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Valuation of Barrier Options in a Black–Scholes Setup with Jump Risk

DIETMAR P. J. LEISEN*
Stanford University, Hoover Institution, Stanford, CA 94305, U.S.A.; E-mail: leisen@Hoover.Stanford.EDU

Abstract. This paper discusses the pitfalls in the pricing of barrier options using approximations of the underlying continuous processes via discrete lattice models. To prevent from numerical deficiencies, the space axis is discretized first, and not the time axis. In a Black–Scholes setup, models with improved convergence properties are constructed: a trinomial model and a randomized trinomial model where price changes occur at the jump times of a Poisson process. These lattice models are sufficiently general to handle options with multiple barriers: the numerical difficulties are resolved and extrapolation yields even more accurate results. In a last step, we extend the Black–Scholes setup and incorporate unpredictable discontinuous price movements. The randomized trinomial model can easily be extended to this case, inheriting its superior convergence properties.

Key words: barrier option, binomial model, lattice-approach, option valuation.

JEL classification: C63, G12, G13.

1. Introduction

Over time, barrier options have become increasingly popular to reduce the cost of plain vanilla options by incorporating individual views of market participants. There are now numerous types of such options; their payoff at maturity depends on whether the asset price path will have (will have not) crossed prespecified boundaries, called "barriers." Conditional on this event, standard barrier options do either pay off a call or a put at the maturity date. Whereas closed-form solutions are available for all standard single barrier options in the Black–Scholes setup, see, e.g., Rubinstein and Reiner (1991) and Carr (1995), for other setups or options with multiple barriers, no closed-form solutions are available, in general. Numerical procedures have to be applied to come up with prices, see, e.g., Kunitomo and Ikeda

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(1992) for analytical approximations in some cases with double barrier options in a Black–Scholes setup.

Empirically, the assumptions of the Black–Scholes setup are questionable: asset prices do not evolve continuously over time and sudden strong price changes occur, not only at "crash" times. We are therefore interested in an extension of the geometric Brownian motion setup for stock prices and turn here to the framework of Merton (1976), adding a compound Poisson process to the Black–Scholes model. In Section 2 we present both setups as well as the barrier option contract in detail.

In this paper we are interested in the "efficient" pricing of barrier options using approximations of the underlying process. Whereas these lattice models yield quite accurate results with plain vanilla options, it is well known that binomial models suffer from numerical deficiencies with barrier options: increasing the refinement, prices converge erratically in a saw-tooth manner to the continuous time price and even high refinements do not ensure adequate accuracy. Many authors addressed this problem and suggested adjustments. Ritchken (1995), and Cheuk and Vorst (1996) constructed trees where the nodes lie on the barrier. Figlewski and Gao (1999) refine the tree further at the barrier. Instead of discretizing time first, as previous approaches did, we start from a discretization of the asset space. Our improvements are related to the finite difference approach (see e.g., Boyle and Tian (1998)). With barrier options this is a straightforward way to ensure nodes are on critical levels by construction. We recover binomial models with the specific refinement of Boyle and Lau (1994). The pitfalls in discretizing the Black–Scholes setup according to CRR are discussed in Section 3.

We take the analysis of the deficiencies of the binomial model as the starting point for our construction of models with improved properties in the following sections. Making use of trinomial models the order of convergence can be increased from 1/2 to 1 and the saw-tooth patterns can be eliminated. Price oscillations coming from the cutoff inherent in the call/put options payoff at maturity, including the odd–even ripple, are removed in our trinomial adjusted model (Section 4).

Discretizing the asset space breaks up the time discretization; in a modification, we allow for random trading, similar to Leisen (1999) for American put options, and Rogers and Stapleton (1998) for barrier options. Whereas the latter did not recognize the numerical deficiencies resulting from the strike at maturity, our trinomial model resolves this problem explicitly. We also differ from their approach by assuming that stock prices change only at the jump times of a *Poisson* process; this results in simple valuation formulas. Our parameters are set in accordance with a convergence theorem of the processes which also ensures convergence of prices. Our model with random jump times driven by a Poisson process allows us in a simple and intuitive way to incorporate jumps (Section 5).

Section 6 discusses the implementation with barrier options and their price accuracy in our framework. It turns out that the randomized model smoothes the convergence structure even better, removes remaining wavy patterns and simulations suggest that extrapolated prices converge with quadratic order. The model has superior convergence properties in the Black–Scholes model and the jump-diffusion setup; it improves on the extension of binomial models to the jump-diffusion setup that was proposed by Amin (1993). Section 7 concludes the paper. All proofs are postponed to the Appendix.

2. The Setup

On a probability space $(\Omega, \mathcal{F}, P)$ we study two setups. Both consist of a bond B with an interest rate r assumed to be constant over time ($B_t = \exp\{rt\}$). Moreover, there is a single risky asset, whose specification does, however, differ according to the setup studied. Subsection 2.1 introduces the Black–Scholes setup, subsection 2.2 incorporates jumps and subsection 2.3 introduces barrier options as used in this paper.

2.1. BLACK–SCHOLES DIFFUSION SETUP

The price of the risky asset is assumed to evolve under the objective measure P according to

$$G_t = G_0 \exp\{\mu^G t + \sigma W_t\}, \tag{1}$$

where W is a standard Wiener process and $\mu^G \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$. The process $(G_t)_t$ is the continuous process known as geometric Brownian motion and has stationary, Gaussian returns. Originally suggested by Samuelson (1965) it has become one of the standard financial models for derivatives pricing, since Black and Scholes (1973) used it in their seminal contribution.

The Black–Scholes setup is a so-called *complete market*, i.e., any contingent claim on the stock can be hedged (see Duffie (1992)). In the language of Harrison and Kreps (1979) this means that there is a unique probability measure Q, equivalent to P, under which $(G_t/B_t)_t$ is a martingale. Such a measure gives rise to a linear pricing operator; it is completely specified for the (B, G) market by $\mu^G = r - \sigma^2/2$.[1] Please note that the no-arbitrage principle is sufficient to price the option, since the option can be perfectly hedged. Preferences do not need to be taken into account explicitly; implicitly, however, we assume that the market is in equilibrium. This enters only to the extent that the no-arbitrage principle is a characteristic of any equilibrium; in incomplete markets, however, this shortcut is no longer feasible and we invoke equilibrium arguments for pricing.

