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The Valuation of Contingent Claims under Portfolio Constraints: Reservation Buying and Selling Prices *

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Abstract. With constrained portfolios contingent claims do not generally have a unique price that rules out arbitrage opportunities. Earlier studies have demonstrated that when there are constraints on the hedge portfolio, a no-arbitrage price interval for any contingent claim exists. I consider the more realistic case where the constraints are imposed on the total portfolio of each investor and define reservation buying and selling prices for contingent claims. I derive properties of these prices, show how they can be computed numerically, and study two simple examples in which the reservation prices and the corresponding hedging strategies are compared to the Black–Scholes setting.

Key words: contingent claims, dynamic programming, incomplete markets, numerical solutions, reservation prices.

JEL classification C63, D52, G11, G13.

1. Introduction

The pricing and hedging of contingent claims is a matter of immense interest and importance to the financial industry. The problem of pricing and hedging a European stock option, where the underlying stock follows a geometric Brownian motion, was solved in the pathbreaking papers of Black and Scholes (1973) and Merton (1973) under the assumption of complete markets. Results for more general contingent claims depending on more generally distributed risky assets followed from the rigorous analysis of Harrison and Kreps (1979) and Harrison and Pliska (1981), also assuming complete markets. In a complete market any contingent claim can be replicated perfectly by some self-financing dynamic portfolio strategy

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and, under a no-arbitrage condition, the price of the contingent claim must equal the cost of investing in this portfolio. It is broadly recognized, however, that financial markets are not perfectly complete. Asset prices are affected by non-tradable factors, and the investors are often restricted in the portfolios they are allowed to hold. Both facts imply that, in general, it is not possible to replicate a contingent claim perfectly.

El Karoui and Quenez (1995) show that in a market with non-traded assets there will be, for any contingent claim, an interval within which the price must be to preclude arbitrage opportunities. Cvitanić and Karatzas (1993) and Karatzas and Kou (1996) extend the analysis to convex constraints on the proportions of wealth invested in the risky assets, while Munk (1997a) has analogous results for the more general case of convex constraints on the dollar investments. The basic idea of these papers is to find the least costly dynamic trading strategy among the strategies that super-replicate the contingent claim, in the sense that the trading strategy yields at least the same payments as the contingent claim. For the special setting of a market with a constant short-term interest rate and constant drift and diffusion coefficients of asset returns Broadie et al. (1998) show that the minimum cost of super-replicating a claim, under constraints, is equal to the price of a related, dominating claim without constraints.

The no-arbitrage price bounds only apply, however, when all investors face the same constraints on their *hedge* portfolio. It seems more reasonable to consider constraints on the *total* portfolio of the investor. An investor whose optimal portfolio without engaging in contingent claims is "close" to the boundary of the set of admissible portfolios may not be allowed to invest in a replicating portfolio, since the total portfolio then will violate the constraints. On the other hand, if the optimal portfolio without engaging in contingent contracts is "far" away from the boundaries of the set of admissible trading strategies, the investor may not be that concerned about the presence of such constraints when trying to hedge a contingent claim.

Because the investors may have different preferences and endowments and hence different optimal portfolios without considering engaging in contingent claims trading, they will typically face different constraints on their hedge portfolio. Consequently, only investor-specific (or to be more precise: preference- and endowment-specific) bounds on the price of the contingent claim can be given.

In this paper I will focus on the concept of a reservation buying and a reservation selling price. The reservation buying price is the highest price the investor is willing to pay for a unit of the contingent claim. The reservation selling price is the lowest price the investor is willing to accept for writing a unit of the contingent claim. The concept of reservation prices has also been addressed in the literature on contingent claims hedging and pricing under transactions costs, see, e.g., Hodges and Neuberger (1989), Davis et al. (1993) and Constantinides and Zariphopoulou

(1999), but the present paper seems to be the first study of reservation prices in a model with portfolio constraints.[1]

Typically the investor-specific bounds will be much tighter than the arbitrage-based bounds which generally leave a very wide range of possible prices of the contingent claim. For instance, an investor writing a European call option on a stock may find it optimal to hold a share of the stock in question in any case and, therefore, she does not have to invest in a super-replicating portfolio to insure herself against bankruptcy. Of course, she will require some compensation for the promise to pay a random amount at some future point in time.

Portfolio constraints arise naturally in the case of real options which in many cases cannot be perfectly replicated by a dynamic trading strategy in traded assets. The reservation buying price is then the adequate valuation tool for such options. The reservation buying price is also relevant for valuing executive stock options. To make the incentives of an executive manager compatible with the preferences of the owners of the company, the manager's compensation scheme often involves options on the stocks of the company. To ensure that the manager cannot trade away the risk imposed this way, the manager is not allowed to trade in the underlying stocks or in stock derivatives. Hence, the manager faces portfolio constraints.

To compute an investor's reservation buying [selling] price of a contingent claim, two stochastic control problems must be solved, namely (1) the optimal consumption/investment problem when the investor has a zero position in the contingent claim, and (2) the optimal consumption/investment problem when she has bought [sold] one unit of the contingent claim. Since analytical solutions of such problems are rare, I discuss how these problems can be solved, and hence how reservation prices of contingent claims can be computed, numerically.

The rest of the paper is organized as follows. In Section 2, I introduce the financial model and define reservation buying and selling prices for an agent by considering the utility maximization problem of the agent with and without contingent claims. In Section 3 some general analytical results on reservation prices are stated and proved. Section 4 discusses general aspects of the computation of reservation buying and selling prices and shows how the utility maximization problems of the agent, and hence the reservation prices and hedging strategies, can be computed numerically. Subsequently, I consider two examples. In both examples, I compute the reservation buying and selling prices and hedging strategies for a European call option on a lognormal risky asset when the investor has a power utility of consumption. In Section 5, I assume that the only constraint on the total portfolio of the investor is that her wealth at all times is non-negative (almost surely). When the investor has a substantial wealth relative to the exercise price of the option, we find, as expected, that the reservation buying and selling prices equal the Black–Scholes price and that the hedging strategy equals the Black–Scholes

[1] The related concept of the *fair price* of a contingent claim was studied in the portfolio constraint set-up of Cvitanić and Karatzas (1993) by Karatzas and Kou (1996). For a definition of the fair price and a short comparison with reservation prices, see Section 3 below.

replicating trading strategy. I also demonstrate that these results do not hold when the wealth of the investor is below or near the exercise price of the option, which of course is due to the non-negative wealth constraint. In Section 6 I superimpose a borrowing constraint in the sense that the investor's position in the riskfree asset has to be non-negative. I study the difference between the reservation prices and hedging strategies in this world and the Black–Scholes world. Finally, Section 7 offers some concluding remarks and directions for future research.

2. The Model and Basic Definitions

2.1. THE FINANCIAL MARKET MODEL

Imagine a financial market where $d+1$ primary assets are traded continuously. There are d risky assets with prices $P(t) = (P_1(t), \ldots, P_d(t))^\top$ having dynamics given by[2]

$$dP(t) = \mathrm{diag}(P(t))\,[b(P(t))\,dt + \sigma(P(t))\,dw(t)]\,, \tag{2.1}$$

where w is a d-dimensional standard Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by w. The remaining asset is an instantaneously riskless asset called *the savings account* with price $P_0(t)$ satisfying

$$dP_0(t) = r(P(t))P_0(t)\,dt, \tag{2.2}$$

where r is the continuously compounded short-term interest rate process. I assume that r is non-negative. Note that I have taken a Markovian set-up in order to apply dynamic programming techniques. For notational simplicity I have assumed time-homogeneity of the market coefficient functions b, σ, and r. These functions are required to satisfy standard conditions for the existence and uniqueness of solutions to the stochastic differential equations above, which is basically a linear growth and a Lipschitz condition, cf. Øksendal (1998, Ch. 5). I assume that the $d \times d$ volatility matrix $\sigma(P)$ has full rank d for all values of P. This implies the dynamic completeness of the market, at least potentially. As explained below, the market can be incomplete due to restrictions on the set of admissible trading strategies.

2.2. SINGLE-AGENT OPTIMIZATION WITHOUT CONTINGENT CLAIMS

A trading strategy in the primary assets is a pair (α, θ) of progressively measurable processes where $\alpha(u)$ denotes the dollar amount invested in the savings account at time u and $\theta(u) = (\theta_1(u), \ldots, \theta_d(u))^\top$ with $\theta_i(u)$ denoting the dollar amount invested at time u in the i'th risky asset. Let $\mathcal{K}$ be a non-empty, closed, convex subset of $\mathbb{R}^{d+1}$. A trading strategy (α, θ) is called $\mathcal{K}$-admissible at time t

[2] For any vector x, $\mathrm{diag}(x)$ is the square matrix with x in the diagonal and zeros off the diagonal.

if $(\alpha(u), \theta(u)) \in \mathcal{K}$, $\ell[t, \infty) \times \mathbb{P}$-almost everywhere[3] and standard integrability conditions are satisfied. $\mathcal{K}$ is called the portfolio constraint set. This way of modeling portfolio constraints was introduced by Cuoco (1997).[4] Also, he argues that the portfolio constraint set $\mathcal{K}$ can be time- and state-dependent so that the basic requirement is that $(\alpha(u), \theta(u)) \in \mathcal{K}(u, X(u), P(u))$ where $X(u)$ is the wealth of the investor at time u. A number of economically interesting portfolio constraints can be modeled with different specifications of $\mathcal{K}$. For example, $\mathcal{K} = \mathbb{R}_+ \times \mathbb{R}^d$ represents a borrowing constraint where the investment in the savings account must be non-negative whereas the investments in the risky assets are unconstrained. As another example the specification $\mathcal{K} = \{(\alpha, \theta) \,|\, \alpha + \theta^\top \mathbf{1} \geq 0\}$ models a liquidity constraint where the total financial investment must stay non-negative. See Cuoco (1997) for further examples. By $\mathcal{P}(t, \mathcal{K})$ we denote the set of $\mathcal{K}$-admissible trading strategies at time t.

An admissible consumption process at time t is a progressively measurable process c satisfying a standard integrability condition and the non-negativity constraint $c(u) \geq 0$, $\forall u \geq t$. The set of admissible consumption processes at time t is denoted by $\mathcal{C}(t)$. Here $c(u)$ represents the rate of consumption at time u. If an investor starts out at time t with wealth $x > 0$, follows a trading strategy $(\alpha, \theta) \in \mathcal{P}(t, \mathcal{K})$ in the primary assets and a consumption process $c \in \mathcal{C}(t)$, her wealth $X^{\alpha,\theta,c}_{x,t}(u) = \alpha(u) + \theta(u)^\top \mathbf{1}$ will follow the process

$$\begin{aligned} X^{\alpha,\theta,c}_{x,t}(u) &= x + \int_t^u \left[\alpha(s) r(P(s)) + \theta(s)^\top b(P(s)) - c(s)\right] ds \\ &\quad + \int_t^u \theta(s)^\top \sigma(P(s))\, dw(s). \end{aligned}$$

A triple (α, θ, c) is called a $\mathcal{K}$-admissible primary trading and consumption strategy at time t with wealth $x > 0$, if $(\alpha, \theta) \in \mathcal{P}(t, \mathcal{K})$ and $c \in \mathcal{C}(t)$. The set of such strategies can depend on the initial wealth level x and the initial prices $P(t) = P$ and is hence denoted by $\mathcal{A}(x, P, t, \mathcal{K})$.

For given wealth x and prices P at time t, the expected remaining life-time utility that the investor gets from choosing a trading and consumption strategy (α, θ, c) is given by

$$J^{\alpha,\theta,c}(x, P, t) = \mathrm{E}_{x,P,t}\left[\int_t^\infty e^{-\beta(s-t)} U(c(s))\, ds\right],$$

where $U : [0, \infty) \to \mathbb{R}$ is a strictly increasing and strictly concave function measuring the utility from consumption and $\mathrm{E}_{x,P,t}[\cdot]$ denotes the expectation operator given $X(t) = x$ and $P(t) = P$.[5] Of course, the investor seeks to choose a trading

[3] Here $\ell[t, \infty)$ denotes the Lebesgue measure on the interval $[t, \infty)$.

[4] Cvitanić and Karatzas (1992) model constraints on the portfolio *weights* in a similar manner.

[5] The assumption of an infinite time horizon is in no way crucial. A fixed, finite time horizon can be handled similarly. It is also possible to allow for an exogenously given stream of income from non-traded assets, but to keep the analysis simple I refrain from doing so.

and consumption strategy that maximizes the expected utility and hence the value function for the investor's problem is

$$V(x, P, t) = \sup_{(\alpha,\theta,c)\in\mathcal{A}(x,P,t,\mathcal{K})} J^{\alpha,\theta,c}(x, P, t). \tag{2.3}$$

Here, I suppress the dependence of the value function V on the portfolio constraint set $\mathcal{K}$. By the assumed time-homogeneity $V(x, P, t) = V(x, P, 0) \equiv V(x, P)$ for all $t \in \mathbb{R}_+$.

2.3. RESERVATION BUYING AND SELLING PRICES

Now suppose that the investor also has the opportunity to buy or sell (write) units of some contingent claim at time t. Assume that the claim expires at time T, is of the European type, and its terminal value is the realization of some non-negative random variable $\varphi(P(T))$. The claim is thus represented by the pair (φ, T). If the investor has a position of $\varepsilon \in \mathbb{R}$ units of the contingent claim, her wealth will jump by $\varepsilon\varphi(P(T))$ at time T. If at time $\tau > t$ she has a wealth of x, follows the trading strategy (α, θ) in the primary assets, and consumes according to the consumption rate process c, her wealth $X^{\alpha,\theta,c}_{x,\tau}(\cdot; \varepsilon)$ will follow the process

$$\begin{aligned} X^{\alpha,\theta,c}_{x,\tau}(u; \varepsilon) &= x + \int_\tau^u \left[\alpha(s)r(P(s)) + \theta(s)^\top b(P(s)) - c(s)\right] ds \\ &\quad + \int_\tau^u \theta(s)^\top \sigma(P(s))\, dw(s) + \varepsilon\varphi(P(T))1_{u\geq T}, \quad \tau \leq u \end{aligned} \tag{2.4}$$

If the total price of the ε contingent claims is z, the liquid wealth of the investor will decrease by z at time t,

$$X^{\alpha,\theta,c}_{x,t}(t+; \varepsilon) = x - z, \quad \forall(\alpha, \theta, c).$$

If the investor takes a position in ε contingent claims at a total price of z at time t, consumes according to the process c, and invests in a trading strategy (α, θ) in the primary assets, her expected utility is given by

$$J^{\alpha,\theta,c}_{\varepsilon,z}(x, P, t) = \mathrm{E}_{x,P,t}\left[\int_t^\infty e^{-\beta(s-t)}U(c(s))\, ds\right]$$

with the wealth dynamics stated above. For a fixed ε and z, a triple (α, θ, c) is $\mathcal{K}$-admissible with time $t < T$ wealth x and prices P, if $(\alpha, \theta) \in \mathcal{P}(t, \mathcal{K})$ and $c \in \mathcal{C}(t)$. The set of such triples is denoted by $\mathcal{A}_{\varepsilon,z}(x, P, t, \mathcal{K})$, so the value function of this problem is

$$V_{\varepsilon,z}(x, P, t) = \sup_{(\alpha,\theta,c)\in\mathcal{A}_{\varepsilon,z}(x,P,t,\mathcal{K})} J^{\alpha,\theta,c}_{\varepsilon,z}(x, P, t). \tag{2.5}$$

Again, the dependence on $\mathcal{K}$ and (φ, T) is suppressed. Note that, since $\mathcal{A}_{0,z}(x, P, t, \mathcal{K}) = \mathcal{A}(x, P, t, \mathcal{K})$, $V_{0,z}(x, P, t) = V(x, P, t)$ for all (x, t) and all z.

