European Finance Review **3:** 389–392, 1999.

Comment on 'The Valuation of Contingent Claims under Portfolio Constraints: Reservation Buying and Selling Prices'

THALEIA ZARIPHOPOULOU
Business School and Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA; E-mail: zariphop@math.wisc.edu

The pricing of derivative securities in the presence of market frictions has always been a question of fundamental importance. The reason is twofold: market frictions are present in numerous practical applications and, in such settings, the classical valuation theories break down entirely. Examples of market frictions include among others, transaction costs, non-traded assets and portfolio constraints. Alternative valuation criteria have been proposed and a variety of methods have been developed in order to define coherent derivative prices and, ultimately, to specify the hedging strategies.

Three main valuation methods have been developed up to date: the super-replication approach, the imperfect-replication method and the utility maximization theory. The super-replication approach looks for hedging strategies that super-replicate, instead of replicating exactly, the payoff of the derivative security. The motivation for such a pricing mechanism comes from the fact that exact replication might result in an infinite derivative price, like for example in the presence of transaction costs (Soner et al., 1995). The imperfect replication method allows for substantial deviation from the dictated Black and Scholes policies but, at the same time, it introduces criteria for tracking the "hedging error" (Leland, 1985).

The utility maximization theory prices contingent claims using an entirely different point of view. The price of the derivative, frequently called the reservation price, is determined by comparing the utility functionals of a single agent with and without the opportunity to employ the derivative security. Three stochastic optimization problems need to be solved in order to specify the reservation write and buy prices. The first problem, without the opportunity to invest in the derivative security, resembles the classical Merton problem of optimal investment and consumption, incorporating at the same time the specific kind of market friction, e.g., portfolio constraints. The other two problems refer specifically to the optimal investment and consumption plans that the writer and the buyer need to specify taking into account their obligations related to the trading of the derivative. In general, these two problems are entirely different because the writer faces the obligation to pay the payoff of the derivative at terminal time as opposed to the buyer who pays for the derivative at the beginning of the trading horizon. In some sense, the utility method is based on the classical principles of stochastic dominance appropriately adapted to accommodate dynamic trading. It was originally introduced by Hodges and Neuberger (Hodges and Neuberger, 1980) for derivative securities with transaction costs and further developed by others (Davis, 1997; Davis et al., 1993).

To introduce some notation, let us first assume that there is no derivative security available and that the wealth process of the investor is denoted by $X_s^{\pi,C}$, $s \geq t > 0$. The indices $\{\pi, C\}$ correspond to the investment strategy and the consumption plan that the investor follows. To keep the presentation simple, it is assumed that trading takes place between a deterministic bond and a stock, with initial price say S. The stock price is modeled as a diffusion process with non-linear dynamics in general.

The value function of the agent is given by

$$V(x, S, t) = \sup_{\mathcal{A}} E\left\{\int_t^{+\infty} e^{-\beta(s-t)} U(C_s) ds / \mathcal{F}_t\right\} \tag{1}$$

where $\mathcal{F}_t$ is the information available at time t, β is a discount factor and U is a given utility function. The set $\mathcal{A}$ is the one of admissible strategies which reflect the particular kind of market friction considered in the model.

A derivative security of European type is introduced with expiration at time T and payoff, say $g(S_T)$. The optimization problem of the derivative's writer is formulated as

$$V_w(x, S, t) = \sup_{\mathcal{A}_w} E\left[\int_t^T e^{-\beta(s-t)} U(C_s) ds + e^{-\beta(T-t)} V(X_T - g(S_T), S_T, T) / \mathcal{F}_t\right]. \tag{2}$$

Observe that the terminal data for V_w coincides with the original value function at wealth reduced by the derivative obligation, i.e., $V_w(x, S, T) = V(x - g(S), S, T)$. This equality and, more generally, the way Equation (2) is set up reflects the Markovian nature of the model together with the European type characteristics of the derivative. The set of admissible policies $\mathcal{A}_w$ is in general different than $\mathcal{A}$ depending on the particular constraints that the writer faces. This issue is further analyzed below.

The reservation state-dependent write price $\widetilde{h}(x, S, t)$ is defined as the input that makes the writer indifferent between writing the available security, provided that she receives $\widetilde{h}$ at the writing of it, or not employing it at all. In other words $\widetilde{h}(x, S, t)$ satisfies, for *all* states (x, S, t),

$$V(x, S, t) = V_w(x + \widetilde{h}(x, S, t), S, t). \tag{3}$$

In the absence of market frictions, the above equality naturally yields a price $\widetilde{h}$ that is wealth independent and coincides with the Black and Scholes classical price. In the presence of frictions, one can show (Constantinides and Zariphopoulou, 1999) that (3) cannot be valid for all (S, t) if the dependence of $\widetilde{h}$ on wealth is removed. This motivates to define the universal reservation write price $\widetilde{C}(S, t)$ via

$$V(x, S, t) \leq V_w(x + \widetilde{C}(S, t), S, t) \tag{4}$$

for all (x, S, t), The minimal possible universal price gives a lower bound for the prices that the writer would accept to write the derivative, in the sense that his value function V_w will universally dominate his optimal payoff if the derivative is not written.