[1] This can be seen from the fact that $E[G_t/G_0] = \exp\{(\mu^G + \sigma^2/2)t\}$, that the martingale requirement is $E[G_t/G_0] = \exp\{rt\}$ and the Girsanov transformation is unique.

2.2. MERTON'S JUMP-DIFFUSION SETUP

Merton (1976) extended the Black–Scholes setup, modeling the risky asset's dynamics under the objective measure P by

$$S_t = G_t \cdot J_t \ , \ \text{ where } J_t = J_0 e^{(\nu - \lambda E[V_i - 1])t} \prod_{i=1}^{N_t} V_i \ , \tag{2}$$

and $(G_t)_t$ as before. $(N_t)_t$ is a Poisson process with constant parameter λ, $(V_i)_i$ a sequence of non-negative iid random variables, $\nu \in \mathbb{R}$. Each of the processes is assumed to be independent of the other and the random variables V_i, $(i = 1, 2, \ldots)$.

S evolves according to G until the next jump time τ of the Poisson process at which N changes from, say, i to $i + 1$. We then observe a percent change $V_i - 1$, i.e., the stock changes value from $S_{\tau-}$ before the jump to $S_{\tau-} \cdot V_i$. So, the two parts can be interpreted as follows: $(G_t)_t$ models the "typical" evolution of the stock under the "normal" arrival of information, whereas $(J_t)_t$ models jumps in the stock prices, due to some rare strong information shock. Since the Poisson process is "memoryless," the expected time until the next shock occurs is equal to $1/\lambda$, independent of current time.

This setup represents an incomplete market; the unforeseeable jump can not hedged. Under the assumption of no-arbitrage there is a multiplicity of equivalent martingale measures (EMM) under which agents do evaluate the remaining risk. Each of these represents a different potential pricing system; we refer to them as pricing measures.

Let us assume for a while that the asset $(J_t)_t$ is traded in the market. The three assets (B, S, J) would then constitute a complete market and they would all have to be martingales under the pricing measure Q. Using the description of the stochastic exponential we derive $E[J_t/J_0] = e^{\nu t}$ (see Protter (1990) or Jacod and Shiryaev (1987)). Any pricing measure would require $E[J_t/J_0] = e^{rt}$; therefore it would involve changing the intensity of the Poisson process to λ_Q and the distributional characteristics of V_i such that under the pricing measure Q holds $\nu - \lambda E[V_i - 1] = r - \lambda_Q E_Q[V_i - 1]$. Using the Girsanov theorem w.r.t. the Wiener process part would then give the pricing measure in the (B, S, J) market as the one after a change of measure for which $\mu^G = -\sigma^2/2$.

As $(J_t)_t$ is, however, not traded, under the pricing measure Q chosen by the market for valuation, only $(S_t)_t$ but $(J_t)_t$ has to be a martingale. The parameter $\tilde{\nu} = \ln E_Q[J_t/J_0]/t - r = \nu - r$ does not have to be equal to zero and can be interpreted as the excess return on the risky process $(J_t)_t$ over the riskless rate; with exogenously fixed parameters λ and (V_i), $\tilde{\nu}$ specifies the risk-premium. Note that, once $\tilde{\nu}$ is fixed, we have $E[S_t/S_0] = \exp\{(\mu^G + \sigma^2/2 + \tilde{\nu} + r)t\}$. So, for a pricing measure we need to apply the Girsanov theorem in such a way that $\mu^G = -\sigma^2/2 - \tilde{\nu}$. As $\tilde{\nu}$ is a "free" parameter, we can treat the problem of choosing an EMM as the specification problem of an appropriate risk-premium in the market.

The description of $(J_t)_t$ (under P) and the choice of Q together with the Girsanov theorem are equivalent. (For a detailed discussion, we refer to Wiesenberg (1998).)

Hamilton (1995), Bates (1996), and Trautmann and Beinert (1995) discuss the statistical inference of the properties of the jump component $(J_t)_t$ in the stock process $(S_t)_t$. Bates (1996) gives statistical evidence that the risk premium ν is non-zero, i.e., the jump-risk is correlated to the market as a whole, and "crashes" need to be taken into account. The mathematical finance literature suggests the use of specific EMM, derived from hedging criteria (see Föllmer and Sondermann (1986) and Föllmer and Schweizer (1991)). Similarly to Merton (1976), we are interested in the case where jumps are firm specific and uncorrelated with the market as a whole. If we assume that the conclusions of the CAPM hold, then this nonsystematic risk has a premium $\tilde{\nu}$ of zero, so that $\nu = r$ under the objective measure P. We do then perform a change of measure which determines the pricing measure as the one leaving V_i unchanged, while $\lambda_Q = \lambda$, and $\mu^G = -\sigma^2/2$. The stock dynamics under the pricing measure is then

$$S_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]\right) t + \sigma W_t\right\} \prod_{i=1}^{N_t} V_i.$$

This is the starting point for our presentation; we subsequently study the evolution of the asset under this risk-neutral probability only. To simplify notation we drop the dependence on it, whenever referring to expectations. While adopting this specific choice for the pricing measure, we would like to point out that our approach could easily incorporate a non-zero $\tilde{\nu}$.

2.3. BARRIER OPTIONS

Barrier options are a type of exotic options where the payoff depends on the crossing of predetermined levels, which may be either discretely or continuously monitored. In the event of a crossing, depending on the specification, a lump-sum (called "rebate") might be paid, another asset might be activated (called "knock-in") or deactivated ("knock-out"). Typical examples are those where the secondary asset is a plain vanilla option, e.g., a down (up) and in (out) barrier option pays this off, if some barrier $H < S_0$ ($H > S_0$) is (not) crossed and nothing otherwise.

We are interested here in continuously monitored constant barrier options, where the final payoff is that of a plain vanilla European option and explicitly allow for multiple barrier options. Our mathematical description of the barrier option payoff uses a "choice variable" Σ that contains all paths ω that activate the terminal payoff: the terminal payoff is then $1_{\omega \in \Sigma} f(S_T)$, where f is the payoff function at maturity, and its price is $E\left[e^{-rT} 1_{\omega \in \Sigma} f(S_T)\right]$.

3. Binomial Pitfalls

This section studies the numerical deficiencies with barrier option pricing in the Black–Scholes setup, i.e. for the model $(G_t)_t$, using binomial models. We focus here on the binomial model of Cox, Ross and Rubinstein (1979) (henceforth CRR).