Because the wealth of the investor follows a controlled diffusion process between time t and the expiration date T, the dynamic programming property implies that

$$V_{\varepsilon,z}(x, P, t+) = \sup_{(\alpha,\theta,c)\in\mathcal{A}_{\varepsilon,z}(x,P,t,\mathcal{K})} \mathrm{E}_{x,P,t}\Big[\int_t^T e^{-\beta(s-t)}U(c(s))\,ds + e^{-\beta(T-t)}V(X_{x,t}^{\alpha,\theta,c}(T-;\varepsilon) + \varepsilon\varphi(P(T)), P(T))\Big]. \tag{2.6}$$

Therefore, I can consider $V_{\varepsilon,z}(\cdot, \cdot, t+)$ the value function for a finite horizon controlled diffusion problem with a running reward given by $U(c)$ and a terminal reward given by the function $(X, P) \mapsto V(X + \varepsilon\varphi(P), P)$. Note that the terminal reward function is itself the value function for a no contingent claim problem.

The size ε of the position in the contingent claim is not a variable of choice in the problem (2.5) since the focus of this paper is not to find the optimal position in the contingent claim for a given price, but rather to find the prices at which the investor is willing to hold a given position in the claim. Starting with wealth x and prices P at time $t < T$, the investor will be willing to sell $\varepsilon \geq 0$ units of the contingent claim (φ, T) at a total price of z, if

$$V_{-\varepsilon,z}(x, P, t) \geq V(x, P, t),$$

i.e., if the wealth transfer z now is a sufficient compensation, in terms of utility, for a commitment to pay $\varepsilon\varphi(P(T))$ at the expiration of the contingent claim. I can therefore define the *ε-reservation selling price* of the investor at time t as the minimum price the investor will accept, i.e.,

$$C_\varepsilon^{\text{sell}}(x, P, t) = \inf\{z \geq 0 \mid V_{-\varepsilon,z}(x, P, t) \geq V(x, P, t)\}. \tag{2.7}$$

The per unit ε-reservation selling price is then $C_\varepsilon^{\text{sell}}(x, P, t)/\varepsilon$.

Conversely, the investor will be willing to buy $\varepsilon \geq 0$ units of the contingent claim (φ, T) at a total price z, if

$$V_{\varepsilon,z}(x, P, t) \geq V(x, P, t),$$

i.e., the investor is better off investing the amount z of initial wealth for the uncertain payment $\varepsilon\varphi(P(T))$ at time T. I define the *ε-reservation buying price* of the investor at time t as the maximum price the investor is willing to pay, i.e.,

$$C_\varepsilon^{\text{buy}}(x, P, t) = \sup\{z \geq 0 \mid V_{\varepsilon,z}(x, P, t) \geq V(x, P, t)\}. \tag{2.8}$$

The per unit ε-reservation buying price is $C_\varepsilon^{\text{buy}}(x, P, t)/\varepsilon$.

Of course, the reservation prices depend on the contingent claim (φ, T), the current wealth x and current prices P, the portfolio constraint set $\mathcal{K}$, and also on the preferences of the investor.

Note the following two important points regarding the definition of the reservation prices. Firstly, in the definition of the reservation buying price of the claim I have implicitly assumed that a possible counterpart in the trade (the seller) will meet the contractual obligations and pay $\varepsilon\varphi(P(T))$ with probability one. (Real options may have no counterpart.) In the definition of the reservation selling price $(\varepsilon < 0)$, the wealth of the seller immediately after the expiration of the option is

$$X(T+;\varepsilon) = X(T-;\varepsilon) + \varepsilon\varphi(P(T)),$$

which may be negative implying the default of the seller. In the numerical sections of the paper I will assume that in the case of default, the buyer of the claim will receive the entire wealth of the seller which is then confined to zero consumption in all future. The value function of the seller upon expiration of the option will then be $V([X(T-;\varepsilon)+\varepsilon\varphi(P(T))]^+, P(T))$, where $x^+ = \max(x, 0)$. Alternatively, the reservation selling price could be defined as the smallest price the seller would accept under the requirement that she is able to meet her contractual obligations with probability one. Then the seller would effectively have to super-replicate her short position. For some constraints, super-replication is impossible and, hence, the reservation selling price would be infinite with such a definition.

Secondly, I have implicitly assumed that the position in the contingent claim is a *buy and hold*. After the transaction at time t the investor cannot change her position in the contingent claim – which indeed is the case for many executive stock options and real options. In order to include the possibility of position adjustments, it is necessary to model the evolution of the price of the contingent claim. However, a unique no-arbitrage price process can only be found if the contingent claim can be perfectly replicated by an admissible trading strategy in the primary assets. If such a replication strategy exists, then the contingent claim is of no use in the utility maximization problem. If the contingent claim is not perfectly replicable, including the claim in the trading strategy may increase the expected utility of the investor. If there is some possibility for subsequent trading the contingent claims, $C_\varepsilon^{\text{buy}}(x, P, t)$ defined above will be a lower bound on the correct reservation buying price, and $C_\varepsilon^{\text{sell}}(x, P, t)$ defined above will be an upper bound on the correct reservation selling price.

3. General Properties of Reservation Prices

In this section some general properties of reservation prices are derived analytically. Damgaard (1999) has a similar analysis for reservation prices in a transactions costs setting. For all the utility maximization problems involved, I shall assume that an optimal policy exists, i.e., that the supremum in the Equations (2.3) and (2.5) is obtained. (For results on the existence of optimal policies in markets with portfolio constraints, the reader is referred to Cvitanić and Karatzas (1992) and Cuoco (1997).) For the special case of no portfolio constraints, the perfectly replicating dynamic trading strategy of the contingent claim is an admissible strategy and, therefore, the reservation selling and the reservation buying prices for any investor are identical to the traditional complete market price. Due to the linearity of market prices, the price of a trading strategy replicating ε contingent claims is equal to ε times the price of replicating one contingent claim, i.e., $C_\varepsilon^{\text{rep}}(P,t) = \varepsilon C_1^{\text{rep}}(P,t)$.

THEOREM 1. With no portfolio constraints, i.e., $\mathcal{K} = \mathbb{R}^{d+1}$, the reservation selling and the reservation buying price are equal and identical to the price of the perfectly replicating portfolio:

$$C_\varepsilon^{\text{sell}}(x,P,t) = C_\varepsilon^{\text{buy}}(x,P,t) = C_\varepsilon^{\text{rep}}(P,t) = \varepsilon C_1^{\text{rep}}(P,t).$$

Proof. Let (α^*, θ^*) be an optimal trading strategy for the problem with no investment in the contingent claim and let (α^r, θ^r) be the trading strategy that perfectly replicates ε contingent claims. Assume that $C_\varepsilon^{\text{buy}}(x,P,t) < C_\varepsilon^{\text{rep}}(P,t)$, then the trading strategy (α, θ) defined by

$$\begin{aligned}(\alpha(s), \theta(s)) &= (\alpha^*(s), \theta^*(s)) - (\alpha^r(s), \theta^r(s)) \\ &\quad + ([C_\varepsilon^{\text{rep}}(P,t) - C_\varepsilon^{\text{buy}}(x,P,t)]e^{\int_t^s r(u)\,du}, 0),\end{aligned}$$

together with an unchanged consumption strategy, is feasible in the "buy ε contingent claim" optimization problem. The time T wealth generated by this strategy is strictly greater than that of the no contingent claim problem. Therefore $V_{\varepsilon, C_\varepsilon^{\text{buy}}(x,P,t)}(x,P,t) > V(x,P,t)$ which is inconsistent with $C_\varepsilon^{\text{buy}}(x,P,t)$ being the reservation buying price. Hence $C_\varepsilon^{\text{buy}}(x,P,t) \geq C_\varepsilon^{\text{rep}}(P,t)$. With $C_\varepsilon^{\text{buy}}(x,P,t) = C_\varepsilon^{\text{rep}}(P,t)$, the strategy (α, θ) defined above is feasible with identical consumption strategy and identical time T wealth as in the no contingent claim problem. Therefore, the reservation buying price equals $C_\varepsilon^{\text{rep}}(P,t)$. Similarly for the reservation selling price. □

With general portfolio constraints the following theorem shows that the reservation selling price is greater than or equal to the reservation buying price. This is essentially due to the concavity of the utility function U and the convexity of the portfolio constraint set $\mathcal{K}$.

THEOREM 2. For any non-empty, closed, convex portfolio constraint set $\mathcal{K} \subseteq \mathbb{R}^{d+1}$ and any $\varepsilon \geq 0$, the reservation selling price is greater than or equal to the reservation buying price:

$$C_\varepsilon^{\text{sell}}(x,P,t) \geq C_\varepsilon^{\text{buy}}(x,P,t).$$

Proof. Take a fixed $\varepsilon \geq 0$. Let $(\alpha^s, \theta^s, c^s)$ be an optimal policy of the problem where ε contingent claims are sold at the total reservation selling price $C_\varepsilon^{\text{sell}}(x, P, t)$, i.e.,

$$\begin{aligned} J^{\alpha^s,\theta^s,c^s}_{-\varepsilon,C_\varepsilon^{\text{sell}}(x,P,t)}(x, P, t) &= J^{\alpha^s,\theta^s,c^s}_{-\varepsilon,C_\varepsilon^{\text{sell}}(x,P,t)}(x + C_\varepsilon^{\text{sell}}(x, P, t), P, t+) \\ &= V_{-\varepsilon,C_\varepsilon^{\text{sell}}(x,P,t)}(x, P, t) = V(x, P, t). \end{aligned}$$

Let $X^s_{x,t}(\cdot)$ denote the wealth process given that the trading and consumption policy $(\alpha^s, \theta^s, c^s)$ is followed and ε units of the contingent claim are sold. Similarly, let $(\alpha^b, \theta^b, c^b)$ be an optimal policy of the problem where ε contingent claims are bought at the reservation buying price, i.e.,

$$\begin{aligned} J^{\alpha^b,\theta^b,c^b}_{\varepsilon,C_\varepsilon^{\text{buy}}(x,P,t)}(x, P, t) &= J^{\alpha^b,\theta^b,c^b}_{\varepsilon,C_\varepsilon^{\text{buy}}(x,P,t)}(x - C_\varepsilon^{\text{buy}}(x, P, t), P, t+) \\ &= V_{\varepsilon,C_\varepsilon^{\text{buy}}(x,P,t)}(x, P, t) = V(x, P, t). \end{aligned}$$

Let $X^b_{x,t}(\cdot)$ denote the wealth process given that the trading and consumption strategy $(\alpha^b, \theta^b, c^b)$ is followed and ε units of the contingent claim are bought.

By convexity of $\mathcal{K}$, the policy

$$(\alpha, \theta, c) = \frac{1}{2}\left(\alpha^s + \alpha^b, \theta^s + \theta^b, c^s + c^b\right)$$

is admissible with initial (time t) wealth

$$\hat{x} = x + \frac{1}{2}\left(C_\varepsilon^{\text{sell}}(x, P, t) - C_\varepsilon^{\text{buy}}(x, P, t)\right)$$

for the problem with no investment in the contingent claim, i.e., for $\varepsilon = 0$. It is easy to show that, given (α, θ, c) and initial wealth $\hat{x}$, the time T wealth $X(T)$ can be written as

$$X(T) = \frac{1}{2}\left(X^s_{x,t}(T+) + X^b_{x,t}(T+)\right).$$

By concavity of the utility function $U(\cdot)$ and the "terminal utility payoff" $X \mapsto V(\cdot, P(T), T)$, it follows that

$$\begin{aligned} J^{\alpha,\theta,c}(\hat{x}, P, t) &\geq \frac{1}{2}\left(V_{-\varepsilon,C_\varepsilon^{\text{sell}}(x,P,t)}(x, P, t) + V_{\varepsilon,C_\varepsilon^{\text{buy}}(x,P,t)}(x, P, t)\right) \\ &= V(x, P, t). \end{aligned}$$

This inequality will only hold if $\hat{x} \geq x$ which implies that $C_\varepsilon^{\text{sell}}(x, P, t) \geq C_\varepsilon^{\text{buy}}(x, P, t)$. $\square$

Next, consider the relation between reservation prices and arbitrage prices based on super-replication arguments. In a setting with proportional transactions costs and no portfolio constraints, Damgaard (1999) has shown that any investor's reservation buying and selling prices are within the interval bounded by the arbitrage buying and selling prices generated by super-replication arguments. To investigate whether the same relation is valid in a portfolio constraint set-up, I focus on the selling prices.

Let y be the cost of implementing a particular trading and consumption strategy (α', θ', c') that super-replicates the contingent claim in the sense that $X_{y,t}^{\alpha',\theta',c'}(T) \geq \varepsilon\varphi(P(T))$ and is feasible so that $(\alpha'(s), \theta'(s)) \in \mathcal{K}$ for all $s \in [t, T]$. Also, let $(\alpha^*, \theta^*, c^*)$ be the optimal policy for the problem with no position in the contingent claim starting with an initial wealth of x. Now, consider the following strategy: Invest in the trading and consumption policy $(\alpha, \theta, c) = (\alpha^*+\alpha', \theta^*+\theta', c^*+c')$ and sell the contingent claim at the reservation selling price $C_\varepsilon^{\text{sell}}(x, P, t)$. Then the time $t+$ wealth is $\hat{x} = x + y - C_\varepsilon^{\text{sell}}(x, P, t)$ and both consumption and terminal wealth are no less than when just investing in $(\alpha^*, \theta^*, c^*)$. Since that policy was optimal with initial wealth x, $\hat{x}$ must be at least as high as x. Hence $y \geq C_\varepsilon^{\text{sell}}(x, P, t)$. Since this applies to all y that finances a super-replicating strategy, $C_\varepsilon^{\text{sell}}(x, P, t)$ must be less than the arbitrage selling price. However, this argument presumes that the combined strategy (α, θ) is $\mathcal{K}$-valued which is not always the case. If the optimal trading strategy is, loosely speaking, close to the boundary of the portfolio constraint set, the least costly super-replicating strategy may not be a feasible hedging strategy. Therefore, the reservation prices with portfolio constraints are not necessarily bounded by the super-replication prices.