Similarly, one may define the value function V_B of the buyer of the derivative via

$$V_B(x, S, t) = \sup_{\mathcal{A}_B} E\left[\int_t^T e^{-\beta(s-t)}U(C_s)ds + e^{-\beta(T-t)}V(x_T + g(S_T), S_T, T)/\mathcal{F}_t\right]. \tag{5}$$

Following the same ideas as before, one could define the universal reservation buy price $\underset{\sim}{C}(S, t)$ as the input that makes the investor to prefer to buy the derivative, and optimally trading at the same time, than not to use the derivative at all. Therefore, $\underset{\sim}{C}(S, t)$ satisfies

$$V(x, S, t) \leq V_B(x - \underset{\sim}{C}(S, t), S, t). \tag{6}$$

From Equations (4) and (6), one sees that the prices $\underset{\sim}{C}(S, t)$ and $\widetilde{C}(S, t)$ define a range in which the trading of the derivative must take place.

The author applies the above approach to price European type derivatives in the presence of portfolio constraints. Some natural properties of the reservation prices are derived which follow from their definition, the optimality in equations (1), (2) and (5) and the concavity properties of the value functions V, V_w and V_B.

The first contribution of the work is that the model allows for portfolio constraints in the total portfolio and not just the candidate hedging portfolio (see for the latter – among others – El Karoui and Quenez, 1995; Karatzas and Kou, 1996; Broadie et al., 1998). This is a valuable generalization for a variety of practical applications. The second contribution of the paper comes from the allowed possibility of default that the writer has. This is appropriately modeled in the way the set of admissible policies $\mathcal{A}_w$ is defined.

The drawbacks of this pricing method are well known. The first difficulty stems from the fact that closed form solutions for V_w and V_B are not in general available, if at all. This, in turn, makes the specification of the reservation prices a formidable task. The author attacks this issue by solving numerically the relevant optimization problems and then, using these numerical results, the reservation prices are specified. The numerical techniques follow closely the existing literature based on Markov chain approximations. The numerical work is very well done and two important cases of portfolio are thoroughly studied: the cases of liquidity and borrowing constraints.

Even though the numerical part of the paper is well crafted and satisfactory results are derived for the derivative prices, it is of fundamental importance to derive analytic expressions for the reservation prices. This is not to say that it is an easy task due to the lack of closed form solutions and the full non-linearity of the relevant problems. On the other hand, it would be useful if one could represent the reservation prices as expectations of terminal payoffs under appropriate measures.

The other crucial drawback of the utility approach is its inability to specify the correct hedging strategies. This is an issue not addressed in the paper and for which very little work has been done in general. A substantial amount of effort must be given in this direction in order to enhance the importance of the utility maximization approach.

References

Broadie, M., Cvitanić, J., and Soner, H. M. (1998) Optimal replication of contingent claims under portfolio constraints, *The Review of Financial Studies* **11**(1), 59–79.

Constantinides, G. M. and T. Zariphopoulou (1999) Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences, *Finance and Stochastics* **3**(3), 345–369.

Davis, M. H. A. (1997) Option prices in incomplete markets. In M. A. H. Dempster and S. R. Pliska (eds.), *Mathematics of Derivative Securities*, Cambridge University Press, Cambridge, UK.

Davis, M. H. A., Panas, V. G. and Zariphopoulou, T. (1993) European option pricing with transaction costs, *SIAM Journal on Control and Optimization* **31**(2), 470–493.

El Karoui, N. and Quenez, M.-C. (1995) Dynamic programming and pricing of contingent claims in an incomplete market, *SIAM Journal on Control and Optimization* **33**(1), 29–66.

Hodges, S. D. and Neuberger, A. (1980) Optimal replication of contingent claims under transaction costs, *Review of Future Markets* **8**, 222–239.

Karatzas, I. and Kou, S. (1996) On the pricing of contingent claims under constraints, *The Annals of Applied Probability* **6**(2), 321–369.

Leland, H. (1985) Option pricing and replication with transaction costs, *Journal of Finance* **40**, 1283–1301.

Soner, H. M., Shreve, S. and Cvitanić, J. (1995) There is no nontrivial hedging portfolio for option pricing with transaction costs, *Annals of Applied Probability* **5**(2), 327–355.