Throughout the paper our aim is to prove weak convergence of the processes,[2] a basic consistency requirement of our discrete approximations.

REMARK 1. Weak convergence of the processes ensures convergence of prices: the put option payoff function at date T is continuous and bounded; therefore, by definition the discrete expected payoff converges to its continuous counterpart. Hence put option prices converge. Similarly we conclude price convergence for barrier options with a put as rebate. For call option rebates the situation is more complicated. General proofs are beyond the scope of the paper; we outline proofs using arbitrage relationships for two examples to explain the idea: for plain vanilla call options we observe that put-call parity holds in the discrete and continuous models and that forward prices are independent of the economic structure. Therefore the convergence of put prices implies those of call prices. In the second example we look at single barrier call options: the payoff of up-and-out options is continuous and bounded, hence convergence of prices follows from weak convergence of the processes. For up-and-in options we make use of the fact that a portfolio consisting of one up-and-out and one up-and-in option has the same payoff as the rebate, here a plain vanilla call. Therefore, convergence of up-and-out option prices together with those of plain vanilla call prices implies convergence of prices for up-and-in options.

The following theorem will be used to prove weak convergence of the continuous sample path components of the processes.

THEOREM 2. Assume $\mu \in \mathbb{R}, \sigma > 0, T > 0$ are constants. For each $n \in \mathbb{N}$ let $(N_t^{(n)})_t$ be a renewal process[3] with renewal function $\eta_n(t)$, $(\bar{R}_{n,i})_i$ a sequence of i.i.d. random variables with jump sizes that are 0 in the probability limit on $[0, T]$ ("vanish"). Denote $\zeta^{(n)}$ the function $\zeta_t^{(n)} = E\Big[\sum_{i=1}^{N_t^{(n)}} \sum_{j=1, j\neq i}^{N_t^{(n)}} \bar{R}_{n,i}\, \bar{R}_{n,j}\Big] - (\eta_n^2(t) - \eta_n(t))E[\bar{R}_{n,i}]^2$, $\bar{X}^{(n)}$ the process $\bar{X}_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} \bar{R}_{n,i}$, and X the process $X_t = \mu t + \sigma W_t$ on $[0, T]$. If

[2] A sequence of real valued càdlàg (right continuous with left hand limits) stochastic processes $\bar{X}^{(n)}$ is converging weakly to a process X, if for every bounded continuous mapping f from the space of càdlàg functions into the real numbers $E[f(\bar{X}^{(n)})] \xrightarrow{n} E[f(X)]$.

[3] For a sequence of i.i.d. non-negative random variables $(\tau_{n,i})_i$ we denote $N_t^{(n)} = \max\left\{n \,\middle|\, \sum_{i=1}^{n} \tau_{n,i} \leq t\right\}$. $\tau_{n,i}$ is called the i-th interarrival time, $\mu_n(t) = E[N_t^{(n)}]$ the renewal function. For an introduction to renewal processes we refer to Feller (1966a), section XI.

$$\sup_{0\le t\le T} \frac{\eta_n(t)}{t} E[\bar{R}_{n,i}] \stackrel{n}{\to} \mu, \quad \sup_{0\le t\le T} \frac{\eta_n(t)}{t} \mathrm{Var}(\bar{R}_{n,i}) \stackrel{n}{\to} \sigma^2, \tag{3}$$

$$\sup_{0\le t\le T} \zeta_t^{(n)} \stackrel{n}{\to} 0, \text{ and for any } 0 < t \le T \ \frac{\eta_n(t)}{t} \stackrel{n}{\to} \infty, \tag{4}$$

then

$$\bar{X}^{(n)} \stackrel{d}{\Rightarrow} X.$$

The following two corollaries lay the ground for the two applications of this theorem that are used in this paper.

COROLLARY 3. For a sequence $\Delta t_n = T/n$, denote $N^{(n)}$ the process $N_t^{(n)} = \left\lfloor \frac{t}{\Delta t_n} \right\rfloor$ on $[0, T]$: If the jump sizes of $\bar{R}_{m,i}$ vanish and

$$\frac{E[\bar{R}_{n,i}]}{\Delta t_n} \to \mu, \quad \frac{\mathrm{Var}(\bar{R}_{n,i})}{\Delta t_n} \to \sigma^2, \tag{5}$$

then the conclusion of Theorem 2 holds.

This corollary is known as Donsker's theorem (see e.g. Billingsley (1968)) and would be sufficient for almost all applications in this paper. Only for randomized models, where we adopt N as a Poisson process, we need the following corollary of Theorem 2:

COROLLARY 4. For a sequence $\lambda_n \to \infty$, corresponding Poisson processes $N^{(n)}$ with that intensity, a sequence $(\bar{R}_{n,i})_i$ with vanishing jump sizes and

$$E[\bar{R}_{n,i}]\lambda_n \to \mu, \text{ and } \mathrm{Var}(\bar{R}_{n,i})\lambda_n \to \sigma^2, \tag{6}$$

the conclusion of Theorem 2 holds.

Let us now look at the CRR model: Discretizing the time axis $[0, T]$ by a refinement n yields a set $\mathcal{T}^n = \{0 = t_{n,0} < t_{n,1} \ldots < t_{n,n} = T\}$ of equidistant dates, i.e., $t_{n,i+1} - t_{n,i} = \Delta t_n = \frac{T}{n}$. For a given Δx_n we model the (per-period) return by

$$\bar{R}_{n,i} \sim \begin{cases} +\Delta x_n & ; \text{ with probability } q_n \\ -\Delta x_n & ; \text{ with probability } 1 - q_n \end{cases}, \tag{7}$$

and the stock process on $[0, T]$ by

$$\bar{G}_t^{(n)} = \prod_{i=1}^{N_t^{(n)}} \exp \bar{R}_{n,i}, \quad \text{with} \quad N_t^{(n)} = \left\lfloor \frac{t}{\Delta t_n} \right\rfloor.$$

This means that the logarithmic asset price evolves in a fixed grid with grid points at distance Δx_n; between two dates the asset price is constant, and at the next date the asset jumps only to the next adjacent grid node. Let us denote by $X_t = \ln G_t$, $\bar{X}_t^{(n)} = \ln \bar{G}_t^{(n)}$ the logarithmic processes of the original process and its approximation. To match the continuous per-period variance, we set

$$\Delta x_n = \sigma \sqrt{\Delta t_n}\ . \tag{8}$$

For pricing, only the evolution of the processes under the risk-neutral probability measure $Q^{(n)}$ matters; it is represented by the probability q_n for an up-move, such that $E[\exp \bar{R}_{n,1}] = \exp\{r\Delta t_n\}$.