The next two results discuss the relation between reservation prices and the number of contingent claims bought or sold.

THEOREM 3. (a) The total reservation buying price $C_\varepsilon^{\text{buy}}(x, P, t)$ is a concavely increasing function of the size ε of the position.

(b) The total reservation selling price $C_\varepsilon^{\text{sell}}(x, P, t)$ is a convexly increasing function of the size ε of the position.

Proof. The reservation buying price is clearly increasing in the number of contingent claims since the payoff is nonnegative. To prove that the relation is concave, take $0 < \varepsilon_1 < \varepsilon_2$. Define $\bar{\varepsilon} = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$ and $\bar{C}^{\text{buy}}(x, P, t) = \lambda C_{\varepsilon_1}^{\text{buy}}(x, P, t) + (1-\lambda)C_{\varepsilon_2}^{\text{buy}}(x, P, t)$. I want to show that, for any $\lambda \in [0, 1]$, $C_{\bar{\varepsilon}}^{\text{buy}}(x, P, t) \geq \bar{C}^{\text{buy}}(x, P, t)$. Let $(\alpha_i, \theta_i, c_i)$ be the optimal policy and $X_i(\cdot)$ the corresponding wealth process for $V_{\varepsilon_i, C_{\varepsilon_i}^{\text{buy}}}(x, P, t)$ for $i = 1, 2$. Define the policy

$$(\bar{\alpha}, \bar{\theta}, \bar{c}) = \lambda(\alpha_1, \theta_1, c_1) + (1-\lambda)(\alpha_2, \theta_2, c_2).$$

Note that

$$\lambda X_1(T+) + (1-\lambda)X_2(T+) = x - \bar{C}^{\text{buy}}(x, P, t) + \int_t^T \left[\bar{\alpha}(s)r(s) + \bar{\theta}(s)^\top b(s) - \bar{c}(s)\right] ds + \int_t^T \bar{\theta}(s)^\top \sigma(s)\, dw(s) + \bar{\varepsilon}\varphi(P(T)),$$

which is equal to the time $T+$ wealth from implementing the policy $(\bar{\alpha}, \bar{\theta}, \bar{c})$ starting with an initial time t wealth of x and buying $\bar{\varepsilon}$ contingent claims at time t at a total price of $\bar{C}^{\text{buy}}(x, P, t)$. This strategy is feasible by convexity of the portfolio constraint set $\mathcal{K}$. By concavity of the utility function and the terminal value function, the expected utility from this strategy is

$$J^{\bar{\alpha},\bar{\theta},\bar{c}}_{\bar{\varepsilon},\bar{C}^{\text{buy}}(x,P,t)}(x, P, t) \geq \lambda V_{\varepsilon_1, C^{\text{buy}}_{\varepsilon_1}(x,P,t)}(x, P, t) + (1-\lambda) V_{\varepsilon_2, C^{\text{buy}}_{\varepsilon_2}(x,P,t)}(x, P, t),$$

and hence $V_{\bar{\varepsilon},\bar{C}^{\text{buy}}(x,P,t)}(x, P, t) \geq V(x, P, t)$. Since $V_{\bar{\varepsilon},z}(x, P, t)$ is a decreasing function of z and, by definition, $C^{\text{buy}}_{\bar{\varepsilon}}(x, P, t)$ is such that $V_{\bar{\varepsilon},C^{\text{buy}}_{\bar{\varepsilon}}}(x, P, t)$ is equal to $V(x, P, t)$, the result follows. Part (b) is proven similarly. □

THEOREM 4. (a) The unit reservation buying price $C^{\text{buy}}_{\varepsilon}(x, P, t)/\varepsilon$ is decreasing in the size ε of the position.

(b) The unit reservation selling price $C^{\text{sell}}_{\varepsilon}(x, P, t)/\varepsilon$ is increasing in the size ε of the position.

Proof. To prove (a), note that part (a) of Theorem 3 and the fact that the total reservation price of a zero position is zero imply that

$$\begin{aligned} C^{\text{buy}}_{\lambda\varepsilon}(x, P, t) &= C^{\text{buy}}_{\lambda\varepsilon+(1-\lambda)0}(x, P, t) \geq \lambda C^{\text{buy}}_{\varepsilon}(x, P, t) + (1-\lambda) C^{\text{buy}}_{0}(x, P, t) \\ &= \lambda C^{\text{buy}}_{\varepsilon}(x, P, t) \end{aligned}$$

for all $\varepsilon \geq 0$ and $0 \leq \lambda \leq 1$. In particular, if $0 < \varepsilon_1 < \varepsilon_2$, it follows from choosing $\lambda = \varepsilon_1/\varepsilon_2$ and $\varepsilon = \varepsilon_2$ that

$$C^{\text{buy}}_{\varepsilon_1}(x, P, t) \geq \frac{\varepsilon_1}{\varepsilon_2} C^{\text{buy}}_{\varepsilon_2}(x, P, t)$$

and hence $C^{\text{buy}}_{\varepsilon_1}(x, P, t)/\varepsilon_1 \geq C^{\text{buy}}_{\varepsilon_2}(x, P, t)/\varepsilon_2$. Part (b) can be proven similarly. □

In a set-up similar to that of the present paper, Karatzas and Kou (1996) define the fair price of a contingent claim as the price at which the value function of the investor is unaffected by an infinitesimal position in the contingent claim. In symbols, the fair price $C^{\text{fair}}(x, P, t)$ is the solution to the equation[6]

$$\left.\frac{\partial V_{z/C^{\text{fair}}(x,P,t),z}(x, P, t)}{\partial z}\right|_{z=0} = 0. \tag{3.1}$$

[6] More correctly, the fair price is defined as the unique *weak* solution of (3.1), whenever such a solution exists. For the precise meaning of weak solution, see Definition 7.2 in Karatzas and Kou (1996).

It is intuitively clear that the fair price must lie between the reservation buying and the reservation selling prices. While the fair price seems to be a good valuation measure of small positions in contingent claims with a payoff that only changes the total wealth of the seller or buyer marginally, the reservation prices are appropriate for valuing contingent claims with a payoff that is likely to significantly change the total wealth. This is, for example, often the case for real options.

4. Numerical Computation of Reservation Prices

4.1. GENERAL COMPUTATIONAL ASPECTS

To compute the reservation buying and selling prices and the corresponding hedging policies, two types of stochastic control problems are to be solved: The *no contingent claim problem* (2.3) associated with the situation where the investor only invests in primary assets, and the *fixed ε problem* (2.5) for $\varepsilon = \pm 1$ associated with the purchase/sale of one unit of the contingent claim. The no contingent claim problem is of the well-known infinite horizon controlled diffusion type, and from (2.6) the fixed ε problem can be solved as a finite horizon controlled diffusion problem with a terminal value function which in itself is given by an infinite horizon controlled diffusion problem.

In the remaining part of the paper it is assumed that the utility function is of the CRRA type $U(c) = c^\gamma$ with $0 < \gamma < 1$, that there is a single risky asset ($d = 1$), and that the market coefficients b, σ, and r are constant. Under these assumptions, the dynamics of the wealth process between the date of the purchase/sale of the contingent claim and its expiration date is

$$dX(t) = [rX(t) + \theta(t)(b - r) - c(t)]\, dt + \theta(t)\sigma\, dw(t),$$

and the dynamics of the price of the risky asset is

$$dP(t) = P(t)\,[b\,dt + \sigma\, dw(t)]\,.$$

Since the price does not enter the wealth dynamics, wealth is the only state variable for the no contingent claim problem. For the fixed ε problem, however, the risky asset price must be included as a state variable due to its influence on the terminal reward through the payment $\varepsilon\varphi(P(T))$ upon expiration of the claim.

For this simple model the value function for the no contingent claim problem and the terminal reward function for the fixed ε problem are known in closed form if the only portfolio constraint is a non-negative wealth constraint, cf. Merton (1969). Assuming $A > 0$, where

$$A = \frac{\beta - r\gamma}{1-\gamma} - \frac{\gamma(b-r)^2}{2\sigma^2(1-\gamma)^2},$$

the value function is given by $V(x) = A^{\gamma-1}x^{\gamma}$, and the optimal controls are given by

$$c(x,t) = Ax, \tag{4.1}$$

$$\theta(x,t) = \frac{b-r}{\sigma^2(1-\gamma)}\,x. \tag{4.2}$$

For more general constraints the solution has to be computed numerically. The appearance of the random variable $\varphi(P(T))$ in (2.6) makes it impossible to solve the fixed ε problem analytically. Therefore, to compute the reservation prices, generally both a finite-horizon and an infinite horizon controlled diffusion problem have to be solved with numerical methods.

Several numerical methods have been suggested for solving continuous-time, continuous-state stochastic control problems as those appearing in this paper. The Markov chain approximation approach is a well-documented and intuitive technique. The approach, which is described in, e.g., Kushner (1990) and Kushner and Dupuis (1992), has been applied to various financial optimization problems by Fitzpatrick and Fleming (1991), Hindy et al. (1997), and Munk (1997a, 1999). The main idea of the method is to approximate the state variables of the problem with a discrete time, discrete state Markov chain, and define a control problem for this Markov chain so that the solution (the value function and optimal controls) is a good approximation to the solution of the original continuous-time, continuous-state controlled diffusion problem. The approach can alternatively be seen as a non-standard finite difference approximation to the Hamilton-Jacobi-Bellman (HJB) equation associated with the control problem.

4.2. SOLUTION OF THE INFINITE-HORIZON PROBLEM

The no contingent claims problem to be solved is

$$V(x) = \sup_{(\alpha,\theta,c)\in\mathcal{A}(x,\mathcal{K})} \mathrm{E}_x\left[\int_0^\infty e^{-\beta t}c(t)^{\gamma}\,dt\right] \tag{4.3}$$

in the setting of Section 2 with the simplifications stated in Section 4.1. To simplify the description of the numerical method consider the case where the investor faces a liquidity constraint in the sense that her wealth has to stay non-negative at all times (a.s.) plus possibly additional constraints. Hence, the portfolio constraint set $\mathcal{K}$ is of the form

$$\mathcal{K} = \left\{(\alpha,\theta)\in\mathbb{R}^{d+1}\,|\,\alpha+\theta^{\top}\mathbf{1}\geq 0\right\}\cap\mathcal{K}' \tag{4.4}$$

for some non-empty, closed, convex set $\mathcal{K}'\subseteq\mathbb{R}^{d+1}$. This allows me to consider zero as a lower (and absorbing) bound for the wealth. The dynamics of wealth is given by

$$dX(t) = [rX(t)+\theta(t)(b-r)-c(t)]\,dt+\theta(t)\sigma\,dw(t), \quad X(0)=x.$$

I approximate the controlled diffusion $X(\cdot)$ on the state space $\mathbb{R}_+ = [0, \infty)$ with a controlled Markov chain $\xi^h = \left(\xi_n^h\right)_{n \in \mathbb{Z}_+}$ on the state space $\mathcal{R}^h = \{0, h, 2h, \dots, \bar{x} \equiv Ih\}$. The Markov chain is controlled by a sequence $(\alpha^h, \theta^h, c^h)$ where $\theta^h = (\theta_n^h)_{n \in \mathbb{Z}_+}$ with $\theta_n^h = \theta^h(\xi_n^h)$, etc. Given θ^h and ξ^h the control α^h follows from the relation $\xi_n^h = \alpha_n^h + \theta_n^h$, therefore α^h is suppressed in the sequel. For numerical reasons the controls are bounded so that $|\theta_n^h| \leq K_\theta \xi_n^h$ and $c_n^h \leq K_c \xi_n^h$. An admissible control $(\alpha^h, \theta^h, c^h)$ must, furthermore, satisfy the constraints $(\alpha_n^h, \theta_n^h) \in \mathcal{K}'$, $c_n^h \geq 0$, and $\xi_n^h \geq 0$.

The evolution of the Markov chain is given by a transition probability function $p(x, x'|\theta^h(x), c^h(x))$, where $x, x' \in \mathcal{R}^h$ and $(\theta^h(x), c^h(x))$ is the currently applied control, denoting the probability of going from state x to state x' in one "time step". For $x \in \{h, 2h, \dots \bar{x} - h\}$, let

$$
\begin{aligned}
p(x, x+h|\theta, c) &= \frac{\frac{1}{2}\sigma^2\theta^2 + h\left(rx + \theta^+(b-r)\right)}{Q^h(x)}, \\
p(x, x-h|\theta, c) &= \frac{\frac{1}{2}\sigma^2\theta^2 + h\left(c + \theta^-(b-r)\right)}{Q^h(x)}, \\
p(x, x|\theta, c) &= 1 - p(x, x+h|\theta, c) - p(x, x-h|\theta, c),
\end{aligned}
$$

where

$$
Q^h(x) = \sigma^2 K_\theta^2 x^2 + h(rx + K_\theta x(b-r) + K_c x)
$$

and $\theta^+ = \max(\theta, 0)$ and $\theta^- = \max(-\theta, 0)$. Since the lower bound of $x = 0$ is absorbing let

$$
p(0, 0|\theta, c) = 1.
$$

At the upper bound, take

$$
\begin{aligned}
p(\bar{x}, \bar{x}-h|\theta, c) &= \frac{\frac{1}{2}\sigma^2\theta^2 + h\left(c + \theta^-(b-r)\right)}{Q^h(\bar{x})}, \\
p(\bar{x}, \bar{x}|\theta, c) &= 1 - p(\bar{x}, \bar{x}-h|\theta, c).
\end{aligned}
$$

All other transition probabilities are zero.

Define the *time interpolation interval* $\Delta t^h(x) = h^2/Q^h(x)$, and let $t_n^h = \sum_{m=0}^{n-1} \Delta t^h(\xi_m^h)$. The value function of the controlled Markov chain is defined as

$$
V^h(x) = \sup_{(\alpha^h, \theta^h, c^h) \in \mathcal{A}^h(x, \mathcal{K})} \mathrm{E}\left[\sum_{n=0}^{\infty} e^{-\beta t_n^h} \left(c_n^h\right)^\gamma \Delta t^h(\xi_n^h) \middle| \xi_0^h = x\right].
$$

It can be shown that $V^h(x) \to V(x)$ as $h \to 0$ and the artificial bound parameters K_θ and K_c go to infinity. The computation of $V^h(\cdot)$ and the corresponding optimal policies is based on the dynamic programming equation

$$V^h(x) = \sup_{\substack{|\theta^h(x)| \le K_\theta x \\ 0 \le c^h(x) \le K_c x \\ (\alpha^h(x), \theta^h(x)) \in \mathcal{K}}} \left\{ e^{-\beta \Delta t^h(x)} \sum_{x' \in \mathcal{R}^h} p(x, x' | \theta^h(x), c^h(x)) V^h(x') + \Delta t^h(x) c^h(x)^\gamma \right\}, \tag{4.5}$$

which, e.g., can be solved with the *policy iteration algorithm*, cf. Rust (1996). The approach is almost equivalent to a particular finite difference approximation to the Hamilton-Jacobi-Bellman (HJB) equation associated with the problem (4.3). This relation is discussed further in Appendix A.

