \\ &= \ln(x_t) + \int_t^s \left(\overline{m}_t - \frac{1}{2}\sigma^2\right)du \\ &= \ln(x_t) + \left(\overline{m}_t - \frac{1}{2}\sigma^2\right)(s-t), \end{aligned} \tag{45}$$

which is the expression used in Appendix A. In order to compute the conditional variance of $\ln(x_s)$, let

$$Y_s = \begin{pmatrix} y_s \\ \overline{m}_s \end{pmatrix} = \begin{pmatrix} \ln x_s + \frac{1}{2}\sigma^2(s-t) \\ \overline{m}_s \end{pmatrix}. \tag{46}$$

Then, by Ito's formula, we have[10]

$$\begin{aligned} dY_s &= \begin{pmatrix} dy_s \\ d\overline{m}_s \end{pmatrix} = \begin{pmatrix} d\ln x_s + \frac{1}{2}\sigma^2 ds \\ d\overline{m}_s \end{pmatrix} \\ &= \begin{pmatrix} \overline{m}_s ds + \sigma d\overline{B}_s \\ \frac{\overline{V}_s}{\sigma} d\overline{B}_s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y_s ds + \begin{pmatrix} \sigma \\ \frac{\overline{V}_s}{\sigma} \end{pmatrix} d\overline{B}_s. \end{aligned} \tag{47}$$

Let $Z(s)$ denote the conditional variance of Y_s, $Z(s) = \mathrm{Var}_t(Y_s)$. Then, $Z(t) = 0$ and Z follows the system of linear first-order differential equations

$$\begin{aligned} \begin{pmatrix} z'_{11} & z'_{12} \\ z'_{21} & z'_{22} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} + \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \sigma \\ \frac{\overline{V}_s}{\sigma} \end{pmatrix} \begin{pmatrix} \sigma & \frac{\overline{V}_s}{\sigma} \end{pmatrix} \\ &= \begin{pmatrix} 2z_{21} + \sigma^2 & z_{22} + \overline{V}_s \\ z_{22} + \overline{V}_s & \frac{\overline{V}_s^2}{\sigma^2} \end{pmatrix}, \end{aligned} \tag{48}$$

where primes denote time derivatives. Using the fact that $\overline{V}_t = \frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t}$, we have

$$\begin{aligned} z_{22}(s) &= \int_t^s \frac{\overline{V}_u^2}{\sigma^2} du = \frac{1}{\sigma^2} \int_t^s \left(\frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}u} \right)^2 du \\ &= -\frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}u} \Bigg|_t^s = \frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t} - \frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}s}, \end{aligned} \tag{49}$$

$$\begin{aligned} z_{12}(s) = z_{21}(s) &= \int_t^s \left(z_{22}(u) + \overline{V}_u \right) du = \int_t^s \left(\frac{1}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t} \right) du \\ &= \frac{s-t}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t} \end{aligned} \tag{50}$$

and

$$\begin{aligned} z_{11}(s) &= \int_t^s \left(2z_{12}(u) + \sigma^2 \right) du = \int_t^s \left(2\frac{u-t}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t} + \sigma^2 \right) du \\ &= \left(\frac{(u-t)^2}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t} + \sigma^2 \right) \Bigg|_t^s = \frac{(s-t)^2}{\frac{1}{\overline{V}_0} + \frac{1}{\sigma^2}t} + \sigma^2(s-t). \end{aligned} \tag{51}$$

[10] The change in the drift term is made for analytical convenience and has no effect on the conditional variance of $\ln(x_s)$.

Thus, we have

$$\mathrm{Var}_t(\overline{m}_s) = z_{22}(s) = \overline{V}_t - \overline{V}_s, \tag{52}$$

$$\mathrm{Cov}_t(\ln(x_s), \overline{m}_s) = z_{12}(s) = \overline{V}_t(s-t) \tag{53}$$

and

$$\mathrm{Var}_t(\ln(x_s)) = z_{11}(s) = \sigma^2(s-t) + \overline{V}_t(s-t)^2, \tag{54}$$

which is the result used in Appendix A.

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