THEOREM 5.[4] If $E\left[\bar{R}_{n,1}\right] = \left(r - \frac{\sigma^2}{2}\right)\Delta t_n$, then $E\left[\exp \bar{R}_{n,1}\right] = \exp\{r\Delta t_n\} + \mathcal{O}(\Delta t_n)$. On the other hand, if the martingale measure condition $E[\exp \bar{R}_{n,1}] = \exp\{r\Delta t_n\}$ holds, then $E\left[\bar{R}_{n,1}\right] = \left(r - \frac{\sigma^2}{2}\right)\Delta t_n + \mathcal{O}(\Delta t_n^{3/2})$. Moreover, in both cases, $\bar{G}^{(n)} \overset{d}{\Rightarrow} G$.

If we are only interested in limiting prices, Theorem 5 leaves us with two alternatives to specify q_n. First, looking at the processes G and $\bar{G}^{(n)}$ we can match their risk-neutral drift $E[\exp \bar{R}_{n,i}] = \exp\{r\Delta t_n\}$. Second, we can take $E\left[\bar{R}_{n,1}\right] = \left(r - \frac{\sigma^2}{2}\right)\Delta t_n$, to match the drift of the logarithmic processes X and $\bar{X}^{(n)}$. In the limit the same processes and thus the same prices will result. Here we will use in the sequel the second alternative to specify the transition probability, and require $E\left[\bar{R}_{n,1}\right] = \left(r - \frac{\sigma^2}{2}\right)\Delta t_n$. Linear transformations reveal that

$$q_n = \frac{1}{2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta t_n}{2\Delta x_n} = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t_n}. \tag{9}$$

Figure 1 presents a pricing example for a down-and-out call with strike $K = 110$ and a barrier at $H = 90$ written on a stock when today's stock price is 100, the volatility is $\sigma = 0.2$ and the interest rate is $r = 0.1$. Convergence to the continuous time solution is not "smooth," it exhibits the typical "saw-tooth" pattern and the "odd-even ripple." Figure 2 depicts the error on a log–log scale. Since the function $1/\sqrt{n}$ is an appropriate upper bounding line, we conclude that the order of convergence is $1/2$. Compared to the pricing of plain vanilla options, where the order is 1 we lose $1/2$. We see that we have quite high errors in the worst case. Furthermore we deduce that we need at least a refinement of $1/\sqrt{n} = 0.01$, i.e. $n = 10000$, to ensure "penny-accuracy."

[4] For $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ we also use of the Landau notation $f = \mathcal{O}(g)$: there exists a constant c, s.t. $|f/g| < c$ on the strictly positive real line.

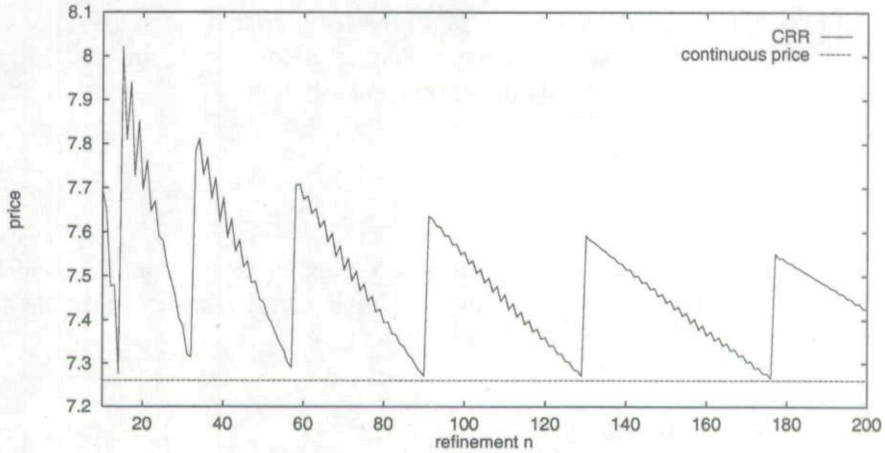

Figure 1. Price picture for the barrier option, depicting the convergence structure.

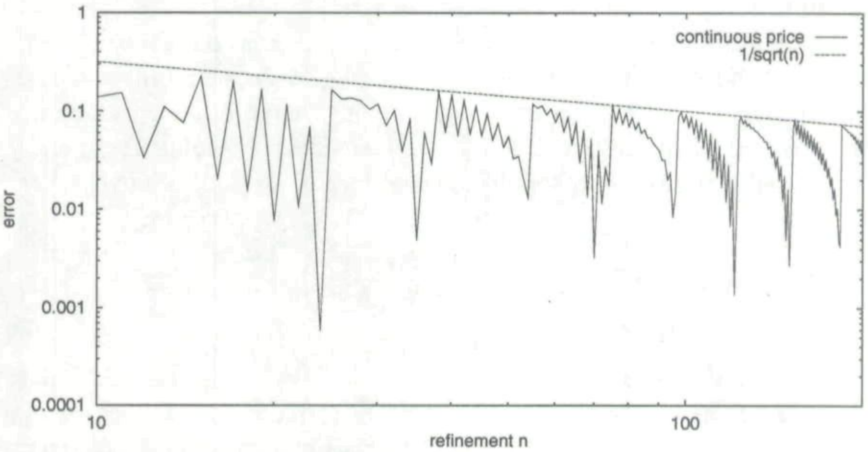

Figure 2. Error picture for the barrier option.

The convergence is slower than in the standard European call and put option case, since in binomial models the whole probability mass is concentrated in the tree-nodes. The difference in probability between two adjacent nodes is known to be of the order $\mathcal{O}(1/\sqrt{n})$.[5] If a node "jumps" over the barrier layer, resulting from an increase in the refinement by one, then, taking the corresponding probability mass with, we will observe a change in the price of the same order in n.