4.3. SOLUTION OF THE FINITE-HORIZON PROBLEM

The finite-horizon problem I want to solve is of the form

$$\begin{aligned} V(x, P, t) \;=\; & \sup_{(\alpha, \theta, c) \in \mathcal{A}(x, t, \mathcal{K})} \mathrm{E}_{x, P, t} \left[\int_t^T e^{-\beta(s-t)} c(s)^\gamma \, ds \right. \\ & \left. + e^{-\beta(T-t)} V\left([X(T) + \varepsilon \varphi(P(T))]^+ \right) \right] \end{aligned} \tag{4.6}$$

where the risky asset price with dynamics

$$dP(s) = P(s) \left[b \, ds + \sigma \, dw(s) \right], \quad P(t) = P,$$

must be added as an additional state variable compared to the infinite-horizon problem. The constraint set is again of the form (4.4). Note that for an investor buying options ($\varepsilon > 0$), zero is an absorbing boundary for wealth between time t and T, but wealth will go from zero to positive at time T in case the option ends up in-the-money. An investor selling options ($\varepsilon < 0$) may end up at time T with insufficient wealth to honor the claim of the owner of the option. In that case, I assume that the owner receives the entire wealth of the seller who is then confined to zero consumption in all future. Obviously, for a seller likely to default on the option, it may be optimal to consume all wealth just before the maturity of the option. Alternatively, various default protection rules could be incorporated in the analysis.

The dynamic programming problem stated above involves a two-dimensional state variable process (x, P) with covariance matrix

$$S(P(t), \theta(t)) = \begin{pmatrix} \sigma^2 \theta(t)^2 & \sigma^2 \theta(t) P(t) \\ \sigma^2 \theta(t) P(t) & \sigma^2 P(t)^2 \end{pmatrix}.$$

Because this matrix is not uniformly diagonally dominated, and since it depends on one of the control variables, $\theta(t)$, in a non-trivial way, it is not possible to approximate (x, P) (or a transformation of (x, P)) by a two-dimensional Markov chain with "sufficient accuracy" (see Kushner and Dupuis (1992) for the precise requirements). The standard procedure for deriving the transition probabilities results in "probabilities" which may be negative for some combinations of states and controls. The Markov chain approximation approach boils down to solving a dynamic programming equation like (4.5), however, and this equation can generally be solved even when some of the $p(\cdot, \cdot|\cdot)$ terms are negative. Therefore, I can still compute an approximation $V^h(x)$ for $x \in \mathcal{R}^h$ in this way although the Markov chain interpretation breaks down. Since the present proofs of the convergence of the Markov chain approximation approach rely heavily on the non-negativity of these probabilities, convergence is not guaranteed. On the other hand, no other well-known numerical methods seem to ensure convergence when applied to the problem studied here. Therefore, I shall stick as closely as possible to the Markov chain approximation approach. The numerical results presented in the following sections are very plausible.

Approximate the state space $\mathbb{R}_+^2$ of the state variable (X, P) with the grid

$$\mathcal{R}_h = \{0, h_x, 2h_x, \ldots, Ih_x \equiv \bar{x}\} \times \left\{0, h_P, 2h_P, \ldots, Jh_P \equiv \bar{P}\right\} \tag{4.7}$$

and partition the time interval $[0, T]$ into $N + 1$ subintervals with the boundary points

$$0, \tau, 2\tau, \ldots, N\tau \equiv T.$$

Following the standard implicit Markov chain approximation procedure for finite-horizon problems, cf. Kushner and Dupuis (1992, Ch. 12), I get the following transition *weights* for transitions from an interior state $(x, P) \in \mathcal{R}_h$ at some time $n\tau$ (for notational ease, the h superscripts on controls, etc., are skipped):

$$p(x, P, n\tau; x, P, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(x, P)}, \tag{4.8a}$$

$$p(x, P, n\tau; x', P', n\tau + \tau|\theta, c) = 0, \quad (x', P') \neq (x, P), \tag{4.8b}$$

$$p(x, P, n\tau; x + h_x, P, n\tau|\theta, c)$$
$$= \frac{\frac{1}{h_x}(rx + \theta^+(b - r)) - \frac{1}{2h_x h_P}|\theta|P\sigma^2 + \frac{1}{2h_x^2}\theta^2\sigma^2}{Q^h(x, P)}, \tag{4.8c}$$

$$p(x, P, n\tau; x - h_x, P, n\tau|\theta, c)$$
$$= \frac{\frac{1}{h_x}(\theta^-(b - r) + c) - \frac{1}{2h_x h_P}|\theta|P\sigma^2 + \frac{1}{2h_x^2}\theta^2\sigma^2}{Q^h(x, P)}, \tag{4.8d}$$

$$p(x, P, n\tau; x, P + h_P, n\tau|\theta, c) = \frac{\frac{1}{h_P} bP - \frac{1}{2h_x h_P}|\theta| P\sigma^2 + \frac{1}{2h_P^2} P^2\sigma^2}{Q^h(x, P)}, \tag{4.8e}$$

$$p(x, P, n\tau; x, P - h_P, n\tau|\theta, c) = \frac{-\frac{1}{2h_x h_P}|\theta| P\sigma^2 + \frac{1}{2h_P^2} P^2\sigma^2}{Q^h(x, P)}, \tag{4.8f}$$

$$p(x, P, n\tau; x + h_x, P + h_P, n\tau|\theta, c) = \frac{\theta^+ P\sigma^2}{2h_x h_P Q^h(x, P)}, \tag{4.8g}$$

$$p(x, P, n\tau; x - h_x, P - h_P, n\tau|\theta, c) = \frac{\theta^+ P\sigma^2}{2h_x h_P Q^h(x, P)}, \tag{4.8h}$$

$$p(x, P, n\tau; x + h_x, P - h_P, n\tau|\theta, c) = \frac{\theta^- P\sigma^2}{2h_x h_P Q^h(x, P)}, \tag{4.8i}$$

$$p(x, P, n\tau; x - h_x, P + h_P, n\tau|\theta, c) = \frac{\theta^- P\sigma^2}{2h_x h_P Q^h(x, P)}, \tag{4.8j}$$

$$p(x, P, n\tau; x, P, n\tau|\theta, c) = 1 - \sum_{(x', P') \in \mathcal{S}(x, P)} p(x, P, n\tau; x', P', n\tau|\theta, c) - \frac{1/\tau}{Q^h(x, P)}, \tag{4.8k}$$

$$p(x, P, n\tau; x', P', n\tau|\theta, c) = 0, \quad (x', P') \notin \mathcal{S}(x, P), \tag{4.8l}$$

where

$$\mathcal{S}(x, P) = \Big\{ (x + h_x, P), (x - h_x, P), (x, P + h_P), (x, P - h_P), (x + h_x, P + h_P), (x + h_x, P - h_P), (x - h_x, P + h_P), (x - h_x, P - h_P) \Big\}.$$

For completeness, the transition weights for boundary states are given in Appendix B.[7]

[7] In the implicit version of the Markov chain approximation approach, time is considered another state variable. Therefore, it is possible for (x, P) to move to another value without changing the time. While this may at first depreciate the intuitive appeal of the Markov chain approximation, it is no different from the underlying approximation involved in the traditional implicit finite difference approach to linear second-order PDEs that are frequently met in option pricing. There is also an explicit version of the Markov chain approximation approach, but this suffers from the same stability problems as the explicit finite difference method for linear second-order PDEs.

To simplify the policy improvement steps of the policy iteration algorithm, again impose artificial bounds on the controls,

$$c_{\varepsilon,z}(x, P, t) \leq K_c x, \qquad |\theta_{\varepsilon,z}(x, P, t)| \leq K_\theta x.$$

Let

$$Q^h(x, P) = \frac{1}{\tau} + \frac{1}{h_x}(r + K_\theta(b - r) + K_c)x + \frac{1}{h_P}bP + \frac{1}{h_x^2}K_\theta^2 x^2\sigma^2 + \frac{1}{h_P^2}P^2\sigma^2. \tag{4.9}$$

The dynamic programming equation for the approximating discrete problem is

$$V_{\varepsilon,z}(x, P, n\tau) = \sup_{\substack{0 \leq c \leq K_c x \\ |\theta| \leq K_\theta x}} \left\{ e^{-\beta\Delta t(x,P)} \sum_{(x',P') \in \mathcal{S}(x,P)} p(x, P, n\tau; x', P', n\tau|\theta, c)V_{\varepsilon,z}(x', P', n\tau) + e^{-\beta\Delta t(x,P)}p(x, P, n\tau; x, P, n\tau + \tau|\theta, c)V_{\varepsilon,z}(x, P, n\tau + \tau) + \Delta t(x, P)c^\gamma \right\}, \quad x \in \mathcal{R}_h, \tag{4.10}$$

where $\Delta t(x, P) = 1/Q^h(x, P)$. The terminal value is

$$V_{\varepsilon,z}(x, P, T) = V\left((x + \varepsilon\varphi(P))^+\right).$$

The problem can again be solved with the policy iteration algorithm.[8] The relation between the "pseudo" Markov chain approximation approach outlined above and

[8] The system (4.10) can be written as a matrix equation of the form $\mathbf{V} = \sup_{\mathbf{c},\boldsymbol{\theta},} \{\mathbf{M}_{\boldsymbol{\theta},\mathbf{c}}\mathbf{V} + \mathbf{d}_{\boldsymbol{\theta},\mathbf{c}}\}$, where $\mathbf{V}$ is the vector $\left(V_{\varepsilon,z}(x, P, n\tau)\right)_{(x,P)\in\mathcal{R}_h}$, $\mathbf{d}_{\boldsymbol{\theta},\mathbf{c}}$ is the vector with entries

$$d(x, P) = e^{-\beta\Delta t(x,P)}p(x, P, n\tau; x, P, n\tau + \tau|\theta, c)V_{\varepsilon,z}(x, P, n\tau + \tau) + \Delta t(x, P)c^\gamma,$$

and $\mathbf{M}_{\boldsymbol{\theta},\mathbf{c}}$ is the $|\mathcal{R}_h| \times |\mathcal{R}_h|$ dimensional [where $|\mathcal{R}_h| = (I + 1)(J + 1)$] band matrix with entries $M(x, P; x', P') = e^{-\beta\Delta t(x,P)}p(x, P, n\tau; x', P', n\tau|\theta, c)$. If the elements of $\mathcal{R}_h$ are enumerated in the order

$$\mathcal{R}_h = \{(0, 0), (h_x, 0), (2h_x, 0), \ldots, (Ih_x, 0), (0, h_P), (h_x, h_P), \ldots, (Ih_x, h_P), \ldots, (0, Jh_P), (h_x, Jh_P), \ldots, (Ih_x, Jh_P)\},$$

the matrix $\mathbf{M}_{\boldsymbol{\theta},\mathbf{c}}$ will be a band matrix of width $2I + 5$. If the elements of $\mathcal{R}_h$ are enumerated in the order

$$\mathcal{R}_h = \{(0, 0), (0, h_P), (0, 2h_P), \ldots, (0, Jh_P), (h_x, 0),$$

a finite difference scheme for the HJB equation associated with the problem is studied in Appendix A.

5. Reservation Prices for a European Call Option: Liquidity Constraints

In this and the following section, the reservation prices for a European call option are studied within the simple constant coefficient, single risky asset model outlined in Section 4.1. The option payoff is $\varphi(P(T)) = (P(T) - \chi)^+$, where χ is the exercise price, and the complete market price is the Black–Scholes price

$$C^{\mathrm{BS}}(P) = P\mathcal{N}(\eta_+) - \chi e^{-rT}\mathcal{N}(\eta_-),$$

where

$$\eta_\pm = \frac{1}{\sigma\sqrt{T}}\log\frac{P}{\chi e^{-rT}} \pm \frac{1}{2}\sigma\sqrt{T},$$

and $\mathcal{N}$ is the cumulative normal distribution function. In this section, the only constraint on the portfolios is the liquidity (or no-bankruptcy) constraint, i.e.,

$$\mathcal{K} = \left\{(\alpha, \theta) \in \mathbb{R}^2 \,|\, \alpha + \theta \geq 0\right\}.$$

In the next section, I shall study the impact of borrowing constraints on the results. The parameter values are taken to be: $\beta = 1, b = 0.1, r = 0.05, \sigma = 0.3, \gamma = 0.5, \chi = 100, T = 0.5$. The constant A is then approximately equal to 1.9222. Take the initial time $t = 0$. For the policy space stopping criterion I take a tolerance of $\varepsilon = 10^{-4}$. The constants K_c and K_θ are set equal to 5. I impose an upper bound on the price of the risky asset at $\bar{P} = 400$.[9] The artificial upper bound on wealth, $\bar{x}$, is varied in the experiments as will appear from the results presented below. The numerical results suggest a relation between I, J, and N of the magnitude $I/J = 50$ and $I/N = 400$. Due to memory limits on the computer, I have therefore taken $I = 2000$, $J = 40$, and $N = 5$.[10]

$$(h_x, h_P), \ldots, (h_x, Jh_P), \ldots, (Ih_x, 0), (Ih_x, h_P), \ldots, (Ih_x, Jh_P)\},$$

the matrix $\mathbf{M}_{\theta,\mathbf{c}}$ will be a band matrix of width $2J + 5$. Since the necessary algebraic operations in the solution of the system of equations is proportional to the square of the band width and the numerical results indicate that I should be much greater than J in order to obtain a high precision, I use the second of these two enumerations.

[9] From Munk (1998) it is known that the numerical results can be relatively imprecise near the artificial upper bounds. Therefore, the upper bound on P is taken to be large relative to the exercise price.

[10] To improve precision I could consider applying the Richardson extrapolation technique. Successful application of this trick requires that the numerical results indicate a smooth convergence for each of the approximation parameters h_x, h_P, and τ, i.e., that a convergence order can be

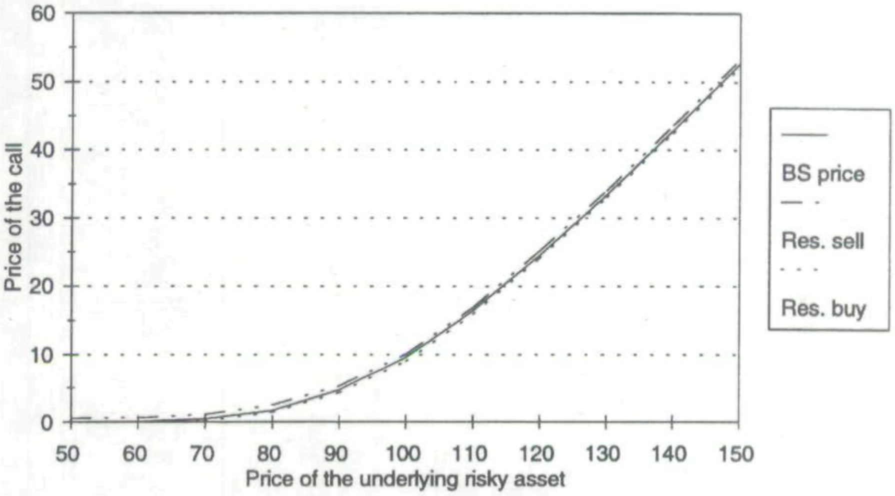

Figure 1. The (time 0) reservation selling price $C^{\text{sell}}(x, P)$ and the reservation buying price $C^{\text{buy}}(x, P)$ compared to the Black–Scholes price $C^{\text{BS}}(P)$ of the European call option starting with a wealth of $x = 1000$. Results from an implementation with $\bar{x} = 4000$.