[5] Feller (1996a), section 7.2 proves that the binomial probability $b(k; n, p)$ for the event where we observe exactly k successes out of n, when the probability of a single success is p, is asymptotically $b(k; n, p) \sim (2\pi npq)^{-1/2} \exp(-\delta_k^2/2npq)$ $(\delta_k = k - np)$. Since the exponential function term is bounded by 1, the probability of each success is of order $\mathcal{O}(1/\sqrt{n})$.

To prevent this, Ritchken (1995) ensured for barrier options that nodes lie exactly on the barrier for any refinement. Similarly Boyle and Lau (1994) argued to take depending on $m \in \mathbb{N}$ only the refinements

$$\left\lfloor \frac{m^2\sigma^2 T}{(\ln S_0/H)^2} \right\rfloor . \tag{10}$$

These refinements are exactly the one's before an entire layer jumps over the barrier, again. In both articles pricing errors are reduced to a size comparable to those of call options.

4. How to discretize properly

This section constructs a trinomial model that ensures nodes lie on *all* barrier lines *and* on the kink of the payoff function for put/call options; it is a first approach to remove the pitfalls of the previous section.

With multiple barriers, we term each barrier line as a "critical line." Discretizing time first, and *then* studying the resulting X-grid does not allow control of the state spacing and runs therefore into problems. Therefore we conclude that the X-axis needs to be discretized first by a refinement m, yielding Δx_m, and then the refinement n_m of the t-axis should be set according to Equation (8) as

$$\Delta t_m = \left(\frac{\Delta x_m}{\sigma}\right)^2, \quad \text{or} \quad n_m = \left\lfloor T\left(\frac{\sigma}{\Delta x_m}\right)^2 \right\rfloor . \tag{11}$$

A down-and-out call option requires $\Delta x_m = \frac{\ln G_0/H}{m}$ to place grid points on the barrier H. Interestingly, this way we recover formula (10) derived by Boyle and Lau (1994) as the resulting refinement n_m in Equation (11). In the symmetrical case where $|G_0 - K| = |G_0 - H|$ this improves the accuracy and reduces the oscillations to a size comparable to European call options. To solve this problem we need to ensure that nodes also lie on the strike.

We will now present theoretically and in full generality a first approach to resolve this difficulty; below we explain it on a concrete example with a single barrier option. We call *critical layer* all barrier lines (possibly many), the strike and the current asset price. Let us denote by $\mathcal{L}$ the ordered set $l_1 < \ldots < l_L$ of critical layers, and then define variables $\Delta x_{m,i} = \frac{l_{i+1}-l_i}{m}$ for $1 \leq i \leq L-1$, and $\Delta x_{m,L} = \Delta x_{m,L-1}$, $\Delta x_{m,0} = \Delta x_{m,1}$. Setting $l_0 = -\infty$, $l_{L+1} = +\infty$, this defines a time-homogeneous grid for the process $\bar{X}^{(m)}$ on the entire real line:[6] for $0 \leq i \leq L$ and $x \in]l_i, l_{i+1}[$ we adopt $\Delta x_m^u(x) = \Delta x_m^d(x) = \Delta x_{m,i}$ and *on* the critical line l_i, for $1 \leq i \leq L$, we take $\Delta x_m^u(l_i) = \Delta x_{m,i}$, and $\Delta x_m^d(l_i) = \Delta x_{m,i-1}$. Finally we define the minimum spacing $\Delta x_m = \min_{0<i<L} \Delta x_{m,i}$, which defines Δt_m according

[6] The stock price process is the exponential; therefore taking only positive values.

to Equation (11), and model the return, depending on some $x \in \mathbb{R}$, by the trinomial random variable

$$\bar{R}_{m,i}(x) \sim \begin{cases} \Delta x_m^u(x) & ; \text{ with probability } p_m(x) \\ 0 & ; \text{ with probability } 1 - p_m(x) - q_m(x) \\ -\Delta x_m^d(x) & ; \text{ with probability } q_m(x) \end{cases} . \tag{12}$$

Intuitively, we take at each grid point x the absolute difference to the next grid point above as $\Delta x_m^u(x)$, and the absolute difference to the next grid point below as $\Delta x_m^d(x)$. $\Delta x_m^u(x)$ and $\Delta x_m^d(x)$ will adopt different values only on the critical lines. A trinomial model has one more degree of freedom in the probability, which allows us to get the variance equal to Δx_m as long as $\max\{\Delta x_m^d, \Delta x_m^u\} \geq \Delta x_m$. Therefore, properly done, the variance can be set right despite the complication that discretizations are not constant and change at critical lines. The choice of $\Delta x_m^u(x)$, $\Delta x_m^d(x)$ gives enough degrees of freedom to place nodes on critical "lines," like the barrier and the strike price, and set the probabilities right.

For $\Delta t_m, n_m$ defined by Equation (11), we call the processes

$$\bar{X}_t^{(m)} = \sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i}\left(\bar{X}_{t-}^{(m)}\right), \quad \bar{G}_t^{(m)} = \exp \bar{X}_t^{(m)}, \text{ with } N_t^{(m)} = \left\lfloor \frac{t}{\Delta t_m} \right\rfloor \tag{13}$$

the *Trinomial Adjusted (TA)* model. Similarly to our previously defined continuous-time binomial model, this process moves along time with price changes at dates, spaced at distance Δt_m. For simplicity we will drop the dependence of the return variable on $\bar{X}_{t-}^{(m)}$ in the sequel, and use only the notation $\bar{R}_{m,i}$.

REMARK 6.

(1) Note, that $\bar{X}_{t-}^{(m)}$ is the asset price *before* the jump; this is the appropriate one to choose, which return variable $\bar{R}_{m,i}(x)$ depending on x drives the jump.

(2) The process is path-dependent; however this dependency is only conditionally on the actual state and is therefore easy to handle in the common backward induction. The model is also recombining and therefore computationally simple; calculations are similar to any standard trinomial model.

Let us explain our approach in detail for a down-and-out call option in Figure 3 where a strike price is at $K = 120$ and a barrier is at $H = 90$: We see that we can distinguish $L = 3$ critical layers and four different ranges: one below the barrier, one between the barrier and the current asset price, one between the strike and the current asset price and one above the strike. In each of the two inner ranges we need a different Δx_m. To place nodes on critical layers, we take between the barrier and the current asset price $\Delta x_{m,1} = |\ln H/S_0|/m$ and between the strike and the current asset price $\Delta x_{m,2} = |\ln K/S_0|/m$. This results in an equidistant discretization between critical layers, i.e. there is a fixed number m of grid points

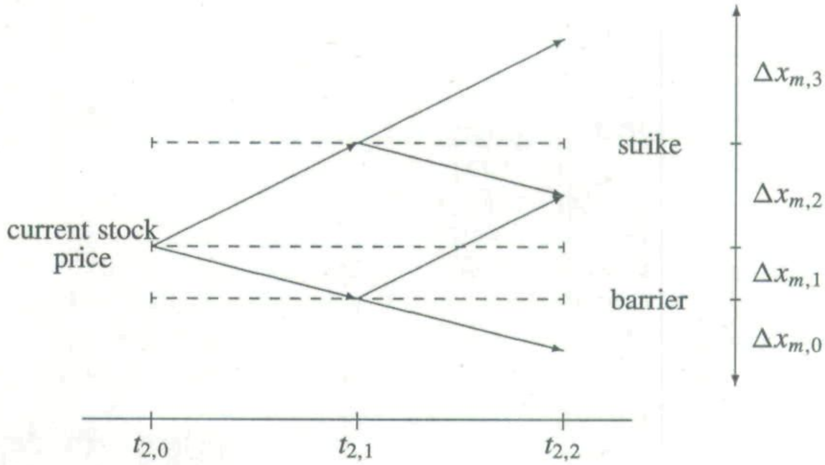

Figure 3. Example dynamics.

between each layer, and defines our grid for the asset evolution. For $\Delta x_{m,3}$ ($\Delta x_{m,0}$), our approach takes $\Delta x_{m,2}$ ($\Delta x_{m,1}$) for simplicity.

A consistency requirement is $\bar{G}^{(m)} \overset{d}{\Rightarrow} G$. Since the exponential function is continuous, this is equivalent to $\bar{X}^{(m)} \overset{d}{\Rightarrow} X$. According to Corollary 3 it is sufficient that the first two moments of the discrete and corresponding continuous-time process match in the limit, i.e.,

$$\frac{E\left[\bar{R}_{m,i}\right]}{\Delta t_m} \overset{m}{\to} r - \frac{\sigma^2}{2}, \quad \text{and} \quad \frac{\operatorname{Var}\left[\bar{R}_{m,i}\right]}{\Delta t_m} \overset{m}{\to} \sigma^2 . \tag{14}$$

We require them to hold with equality:

$$\Delta x_m^u p_m - \Delta x_m^d q_m = \left(r - \frac{\sigma^2}{2}\right) \Delta t_m \text{ , and } (\Delta x_m^u)^2 p_m + (\Delta x_m^d)^2 q_m = \sigma^2 \Delta t_m .$$

Linear transformations yield

$$p_m = \frac{\left(r - \frac{\sigma^2}{2}\right) \Delta x_m^d + \sigma^2}{\Delta x_m^u (\Delta x_m^u + \Delta x_m^d)} \Delta t_m \text{ , and } q_m = \frac{-\left(r - \frac{\sigma^2}{2}\right) \Delta x_m^u + \sigma^2}{\Delta x_m^d (\Delta x_m^u + \Delta x_m^d)} \Delta t_m .$$

THEOREM 7. For sufficiently high refinements, the parameters $p_m, q_m, 1 - p_m - q_m \in [0, 1]$, i.e. are transition probabilities, and the processes $\bar{X}^{(m)} \overset{d}{\Rightarrow} X$ and $\bar{G}^{(m)} \overset{d}{\Rightarrow} G$ converge in distribution to their continuous counterparts.

As explained in Remark 1 this ensures convergence of discrete to continuous prices. Table I gives a pricing example for a barrier option using CRR, TA, and the modified Richardson extrapolation rule (see Leisen (1998))

$$\pi^e(m) = \frac{n_{m+1} \cdot \pi(m+1) - n_m \cdot \pi(m)}{n_{m+1} - n_m}, \tag{15}$$

Table I. Pricing example with S = 100, K = 110, H = 85, T = 1, r = 0.1, $\sigma = 0.2$. The continuous time price is 7.97888.

m	n_m	$\pi_{\mathrm{CRR}}(m)$	$\pi_{\mathrm{TA}}(m)$	$\pi^e_{\mathrm{TA}}(m)$	$\pi_{\mathrm{RT}}(m)$	$\pi^e_{\mathrm{RT}}(m)$
1	8	8.44693	7.89878	7.93419	6.60928	8.14613
2	34	8.17859	7.92586	7.92367	7.78452	7.97775
3	78	8.04017	7.92463	7.98542	7.89352	7.97775
4	140	7.97567	7.95155	7.98644	7.93082	7.97837
5	220	8.03844	7.96423	7.97371	7.94811	7.97904
6	316	8.02729	7.96711	7.97778	7.95751	7.97889
7	430	7.97612	7.96994	7.98570	7.96318	7.97884
8	562	8.00860	7.97364	7.97786	7.96686	7.97883

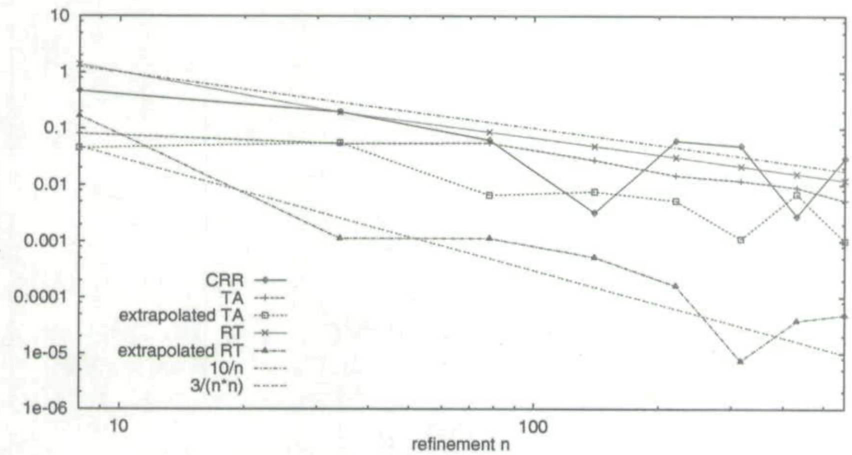

Figure 4. Error picture for the barrier option.

iterating $m = 1, \ldots, 8$. The RT model will be introduced in the following section. Figure 4 contains the error picture for the same parameter constellation as in Table I. However, we look at a barrier struck at $H = 85$, in order to demonstrate that our method is capable of treating any case, including the nonsymmetric case $|H - G_0| \neq |K - G_0|$ efficiently. Here we take only specific refinements; we see that the convergence structure is much smoother than in Figure 2. However it is not sufficiently smooth to apply extrapolation, which therefore not depict. For TA, in comparison to CRR, errors are drastically reduced and extrapolation gives quickly "penny-accuracy." We do also observe that the convergence structure is fairly smooth.