In Figure 1 I compare the numerically computed reservation selling price $C^{\text{sell}}(x, P)$ and reservation buying price $C^{\text{buy}}(x, P)$ with the Black–Scholes price $C^{\text{BS}}(P)$ for an investor with an initial wealth of $x = 1000$. Over the entire range of P's shown in the figure the reservation selling price is higher than the Black–Scholes price which again is higher than the reservation buying price.[11]

The percentage distance between the reservation prices and the Black–Scholes call price is depicted in Figure 2. For relatively small values of the price of the underlying asset the percentage difference is substantial, although the absolute difference is relatively small.

Next, I consider the implicit hedge strategies of the contingent claim, i.e., the difference in the optimal trading and consumption strategies with and without the given position in the contingent claim. Let $\theta_{\varepsilon,z}(x, P, u)$ and $c_{\varepsilon,z}(x, P, u)$ denote the optimal investment and consumption policies given wealth x and prices P at time u when the investor took a position of ε in the contingent claim at time 0 at a total price of z. Let $\theta(x, P, u)$ and $c(x, P, u)$ denote the optimal policies without engaging in contingent claims. The number of risky assets in the implicit hedge of a

estimated with some confidence. However, it was not possible to give a precise estimate of the order of convergence with respect to h_P, presumably because h_P was too big in the implementations. As explained in the text, I should be large compared to J, i.e., h_x small compared to h_P, in order that the contribution of these two approximation parameters are of the same magnitude. Richardson extrapolation can therefore not be recommended without further analysis.

[11] The numerically computed value functions are only defined on the grid $\mathcal{R}_h$. In the computation of the reservation prices, linear interpolation is applied.

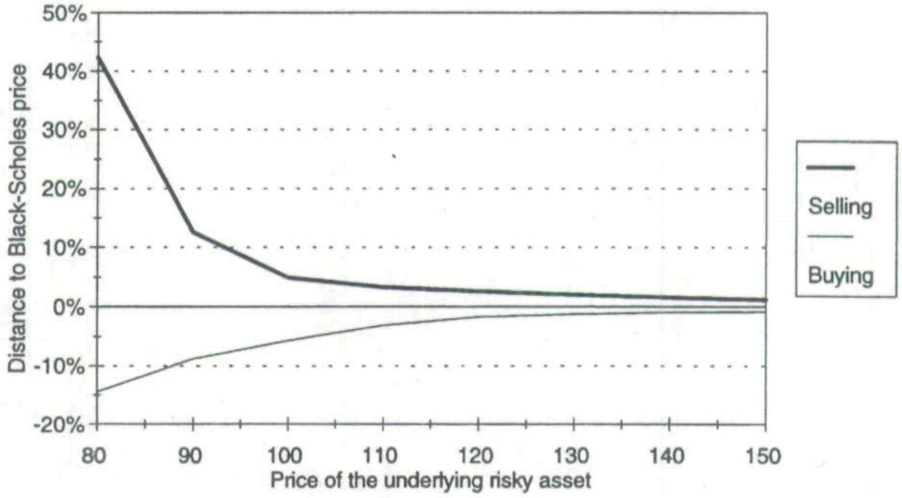

Figure 2. The percentage distance of the reservation selling price $C^{\text{sell}}(x, P)$ and the reservation buying price $C^{\text{buy}}(x, P)$ to the Black–Scholes price $C^{\text{BS}}(P)$ of the European call option starting with a wealth of $x = 1000$. Results from an implementation with $\bar{x} = 4000$.

purchase and sale, respectively, of one unit of the contingent claim at the investor's reservation price is then given by

$$\begin{aligned}\Delta_{\text{buy}}(x, P, u) &= \frac{1}{P(u)}\left(\theta_{1,C^{\text{buy}}(x,P)}(x, P, u) - \theta(x, P, u)\right),\\ \Delta_{\text{sell}}(x, P, u) &= \frac{1}{P(u)}\left(\theta_{-1,C^{\text{sell}}(x,P)}(x, P, u) - \theta(x, P, u)\right),\end{aligned}$$

and the corresponding hedge consumption rate is given by

$$\begin{aligned}\Delta^{c}_{\text{buy}}(x, P, u) &= c_{1,C^{\text{buy}}(x,P)}(x, P, u) - c(x, P, u),\\ \Delta^{c}_{\text{sell}}(x, P, u) &= c_{-1,C^{\text{sell}}(x,P)}(x, P, u) - c(x, P, u).\end{aligned}$$

Here, I focus on the immediate hedge, i.e., the changes in the policies at time 0. For the present problem, $\theta(x, P)$ and $c(x, P)$ are known from the solution to Merton's problem, cf. (4.1)–(4.2). Figure 3 shows the number of risky assets in the hedge portfolio of the investor. To be more precise, Δ_{sell} and $-\Delta_{\text{buy}}$ are graphed, together with the Black–Scholes Delta $\Delta_{\text{BS}} = \mathcal{N}(\eta_+)$ which is the number of units of the underlying asset in the perfectly replicating portfolio of the call option in the Black–Scholes world. For a wide range of prices of the underlying asset, the three curves are almost identical, but for relatively low prices of the underlying asset the selling and buying hedges miss Δ_{BS} considerably. This is more obvious when looking at Figure 5 which shows the percentage difference between the selling and buying hedges and Δ_{BS}, i.e., $(\Delta_{\text{sell}} - \Delta_{\text{BS}})/\Delta_{\text{BS}}$ and $(\Delta_{\text{buy}} + \Delta_{\text{BS}})/\Delta_{\text{BS}}$.

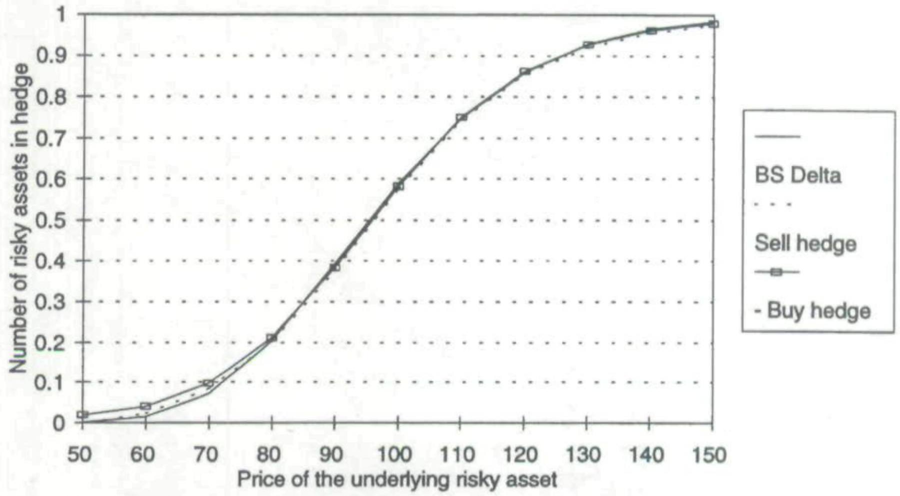

Figure 3. The selling and buying hedge portfolios for the European call compared with the Black–Scholes Delta. Results for an initial wealth of $x = 1000$ from an implementation with $\bar{x} = 4000$.

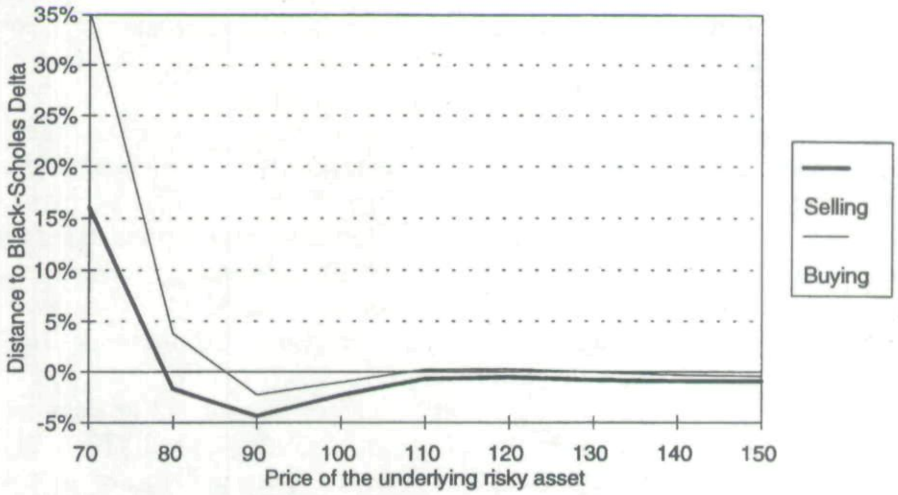

Figure 4. The percentage distance of the selling and buying hedge portfolios to the Black–Scholes hedge portfolio. Results for an initial wealth of $x = 1000$ from an implementation with $\bar{x} = 4000$.

The hedge consumption rates, Δ^c_{sell} and Δ^c_{buy}, are both slightly negative for an initial wealth of $x = 1000$ for all levels of the price of the underlying asset, but seen as a fraction of the total consumption rate the hedge consumption is insignificant.

Altogether, these results show that, as expected, the investor hedges a long or a short call option by selling or buying the Black–Scholes replicating portfolio and keeping consumption unchanged. The differences between the numerically com-

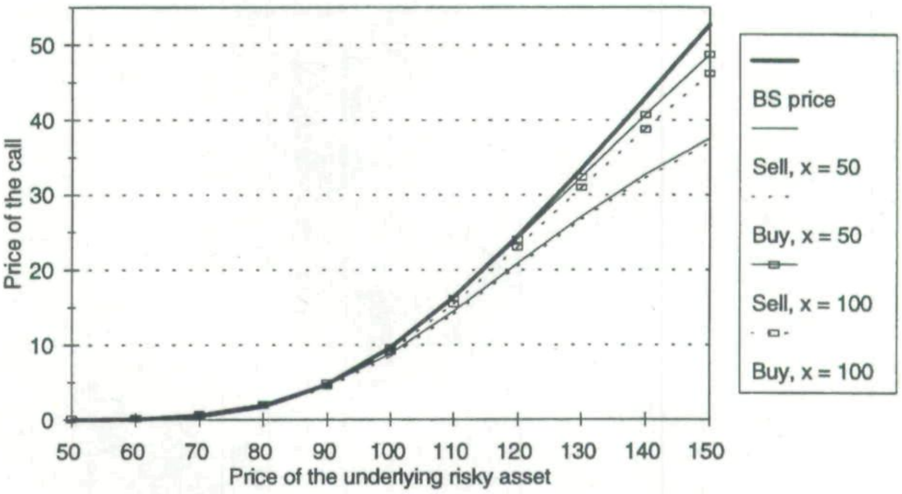

Figure 5. The reservation selling price $C^{\text{sell}}(x, P)$ and reservation buy price $C^{\text{buy}}(x, P)$ of the European call option starting with a wealth of x compared to the Black–Scholes price $C^{\text{BS}}(P)$. Results from an implementation with $\bar{x} = 400$.

puted prices and hedge portfolios and the Black–Scholes price and hedge portfolio must be attributed to approximation errors of the numerical approach.

The results discussed above were derived for an investor with a high initial wealth as compared to the exercise price, which I implicitly assume that the present price of the underlying asset is near. The same conclusions are not valid, however, if the initial wealth is of the same magnitude as the price of the underlying asset. Figure 5 shows the reservation selling and reservation buying price of the investor starting with a wealth of $x = 50$ and $x = 100$, respectively, compared to the Black–Scholes price. For relatively high prices of the underlying risky asset, both reservation prices are significantly lower than the Black–Scholes price. The lower the initial wealth x, the higher the difference.

In Figure 6, I compare the implicit hedge portfolio of the investor with the Black–Scholes hedge when her initial wealth is relatively small. The number of risky assets the investor buys when she writes the option at her reservation selling price is much smaller than the Black–Scholes Delta. In fact, for high prices of the underlying asset, the investor will *sell* a fraction of the underlying risky asset when she sells the call option. If the investor buys the call at her reservation buying price, she will sell a fraction of the underlying asset, but a much smaller fraction than that prescribed by the Black–Scholes hedge.

With a relatively high initial wealth the consumption policy of the investor is roughly unaffected by the selling or buying of the call option. From Figure 7, it can be seen that this is certainly not the case when the initial wealth of the investor is low compared to the price of the underlying risky asset. If she sells the call at her reservation selling price, she will consume more than before. If she buys the call at her reservation buying price, she will consume less than before. To get a

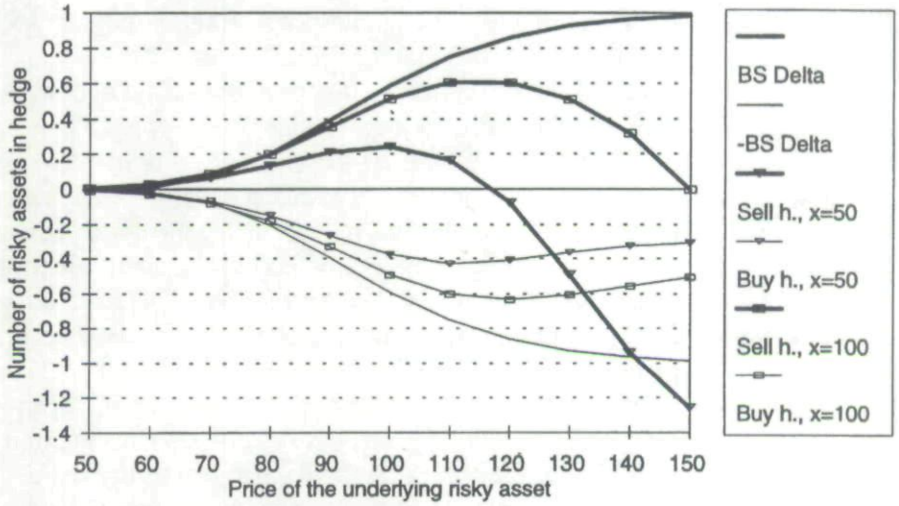

Figure 6. The selling and buying hedge portfolios for the European call compared to the Black–Scholes Delta. Results from an implementation with $\bar{x} = 400$.