5. Randomization and Jump–diffusions

We will now randomize the previous model. This will be an easy and straightforward way to incorporate the additional jumps which characterize the jump-diffusion model in difference to the Black–Scholes setup. It turns out, in accordance to Leisen (1999), that such a randomized model yields even better convergence results.

In the previous section we looked at the TA model which was driven by a process $N^{(m)}$ jumping upward in unit integers at equidistant dates. Here, we start with a sequence of Poisson processes $N^{(m)} = (N_t^{(m)})_{t\geq 0}$ where $N^{(m)}$ has parameter λ_m. The process $N^{(m)}$ is described by interarrival times $\tau_{m,i}$, i.e. independent exponentially distributed random variables with parameter $1/\lambda_m$ by setting $N_t^{(m)} = \max\left\{n \,\middle|\, \sum_{i=1}^n \tau_{m,i} \leq t\right\}$. We adopt $\Delta x_{m,l}$ $(l = 1, \ldots, L)$ to place nodes on critical "lines," like the barrier and the strike price, and the corresponding definition of $\bar{R}_{m,i}$, both exactly as described in the previous section. (The transition probabilities will however differ and defined below.) Then, study the discrete processes $\bar{X}_t^{(m)}$ and $\bar{G}_t^{(m)}$, defined similar to (13) by

$$\bar{X}_t^{(m)} = \sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i}\ ,\ \bar{G}_t^{(m)} = \exp \bar{X}_t^{(m)}\ .$$

Similarly to Leisen (1999) we take

$$\lambda_m = \left(\frac{\sigma}{\Delta x_m}\right)^2 .$$

This choice is motivated by observation that $1/\lambda_m = E[\tau_{m,i+1} - \tau_{m,i}]$, which corresponds to Δt_m in Equation (11). To fulfill Corollary 4 we require

$$E\left[\bar{R}_{m,i}\right]\lambda_m \xrightarrow{m} r - \frac{\sigma^2}{2}\ ,\ \text{ and } \operatorname{Var}\left[\bar{R}_{m,i}\right]\lambda_m \xrightarrow{m} \sigma^2\ . \tag{16}$$

We require this to be fulfilled with equality, i.e.,

$$\Delta x_m^u p_m - \Delta x_m^d q_m = \frac{r - \frac{\sigma^2}{2}}{\lambda_m}\ ,\ (\Delta x_m^u)^2 p_m + (\Delta x_m^d)^2 q_m = \frac{\sigma^2}{\lambda_m}\ ,$$

or equivalently

$$p_m = \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta x_m^d + \sigma^2}{\lambda_m \Delta x_m^u (\Delta x_m^u + \Delta x_m^d)}\ ,\ \text{ and } q_m = \frac{-\left(r - \frac{\sigma^2}{2}\right)\Delta x_m^u + \sigma^2}{\lambda_m \Delta x_m^d (\Delta x_m^u + \Delta x_m^d)}\ .$$

THEOREM 8. For the RT model as constructed above, and sufficiently high refinements the parameters $p_m, q_m, 1 - p_m - q_m \in [0, 1]$, i.e. are transition probabilities. Moreover, the discrete asset price process converges in distribution to its continuous counterpart: $\bar{G}^{(m)} \overset{d}{\Rightarrow} G$.

We call this model the *Randomized Trinomial (RT)* model. Note that the Poisson process $N^{(m)}$ is stationary, which will make valuations in the next section especially easy to perform. The only difference to the process studied in the last section is the driving process. Whereas there it was $\lfloor t/\Delta t_m \rfloor$, here it is the Poisson process $N_t^{(m)}$ with parameter λ_m.

This model can now be used to construct easily an approximation in the jump-diffusion setup of Equation (2). The following lemma is the basis of our approximation:

LEMMA 9. For two independent Poisson processes N^1, N^2 with intensities λ_1, λ_2, two sequences of iid random variables $(U_n^1)_n$, $(U_n^2)_n$, independent of each other and the two Poisson process, denote $N = N^1 + N^2$,

$$U_n \sim \begin{cases} U_n^1 & ; \text{ with probability } \frac{\lambda^1}{\lambda^1+\lambda^2} \\ U_n^2 & ; \text{ with probability } \frac{\lambda^2}{\lambda^2+\lambda^2} \end{cases} . \tag{17}$$

Then:

$$\sum_{i=1}^{N} U_i \overset{d}{=} \sum_{i=1}^{N^1} U_i^1 + \sum_{i=1}^{N^2} U_i^2. \tag{18}$$

We start with a sequence $(N^{(m)}, \bar{R}_m)_m$ defined as above, only adjusting the probability to meet the requirement $E[\bar{R}_{m,1}] = r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]$, i.e.

$$p_m = \frac{\left(r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]\right) \Delta x_m^d + \sigma^2}{\lambda_m \Delta x_m^u (\Delta x_m^u + \Delta x_m^d)} ,$$

and

$$q_m = \frac{-\left(r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]\right) \Delta x_m^u + \sigma^2}{\lambda_m \Delta x_m^d (\Delta x_m^u + \Delta x_m^d)} .$$

Then we define the process

$$\bar{N}^{(m)} = N + N^{(m)} , \tag{19}$$

which is a Poisson process with intensity $\bar{\lambda}_m = \lambda + \lambda_m$, the sequence of random variables

$$\bar{R}'_{m,i} \sim \begin{cases} \ln V_i & ; \text{ with probability } \frac{\lambda}{\lambda+\lambda_m} \\ \bar{R}_{m,i} & ; \text{ with probability } \frac{\lambda_m}{\lambda+\lambda_m} \end{cases} , \tag{20}$$

and the processes

$$\bar{Y}_t^{(m)} = \sum_{i=1}^{\bar{N}_t^{(m)}} \bar{R}'_{m,i} \text{ , and } \qquad \bar{S}_t^{(m)} = S_0 \exp\left(\bar{Y}_t^{(m)}\right) . \tag{21}$$