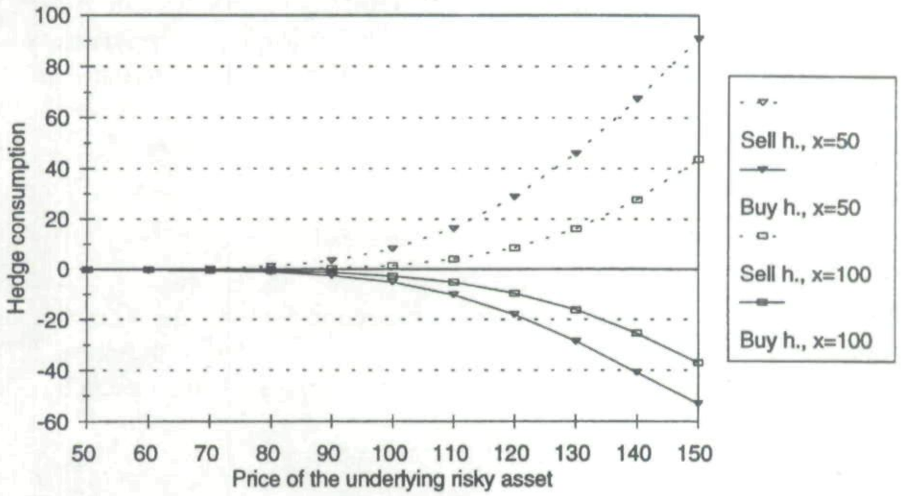

Figure 7. The selling and buying hedge consumption rates for the European call. Results from an implementation with $\bar{x} = 400$.

feeling of the magnitude of the change in consumption compare with the optimal consumption rate without engaging in contingent claims which is Ax where $A \approx 1.9222$.

So if the wealth of the investor is small relative to the price of the underlying asset, the behavior of the investor is as follows. If she sells the call, she will consume more and buy a fraction of the underlying asset smaller than the Black–Scholes Delta (in some instances even a negative fraction). If she buys the call, she will consume less and sell a fraction of the underlying asset smaller than the Black–

Scholes Delta. Both her reservation selling and her reservation buying prices are significantly smaller than the Black–Scholes price.

The explanation behind this is, of course, the presence of the non-negative liquid wealth constraint and the assumed implications of zero wealth discussed in the beginning of Section 4.3. When the investor sells a deep-in-the-money option with low initial wealth, there will be a considerable probability of default at the expiration date of the option. Therefore, she will not hedge the option completely, and she will be more willing to consume out of her wealth before that date. In fact for an initial wealth of 50, she will sell part of her position in the underlying asset to increase her consumption even more. By selling the option, a near default investor gets an opportunity to consume at the expense of the buyer. Therefore, she will accept a relatively low price for the option. If she buys a deep-in-the-money option with low initial wealth, she will have to give up consumption in the period before the expiration of the option where the marginal utility she gets from the consumption is very high. Instead, she will, probably, be able to increase her consumption later on because of the prospective exercise value of the option, but since her wealth at that point in time is likely to be higher than her initial wealth, and because of her high time preference rate, the prospective future increase in consumption is not that valuable to the investor. Hence, her reservation buying price is relatively low compared to the case where her initial wealth is relatively high. Her reservation buying price is anyhow bounded from above by her initial wealth.

6. Reservation Prices for a European Call Option: Borrowing Constraints

In this section I shall compute reservation prices of the European call option with the same assumptions and parameters as in Section 5, except that I now impose the additional portfolio constraint that the investor is not allowed to borrow funds by shorting the savings account. The portfolio constraint set is then given by

$$\mathcal{K} = \left\{(\alpha, \theta) \in \mathbb{R}^2 \,|\, \alpha + \theta \geq 0\right\} \cap (\mathbb{R}_+ \times \mathbb{R}).$$

Of course, the borrowing constraint implies that the amount invested in the risky asset is bounded from above by the wealth of the agent,

$$\theta(t) \leq X_x^{\alpha,\theta,c}(t; \varepsilon, z), \quad \forall t \geq 0, \tag{6.1}$$

or in other words that the fraction of wealth invested in the risky asset cannot exceed one. Denote the value function under the borrowing constraint by $\bar{V}$ and use similar notation for the optimal policies.

As before, I shall derive the reservation prices by comparing the value functions when one contingent claim is sold/bought, $\bar{V}_{\pm 1,z}(x, P)$, with the no-options value function $\bar{V}(x)$. Knowledge of $\bar{V}(x)$ is also needed to start the backwards procedure for the computation of $\bar{V}_{\pm 1,z}(x, P)$. Unfortunately, the value function $\bar{V}(x)$ under the portfolio constraint introduced above is not known, cf. the literature review in Munk (1997b). Therefore, I must first numerically compute

$$\bar{V}(x) = \sup_{(\alpha,\theta,c)\in\bar{\mathcal{A}}(x)} \mathrm{E}_x\left[\int_0^\infty e^{-\beta t} c(t)^\gamma \, dt\right]$$

where

$$\bar{\mathcal{A}}(x) = \left\{(\alpha, \theta, c) \,\middle|\, X_x^{\alpha,\theta,c}(t) \geq 0 \quad \text{and} \quad \theta(t) \leq X_x^{\alpha,\theta,c}(t), \quad \forall t \geq 0\right\}.$$

Here, the procedure outlined in Section 4.2 can be applied. Because $b > r$, the optimal risky investment without options will be non-negative, hence $\theta^- = 0$. Furthermore, I can use

$$Q^h(x) = \sigma^2 x^2 + hx(b + K),$$

where K again is an artificial upper bound on the relative consumption rate c/x, as the denominator in the transition probabilities.

To compute the value functions $\bar{V}_{\pm 1,z}(x, P)$, I proceed as in Section 4.3, but now with the constraint

$$-K_\theta x \leq \theta_{\varepsilon,z}(x, P, t) \leq x$$

on the risky investment. With $K_\theta \geq 1$, I can apply exactly the same denominator $Q^h(x, P)$ as in (4.9) and the same set of transition probabilities as in (4.8).[12] Note that the numerically computed value function with no options is used both as the starting point in the backwards iterative computation of the value function with options at time zero *and* in the actual computation of the reservation selling and reservation buying prices.

The optimal risky investment without portfolio constraints is given by (4.2), and with the assumed parameter values

$$\frac{b - r}{\sigma^2(1 - \gamma)} \approx 1.1111.$$

Therefore, the constraint (6.1) is binding. As expected, the numerical computations show that the optimal investment policy under the constraint is to invest the entire wealth in the risky asset,

$$\bar{\theta}(x, P) = x,$$

[12] It is tempting to set $K_\theta = 1$ since I know that with no options the optimal risky investment is always positive, so the lower bound $-K_\theta x$ is superfluous. But with options there are circumstances under which this is not the case. For very low levels of initial wealth, the optimal risky investment of an investor writing an option will, in fact, be negative and can even be below $-x$.

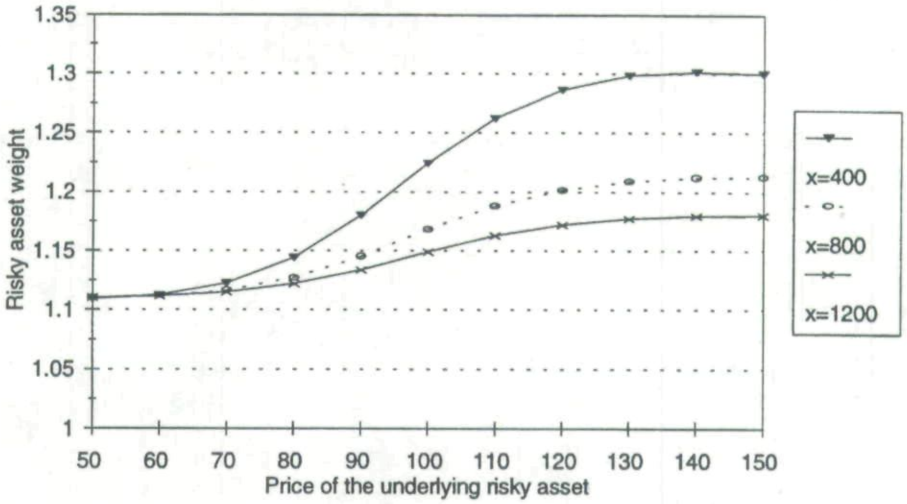

Figure 8. The fraction of wealth optimally invested in the risky asset, $\theta_{-1,C^{\text{sell}}(x,P)}(x, P)$, when one call option is written at the reservation selling price $C^{\text{sell}}(x, P)$ and there is no borrowing constraint.

i.e., to be at the boundary of the set of admissible policies. The constrained optimal consumption rate is not substantially different from the unconstrained rate given in (4.1), and the value function is only slightly reduced due to the borrowing constraint.

When the investor writes an option, she would like to buy a fraction of the risky asset to hedge his position. As discussed in Section 5, in the unconstrained case, he buys a number of units of the risky asset equal to the Black–Scholes Delta except for combinations of a low wealth and a high price of the underlying asset where the results are influenced by bankruptcy considerations. I shall ignore such combinations in the following discussion to separate the bankruptcy effects from the borrowing constraint effects. Figure 8 shows the fraction of wealth optimally invested in the risky asset, $\theta_{-1,C^{\text{sell}}(x,P)}(x, P)/x$, when there is no borrowing constraint and the investor has written a call option at her reservation selling price $C^{\text{sell}}(x, P)$. Obviously, the borrowing constraint is even more restrictive when the investor has written an option.

Under the borrowing constraint the amount optimally invested in the risky asset after selling the call on the asset at a price of z is

$$\bar{\theta}_{-1,z}(x, P) = \bar{\theta}_{-1,z}(x + z, P, 0+) = x + z.$$

The hedge portfolio of the option writer consists of

$$\bar{\Delta}_{\text{sell}}(x, P) \;=\; \frac{1}{P}\left[\bar{\theta}_{-1,\bar{C}^{\text{sell}}(x,P)}(x + \bar{C}^{\text{sell}}(x, P), P, 0+) - \bar{\theta}(x, P)\right]$$

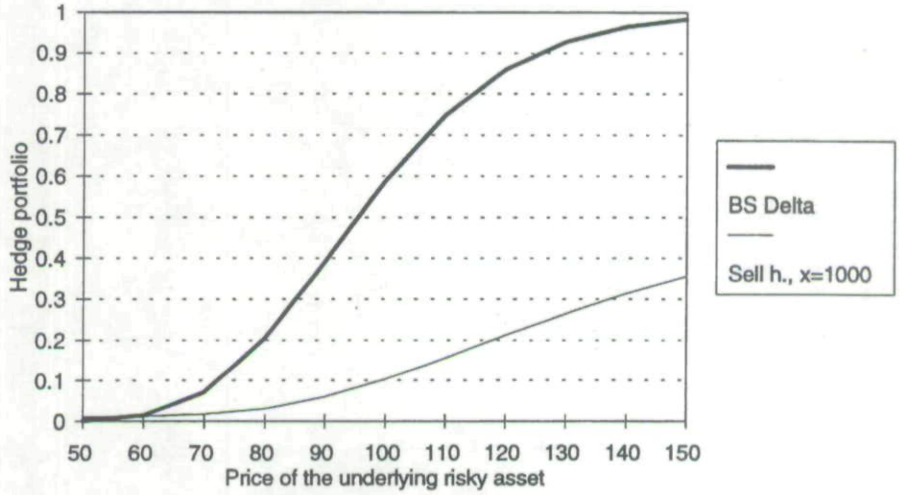

Figure 9. The number of units of the underlying risky asset in the hedge of a written call option, $\bar{\Delta}_{\text{sell}}(x, P)$, when the investor faces a borrowing constraint.

$$
\begin{aligned}
&= \frac{1}{P}\left[x + \bar{C}^{\text{sell}}(x, P) - x\right] \\
&= \frac{\bar{C}^{\text{sell}}(x, P)}{P}
\end{aligned}
$$

units of the underlying risky asset, i.e., the investor is only able to increase his risky investment by exactly the proceeds from selling the option. Figure 9 depicts the number of risky assets in the hedge portfolio of the investor, $\bar{\Delta}_{\text{sell}}(x, P)$, when the initial wealth is $x = 1000$. Since the reservation selling price $\bar{C}^{\text{sell}}(x, P)$ does not vary significantly with x, neither will the hedge portfolio (again ignoring combinations of low wealth and high prices of the underlying asset). The Black–Scholes Delta Δ_{BS} is also shown in the figure. As in the unconstrained case, the optimal consumption rate when the investor has written an option is not significantly different from the no-options case. Contrary to the unconstrained case, the investor does not borrow funds to finance the purchase of the underlying asset hedging the writing of the call simply because the constraint makes it impossible to borrow funds.

Obviously, it is impossible to hedge the written option perfectly and, therefore, the investor requires a higher price for the option before she is willing to sell it, i.e., the reservation selling price $\bar{C}^{\text{sell}}(x, P)$ is higher in the constrained case than in the unconstrained case. In Figure 10, I have graphed the percentage increase in the reservation selling price relative to the unconstrained case, $(\bar{C}^{\text{sell}}(x, P) - C^{\text{sell}}(x, P))/C^{\text{sell}}(x, P)$. As can be seen from Figure 8, the borrowing constraint is more restrictive the lower the initial wealth x is. Therefore, the increase in the reservation selling price relative to the unconstrained case should be decreasing in initial wealth, and this is exactly what the results show.

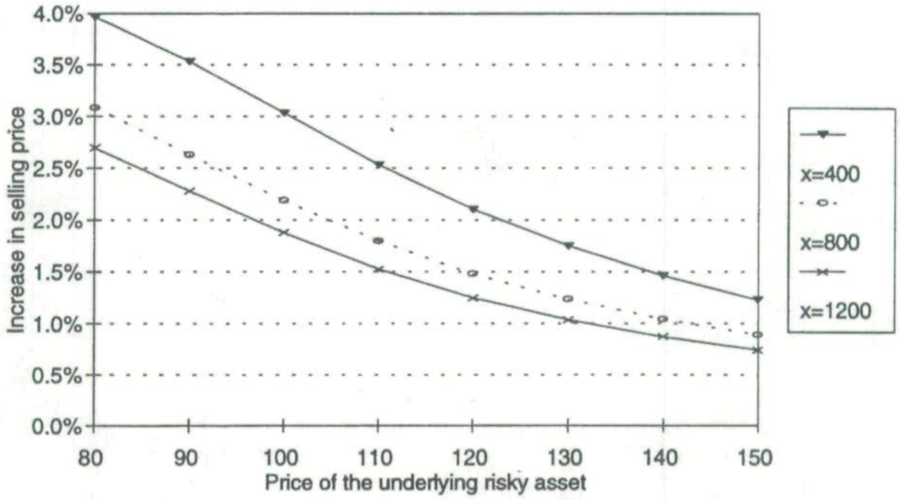

Figure 10. The percentage increase in the reservation selling price caused by the borrowing constraint, $(\bar{C}^{\mathrm{sell}}(x, P) - C^{\mathrm{sell}}(x, P))/C^{\mathrm{sell}}(x, P)$.

When the investor buys the option, she wants to hedge her position by selling a fraction of the underlying risky asset. The unconstrained optimal risky investment when buying the call will therefore be lower than in the no-options case. This can be seen from Figure 11 which shows the fraction of wealth optimally invested in the risky asset, $\theta_{1,C^{\mathrm{buy}}(x,P)}(x, P)/x$, when one call is bought at the reservation buying price $C^{\mathrm{buy}}(x, P)$ and there is no borrowing constraint. The lower the initial wealth and the higher the price of the underlying asset, the less restrictive the constraint. In fact, for combinations of relatively low initial wealth and relatively high prices of the underlying asset the amount optimally invested in the risky asset is lower than the initial wealth. In these situations it is to be expected that the constraint (6.1) will be non-binding. This is confirmed by Figure 12 which shows the fraction of wealth optimally invested in the risky asset, $\bar{\theta}_{-1,\bar{C}^{\mathrm{buy}}(x,P)}(x, P)/x$, under the borrowing constraint when one call option is bought at the reservation buying price $\bar{C}^{\mathrm{buy}}(x, P)$. This fraction is practically indistinguishable from the minimum of the unconstrained optimal risky asset weight $\theta_{-1,C^{\mathrm{buy}}(x,P)}(x, P)/x$ and the upper bound of 1.

For the combinations of initial wealth and risky asset price where the optimal risky investment when buying a call is at the upper bound, the hedge portfolio of the investor consists of selling $\bar{C}^{\mathrm{buy}}(x, P)/P$ units of the underlying asset. The proceeds from this sale finance the purchase of the call option exactly. For wealth/price combinations for which the optimal risky asset weight when buying an option is below one, the hedge portfolio consists of selling a larger fraction of the underlying asset than necessary to buy the option. This is evident from Figure 13. For an initial wealth of $x = 1200$ and the range of prices shown, it is optimal to invest the entire wealth in the risky asset, even when the investor is long one

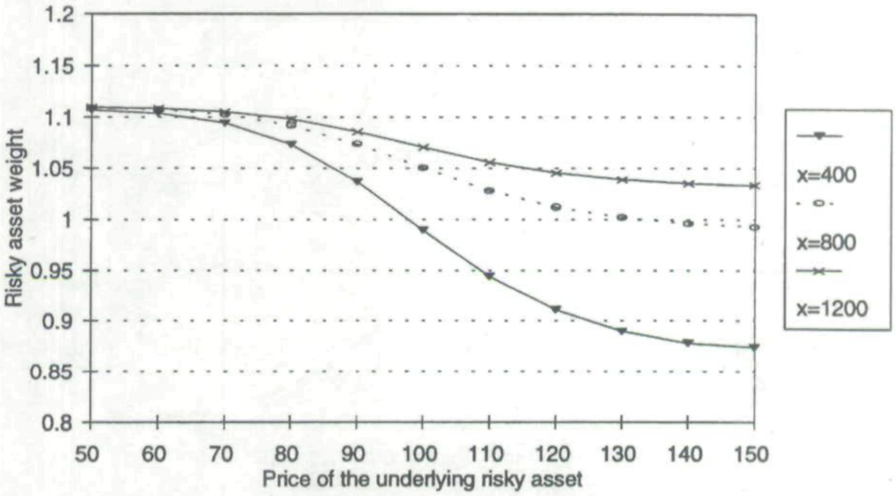

Figure 11. The fraction of wealth optimally invested in the risky asset, $\theta_{1,C^{\text{buy}}(x,P)}(x, P)/x$, when one call option is bought and there is no borrowing constraint.

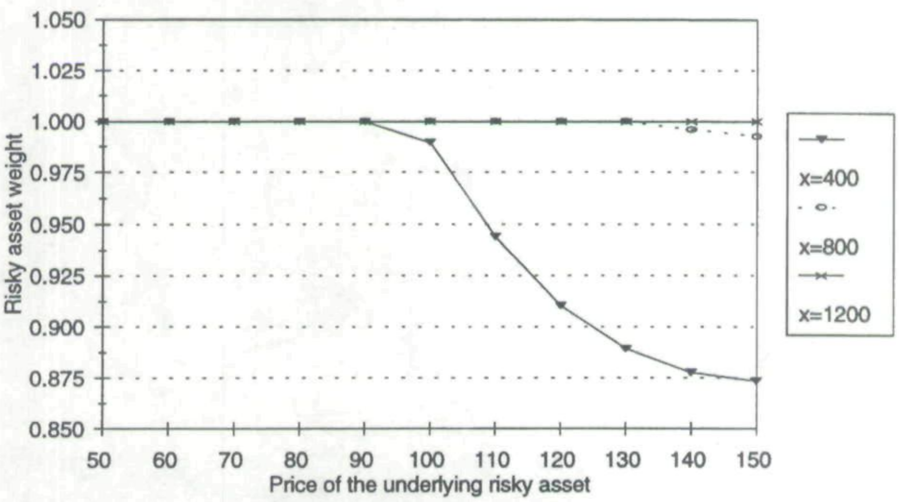

Figure 12. The fraction of wealth optimally invested in the risky asset when one call option is bought and the investor faces a borrowing constraint.

call. This is also true for larger values of x, and since the reservation buying price $\bar{C}^{\text{buy}}(x, P)$ does not vary significantly with x in this range, neither will the hedge portfolio, $\bar{C}^{\text{buy}}(x, P)/P$. The graphs corresponding to an initial wealth of $x = 400$ and $x = 800$ follow the $x = 1200$ line until a certain risky asset price. Above this price the portfolio constraint is not binding when buying an option, and therefore the number of risky assets sold in the hedge exceeds $\bar{C}^{\text{buy}}(x, P)/P$. Still, it will be smaller than the Black–Scholes Delta. The optimal consumption rate is again practically unchanged relative to the unconstrained problem.

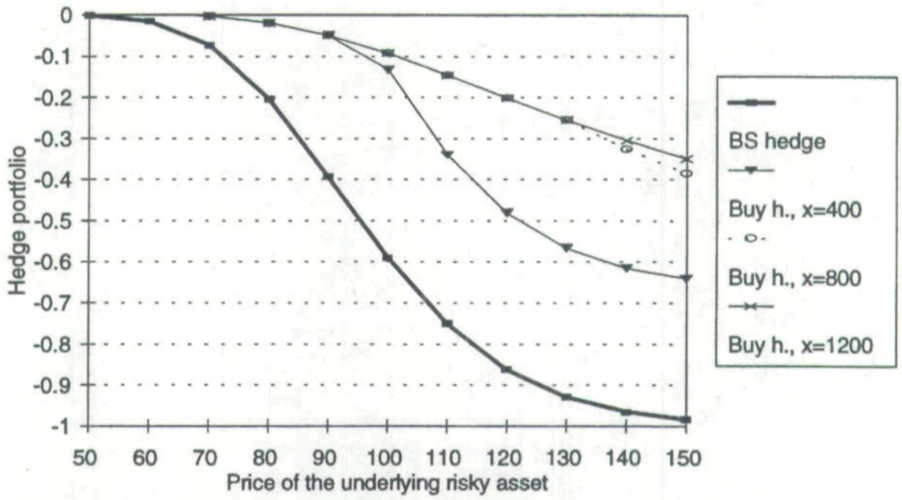

Figure 13. The number of units of the underlying risky asset in the hedge of a bought call option, $\bar{\Delta}_{\text{buy}}(x, P)$, when the investor faces a borrowing constraint.

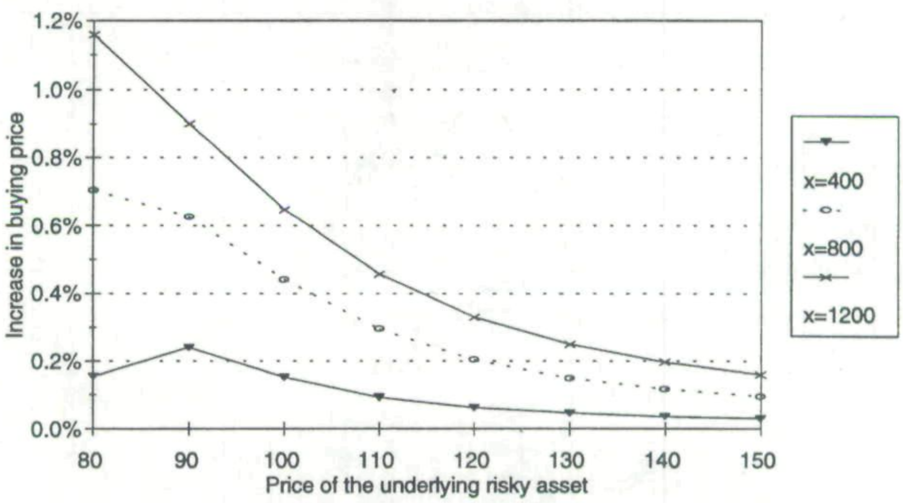

Figure 14. The increase in the reservation buying price caused by the borrowing constraint.

Based on the preceding considerations I expect that the reservation buying price will increase relative to the unconstrained case, and that the increase will be largest for high values of wealth and low risky asset prices since the borrowing constraint is most restrictive for such wealth/price combinations. This is confirmed by Figure 14 which shows the percentage increase in the reservation buying price, $(\bar{C}^{\text{buy}}(x, P) - C^{\text{buy}}(x, P))/C^{\text{buy}}(x, P)$.

Finally, I compare the reservation selling and buying prices with the no-arbitrage price bounds of Cvitanić and Karatzas (1993) and Munk (1997a). These bounds are valid when the portfolio constraints are imposed on the hedge portfolio only. The Black–Scholes hedge of a written European call does not involve borrowing, therefore, the arbitrage buying price equals the Black–Scholes price. The Black–Scholes hedge of a bought European call does involve borrowing, therefore, the arbitrage selling price is higher than the Black–Scholes price. In fact, Cvitanić and Karatzas (1993, Ex. 7.2) have shown that the arbitrage selling price is equal to $P(0)$. In other words, the least expensive super-replicating portfolio consists of buying the underlying asset. Obviously, the reservation buying and selling prices form a much narrower interval for the price of the option than the arbitrage buying and selling prices do. I emphasize that the reservation prices are utility and endowment specific whereas the no-arbitrage prices hold for all investors. On the other hand, the reservation prices are computed under constraints on the total portfolio of the investor whereas the no-arbitrage prices are computed under constraints on the hedge portfolio only.

7. Concluding Remarks

This paper has discussed how contingent claims can be valued under portfolio constraints with utility-based reservation prices and has demonstrated how such prices can be computed numerically for European-style contingent claims. Two numerical examples in a single risky asset set-up with constant coefficients and power utility were considered. First, the comparatively simple problem where there are no constraints on the portfolio chosen by the agent other than a non-negative wealth constraint. In this simple set-up, the reservation selling and buying prices for a European call option on the risky asset are indistinguishable from the Black–Scholes price, except for extremely low wealth where bankruptcy considerations give rise to special hedging strategies.

In the second example, a borrowing constraint was imposed on the total portfolio of the investor. Due to the constraint it is not possible for the investor to hedge the option as in the Black–Scholes model, and, hence, the reservation prices of the option increase, although not by much. Compared to the case where the constraints are imposed on the hedge portfolio only, I got a much smaller price interval for the contingent claim.

I leave the reader plenty of room for further research. From an economic point of view, a vast number of interesting option valuation problems can be studied within the framework of the present paper. One obvious application is the valuation of non-replicable real options which is tremendously important for long-term capital budgeting. Another application is the valuation of management stock options where the manager is restricted not to take positions in the stock of the company. It would be interesting, for example, to see how the value of such an option is affected by the manager's possibility to invest in assets correlated with the stock. Note, however, that executive stock options are typically American options in which case a consumption/portfolio strategy and an exercise strategy are to be determined simultaneously. This complicates the numerical solution considerably.

From a numerical point of view, the major challenge is the *curse of dimensionality*. As for many other numerical methods for stochastic control problems, the approach of this paper cannot be applied to problems with more than three or maybe four state variables. While several approaches have been suggested in the literature, see, e.g., Reiter (1999), one apparently unexplored approach is to use an ADI-type technique. The ADI (Alternating Direction Implicit) method is a well-known and well-documented method for the numerical solution of multi-dimensional linear partial differential equations. Loosely speaking the ADI approach handles the state variables one at a time, and therefore it does not challenge standard computers' memory capacity. The applicability of the idea to stochastic control problems remains an open question.

Appendix A: The Markov Chain Approximation Approach and the HJB Equation

This appendix provides details on the close relation between the Markov chain approximation approach and finite difference solution of the HJB equation associated with the continuous-time, continuous-state control problem. First, consider the infinite-horizon problem (4.3). The corresponding HJB equation is the highly non-linear second order ordinary differential equation

$$\beta V(x) = \sup_{(\alpha,\theta,c)\in\mathcal{K}\times\mathbb{R}_+} \left\{\mathcal{D}_{\theta,c} V(x) + c^\gamma\right\}, \tag{A.1}$$

where

$$\mathcal{D}_{\theta,c} V(x) = V'(x)\left(rx + \theta(b-r) - c\right) + \frac{1}{2}V''(x)\sigma^2\theta^2.$$

The dynamic programming Equation (4.5) coming from the Markov chain approximation approach can be rewritten as

$$\frac{1-e^{-\beta\Delta t^h(x)}}{\Delta t^h(x)} V^h(x) = \sup_{\substack{(\alpha,\theta)\in\mathcal{K}\\ |\theta|\le K_\theta x\\ 0\le c\le K_c x}} \left\{e^{-\beta\Delta t(x)}\mathcal{D}^h_{\theta,c} V^h(x) + c^\gamma\right\},$$

where

$$\mathcal{D}^h_{\theta,c} V^h(x) = D^+V^h(x)\left(rx + \theta^+(b-r)\right) - D^-V^h(x)\left(c + \theta^-(b-r)\right) + \frac{1}{2}D^2V^h(x)\sigma^2\theta^2,$$

with

$$D^+V^h(x) = \frac{V^h(x+h) - V^h(x)}{h}, \qquad D^-V^h(x) = \frac{V^h(x) - V^h(x-h)}{h},$$

$$D^2V^h(x) = \frac{V^h(x+h) - 2V^h(x) + V^h(x-h)}{h^2}.$$

Next, consider the finite horizon control problem (4.6). The associated HJB equation is the highly non-linear second order partial differential equation

$$\beta V(x, P, t) = \sup_{(\alpha,\theta,c)\in\mathcal{K}\times\mathbb{R}_+} \left\{ \frac{\partial V}{\partial t}(x, P, t) + \mathcal{D}_{\theta,c} V(x, P, t) + c^\gamma \right\}$$

with the terminal condition $V(x, P, T) = V(x + \varepsilon\varphi(P))$. Here,

$$\begin{aligned} \mathcal{D}_{\theta,c} V(x, P, t) &= \frac{\partial V}{\partial x}(x, P, t)\,(rx + \theta(b-r) - c) + \frac{\partial V}{\partial P}(x, P, t) bP \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x, P, t)\sigma^2\theta^2 + \frac{\partial^2 V}{\partial x \partial P}(x, P, t)\theta P\sigma^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2. \end{aligned}$$

The "pseudo" Markov chain approximation approach yields the dynamic programming Equation (4.10) which can be rewritten as the implicit finite difference approximation

$$\begin{aligned} &\frac{1 - e^{-\beta\tau}}{\tau} V^{h,\tau}(x, P, t) \\ &= \sup_{\substack{(\alpha,\theta)\in\mathcal{K} \\ |\theta|\le K_\theta x \\ 0\le c\le K_c x}} \left\{ D_t V^{h,\tau}(x, P, t) + e^{-\beta\tau}\mathcal{D}^h_{\theta,c} V^{h,\tau}(x, P, t) + c^\gamma \right\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}^h_{\theta,c} V^{h,\tau}(x, P, t) &= D_x^+ V^{h,\tau}(x, P, t)\left(rx + \theta^+(b-r)\right) \\ &\quad - D_x^- V^{h,\tau}(x, P, t)\left(c + \theta^-(b-r)\right) \\ &\quad + D_P^+ V^{h,\tau}(x, P, t) bP + \frac{1}{2} D_x^2 V^{h,\tau}(x, P, t)\sigma^2\theta^2 \\ &\quad + \frac{1}{2} D_P^2 V^{h,\tau}(x, P, t)\sigma^2 P^2 + D_{xP}^+ V^{h,\tau}(x, P, t)\theta^+ P\sigma^2 \\ &\quad - D_{xP}^- V^{h,\tau}(x, P, t)\theta^- P\sigma^2, \end{aligned}$$

and the finite difference operators are given by

$$D_t V^{h,\tau}(x, P, t) = \frac{V^{h,\tau}(x, P, t+\tau) - V^{h,\tau}(x, P, t)}{\tau},$$

$$D_x^+ V^{h,\tau}(x, P, t) = \frac{V^{h,\tau}(x + h_x, P, t) - V^{h,\tau}(x, P, t)}{h_x},$$

$$D_x^- V^{h,\tau}(x, P, t) = \frac{V^{h,\tau}(x, P, t) - V^{h,\tau}(x - h_x, P, t)}{h_x},$$

$$D_P^+ V^{h,\tau}(x, P, t) = \frac{V^{h,\tau}(x, P + h_P, t) - V^{h,\tau}(x, P, t)}{h_P},$$

$$D_x^2 V^{h,\tau}(x, P, t) = \frac{V^{h,\tau}(x + h_x, P, t) - 2V^{h,\tau}(x, P, t) + V^{h,\tau}(x - h_x, P, t)}{h_x^2},$$

$$D_P^2 V^{h,\tau}(x, P, t) = \frac{V^{h,\tau}(x, P + h_P, t) - 2V^{h,\tau}(x, P, t) + V^{h,\tau}(x, P - h_P, t)}{h_P^2},$$

and

$$\begin{aligned} &D_{xP}^+ V^{h,\tau}(x, P, t) \\ &\quad = \frac{1}{2h_x h_P}\Big\{2V^{h,\tau}(x, P, t) + V^{h,\tau}(x + h_x, P + h_P, t) \\ &\qquad + V^{h,\tau}(x - h_x, P - h_P, t) \\ &\qquad - V^{h,\tau}(x + h_x, P, t) - V^{h,\tau}(x - h_x, P, t) \\ &\qquad - V^{h,\tau}(x, P + h_P, t) - V^{h,\tau}(x, P - h_P, t)\Big\}, \end{aligned}$$

$$\begin{aligned} &D_{xP}^- V^{h,\tau}(x, P, t) \\ &\quad = -\frac{1}{2h_x h_P}\Big\{2V^{h,\tau}(x, P, t) + V^{h,\tau}(x + h_x, P - h_P, t) \\ &\qquad + V^{h,\tau}(x - h_x, P + h_P, t) \\ &\qquad - V^{h,\tau}(x + h_x, P, t) - V^{h,\tau}(x - h_x, P, t) \\ &\qquad - V^{h,\tau}(x, P + h_P, t) - V^{h,\tau}(x, P - h_P, t)\Big\}. \end{aligned}$$

Appendix B: Boundary Cases for the Finite-Horizon Problem

In this appendix I consider the boundary cases for the numerical solution of the finite-horizon utility maximization problem studied in Section 4.3. If wealth reaches zero at some point in time, it will remain zero. The only exception is at the maturity date of the contingent claim in case the investor has bought the claim. Since this event only affects the terminal value function in the control problem, I can consider $X = 0$ an absorbing boundary. Similarly, the price of the underlying risky asset has an absorbing lower boundary at zero. Also note that if wealth is zero, the only admissible, and hence the optimal, policy is

$$c_{\varepsilon,z}(0, P, t) = 0, \qquad \theta_{\varepsilon,z}(0, P, t) = 0, \quad 0 \le P \le \bar{P}.$$

I therefore take the following transition weights

$$p(0, 0, n\tau; 0, 0, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(0, 0)} = 1,$$

$$p(0, P, n\tau; 0, P, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(0, P)},$$

$$p(0, P, n\tau; 0, P + h_P, n\tau|\theta, c) = \frac{\frac{1}{h_P}bP + \frac{1}{2h_P^2}P^2\sigma^2}{Q^h(0, P)},$$

$$p(0, P, n\tau; 0, P - h_P, n\tau|\theta, c) = \frac{1}{2h_P^2 Q^h(0, P)}P^2\sigma^2,$$

$$p(0, P, n\tau; 0, P, n\tau|\theta, c) = 1 - \frac{\frac{1}{\tau} + \frac{1}{h_P}bP + \frac{1}{h_P^2}P^2\sigma^2}{Q^h(0, P)}.$$

When the price of the risky asset hits zero, it stays there. In that case I require a risky investment of zero,

$$\theta_{\varepsilon,z}(x, 0, n\tau) = 0.$$

The transition weights are

$$p(x, 0, n\tau; x, 0, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(x, 0)},$$

$$p(x, 0, n\tau; x + h_x, n\tau|\theta, c) = \frac{rx}{h_x Q^h(x, 0)},$$

$$p(x, 0, n\tau; x - h_x, n\tau|\theta, c) = \frac{c}{h_x Q^h(x, 0)},$$

and

$$p(x, 0, n\tau; x, 0, n\tau|\theta, c) = 1 - \frac{\frac{1}{\tau} + \frac{1}{h_x}(rx + c)}{Q^h(x, 0)}.$$

At the artificial upper boundaries $x = \bar{x}$ and $P = \bar{P}$, let

$$p(\bar{x}, \bar{P}, n\tau; \bar{x}, \bar{P}, n\tau + \tau | \theta, c) = \frac{1/\tau}{Q^h(\bar{x}, \bar{P})},$$

$$p(\bar{x}, \bar{P}, n\tau; \bar{x} - h_x, \bar{P}, n\tau | \theta, c)$$

$$= \frac{\frac{1}{h_x}(\theta^-(b-r) + c) - \frac{1}{2h_x h_P}\theta^+ \bar{P}\sigma^2 + \frac{1}{2h_x^2}\theta^2\sigma^2}{Q^h(\bar{x}, \bar{P})},$$

$$p(\bar{x}, \bar{P}, n\tau; \bar{x}, \bar{P} - h_P, n\tau | \theta, c) = \frac{-\frac{1}{2h_x h_P}\theta^+ \bar{P}\sigma^2 + \frac{1}{2h_P^2} P^2\sigma^2}{Q^h(\bar{x}, \bar{P})},$$

$$p(\bar{x}, \bar{P}, n\tau; \bar{x} - h_x, \bar{P} - h_P, n\tau | \theta, c) = \frac{\theta^+ \bar{P}\sigma^2}{2h_x h_P Q^h(\bar{x}, \bar{P})},$$

$$p(\bar{x}, \bar{P}, n\tau; \bar{x}, \bar{P}, n\tau | \theta, c) = 1 - \sum_{(x', P') \in \mathcal{S}(\bar{x}, \bar{P})} p(\bar{x}, \bar{P}, n\tau; x', P', n\tau | \theta, c)$$

$$- \frac{1/\tau}{Q^h(\bar{x}, \bar{P})},$$

where

$$\mathcal{S}(\bar{x}, \bar{P}) = \left\{(\bar{x} - h_x, \bar{P}), (\bar{x}, \bar{P} - h_P), (\bar{x} - h_x, \bar{P} - h_P)\right\}.$$

When x is at the upper bound, but P is interior, the transition weights are

$$p(\bar{x}, P, n\tau; \bar{x}, P, n\tau + \tau | \theta, c) = \frac{1/\tau}{Q^h(\bar{x}, P)},$$

$$p(\bar{x}, P, n\tau; \bar{x} - h_x, P - h_P, n\tau | \theta, c) = \frac{\theta^+ P\sigma^2}{2h_x h_P Q^h(\bar{x}, P)},$$

$$p(\bar{x}, P, n\tau; \bar{x} - h_x, P + h_P, n\tau | \theta, c) = \frac{\theta^- P\sigma^2}{2h_x h_P Q^h(\bar{x}, P)},$$

$$p(\bar{x}, P, n\tau; \bar{x} - h_x, P, n\tau | \theta, c)$$

$$= \frac{\frac{1}{h_x}(\theta^-(b-r) + c) - \frac{1}{2h_x h_P}|\theta| P\sigma^2 + \frac{1}{2h_x^2}\theta^2\sigma^2}{Q^h(\bar{x}, P)},$$

$$p(\bar{x}, P, n\tau; \bar{x}, P - h_P, n\tau|\theta, c) = \frac{-\frac{1}{2h_x h_P}\theta^+ P\sigma^2 + \frac{1}{2h_P^2}P^2\sigma^2}{Q^h(\bar{x}, P)},$$

$$p(\bar{x}, P, n\tau; \bar{x}, P + h_P, n\tau|\theta, c)$$

$$= \frac{\frac{1}{h_P}bP - \frac{1}{2h_x h_P}\theta^- P\sigma^2 + \frac{1}{2h_P^2}P^2\sigma^2}{Q^h(\bar{x}, P)},$$

$$p(\bar{x}, P, n\tau; \bar{x}, P, n\tau|\theta, c) = 1 - \sum_{(x', P') \in \mathcal{S}(\bar{x}, P)} p(\bar{x}, P, n\tau; x', P', n\tau|\theta, c)$$

$$- \frac{1/\tau}{Q^h(\bar{x}, P)},$$

where

$$\mathcal{S}(\bar{x}, P) = \{(\bar{x} - h_x, P - h_P), (\bar{x} - h_x, P), (\bar{x} - h_x, P + h_P),$$

$$(\bar{x}, P - h_P), (\bar{x}, P + h_P)\}.$$

When P is at the upper bound, but x is interior,

$$p(x, \bar{P}, n\tau; x, \bar{P}, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(x, \bar{P})},$$

$$p(x, \bar{P}, n\tau; x + h_x, \bar{P}, n\tau|\theta, c)$$

$$= \frac{\frac{1}{h_x}(rx + \theta^+(b - r)) - \frac{1}{2h_x h_P}\theta^- \bar{P}\sigma^2 + \frac{1}{2h_x^2}\theta^2\sigma^2}{Q^h(x, \bar{P})},$$

$$p(x, \bar{P}, n\tau; x - h_x, \bar{P}, n\tau|\theta, c)$$

$$= \frac{\frac{1}{h_x}(\theta^-(b - r) + c) - \frac{1}{2h_x h_P}\theta^+ \bar{P}\sigma^2 + \frac{1}{2h_x^2}\theta^2\sigma^2}{Q^h(x, \bar{P})},$$

$$p(x, \bar{P}, n\tau; x, \bar{P} - h_P, n\tau|\theta, c) = \frac{-\frac{1}{2h_x h_P}|\theta|\bar{P}\sigma^2 + \frac{1}{2h_P^2}\bar{P}^2\sigma^2}{Q^h(x, \bar{P})},$$

$$p(x, \bar{P}, n\tau; x - h_x, \bar{P} - h_P, n\tau|\theta, c) = \frac{\theta^+ \bar{P}\sigma^2}{2h_x h_P Q^h(x, \bar{P})},$$

$$p(x, \bar{P}, n\tau; x + h_x, \bar{P} - h_P, n\tau|\theta, c) = \frac{\theta^- \bar{P}\sigma^2}{2h_x h_P Q^h(x, \bar{P})},$$

$$p(x, \bar{P}, n\tau; x, \bar{P}, n\tau|\theta, c) = 1 - \sum_{(x', P') \in \mathcal{S}(x, \bar{P})} p(x, \bar{P}, n\tau; x', P', n\tau|\theta, c) - \frac{1/\tau}{Q^h(x, \bar{P})},$$

where

$$\mathcal{S}(x, \bar{P}) = \{(x + h_x, \bar{P}), (x - h_x, \bar{P}), (x, \bar{P} - h_P), (x - h_x, \bar{P} - h_P), (x + h_x, \bar{P} - h_P)\}.$$

When x is at the upper bound and P is at the lower, the transition weights are

$$p(\bar{x}, 0, n\tau; \bar{x}, 0, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(\bar{x}, 0)},$$

$$p(\bar{x}, 0, n\tau; \bar{x} - h_x, 0, n\tau|\theta, c) = \frac{c}{h_x Q^h(\bar{x}, 0)},$$

$$p(\bar{x}, 0, n\tau; \bar{x}, 0, n\tau|\theta, c) = 1 - \frac{\frac{1}{\tau} + \frac{1}{h_x} c}{Q^h(\bar{x}, 0)}.$$

The risky investment $\theta_{\varepsilon,z}(\bar{x}, 0, n\tau)$ is set equal to zero.

Finally, when P is at the upper bound and x is at the lower,

$$p(0, \bar{P}, n\tau; 0, \bar{P}, n\tau + \tau|\theta, c) = \frac{1/\tau}{Q^h(0, \bar{P})},$$

$$p(0, \bar{P}, n\tau; 0, \bar{P} - h_P, n\tau|\theta, c) = \frac{\bar{P}^2\sigma^2}{2h_P^2 Q^h(0, \bar{P})},$$

$$p(0, \bar{P}, n\tau; 0, \bar{P}, n\tau|\theta, c) = 1 - \frac{\frac{1}{\tau} + \frac{1}{2h_P^2}\bar{P}^2\sigma^2}{Q^h(0, \bar{P})}.$$

The only admissible, and hence the optimal, controls in this case are

$$c_{\varepsilon,z}(0, \bar{P}, n\tau) = 0, \qquad \theta_{\varepsilon,z}(0, \bar{P}, n\tau) = 0.$$

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