Lemma 9 states that the sum of two compound Poisson processes, properly adjusted, is a compound Poisson process again; looking at the jump times of the Poisson process $\bar{N}^{(m)}$, we expect to observe jumps resulting from N in $\frac{\lambda}{\lambda+\lambda_m}$ and those resulting from $N^{(m)}$ in $\frac{\lambda_m}{\lambda+\lambda_m}$ of the cases. Therefore

$$\bar{Y}_t^{(m)} \stackrel{d}{=} \sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i} + \sum_{i=1}^{N_t} \ln(V_i). \tag{22}$$

Let us define the process

$$Y_t = \ln(S_t/S_0) = \left(r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]\right) t + \sigma W_t + \sum_{i=1}^{N_t} \ln(V_i). \tag{23}$$

The counterpart to Theorem 8 is:

THEOREM 10. For the sequence of discrete processes $(\bar{Y}^{(m)})_m$ defined by equations (19)–(21) above, weak convergence holds: $\bar{Y}^{(m)} \stackrel{d}{\Rightarrow} Y$ and $\bar{S}^{(m)} \stackrel{d}{\Rightarrow} S$.

6. Implementing barrier option valuation in the randomized models

Since the structure in the randomized models differs only in the specification of the return variable, we will treat pricing in those models for a general $(\bar{R}''_{m,i})$ which is then either $(\bar{R}_{m,i})$ or $(\bar{R}'_{m,i})$. Due to the independence of $\bar{N}^{(m)}$ and the random variables in the sequence $\bar{R}''_{m,i}$, the value V_m of a barrier option is given by

$$\begin{aligned} V_m &= e^{-rT} E\left[1_\Sigma f(\bar{S}_T)\right] = e^{-rT} E\left[E[1_\Sigma f(\bar{S}_T)|\bar{N}_T^{(m)}]\right] \\ &= e^{-rT} \sum_{n=0}^{\infty} E\left[1_\Sigma f(\bar{S}_T)|\bar{N}_T^{(m)} = n\right] \cdot P\left[\bar{N}_T^{(m)} = n\right] . \end{aligned}$$

This splits up the valuation task, conditioning first on $\bar{N}_T^{(m)}$ and then averaging over all possible values. Let us now denote by Σ_n the "choice variable" corresponding to Σ in the n–step tree with return modeled by $\bar{R}''_{m,i}$, and define

$$\Phi_n^{(m)} = E\left[1_\Sigma f(\bar{S}_T)|\bar{N}_T^{(m)} = n\right] = E\left[1_{\Sigma_n} f(\bar{S}_T)|\bar{N}_T^{(m)} = n\right]$$

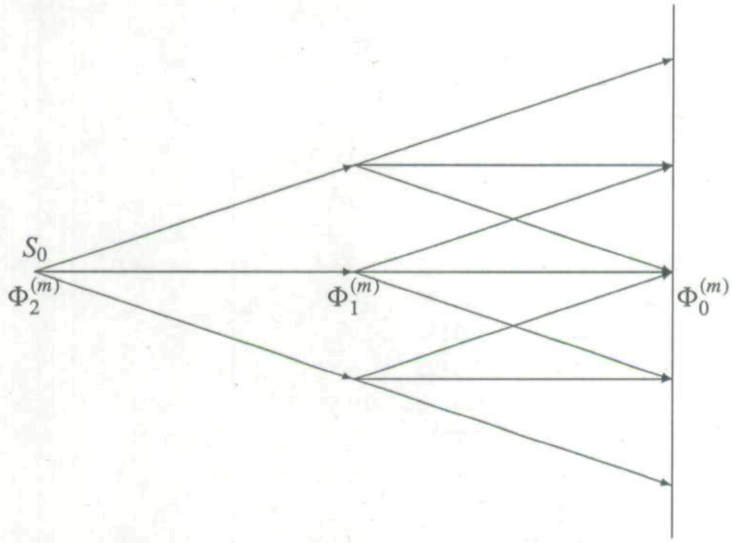

Figure 5. The parameters $\Phi_n^{(m)}$ as intermediate calculations in the trinomial grid.

Therefore

$$V_m = e^{-(r+\bar{\lambda}_m)T} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}_m T)^n}{n!} \Phi_n^{(m)}. \tag{24}$$

For implementations, we have to cut off the infinite sum in Equation (24) at an appropriate γ_m. Here, we are interested in a cut-off with $\lim_{m\to\infty} P[N_T^{(m)} \in \{0, \dots, \gamma_m\}] = 1$. Due to the central limit theorem for renewals[7]

$$\frac{N_T^{(m)} - \bar{\lambda}_m T}{\sqrt{\bar{\lambda}_m T}} \stackrel{d}{\Rightarrow} \mathcal{N}(0, 1) ,$$

so that setting $\gamma_m = 2\lfloor \bar{\lambda}_m T \rfloor$ is *one* choice fulfilling our requirement. In the sequel we adopt as our cut-off

$$V_m \approx e^{-(r+\bar{\lambda}_m)T} \sum_{n=0}^{2\lfloor \bar{\lambda}_m T \rfloor} \frac{(\bar{\lambda}_m T)^n}{n!} \cdot \Phi_n^{(m)} . \tag{25}$$

The value $\Phi_n^{(m)}$ can be interpreted as the value calculated by backward–induction in an n–step tree grid with return $(\bar{R}''_{m,i})_i$, if we *do not* perform discounting (see figure 6 for $\bar{R}''_{m,i} = \bar{R}_{m,i}$). Therefore, calculating $\Phi_{\tilde{n}}^{(m)}$ for any $\tilde{n}$ yields $\Phi_{n'}^{(m)}$ for

[7] Feller (1966b), section XI.5 states that for a renewal N the central limit theorem holds in the usual form with mean (variance) of N equal to t/μ, $(t\sigma^2/\mu^3)$. (μ, σ are here the mean, variance of the interarrival times; for a Poisson process we therefore have $\mu = \sigma^2 = 1/\lambda$.)

Table II. Pricing example in a "ruin" setup with S = 100, K = 110, H = 85, T = 1, r = 0.1, $\sigma = 0.2$, $\lambda = 0.1$.

m	n_m	$\pi_{\mathrm{RT}}(m)$	$\pi^e_{\mathrm{RT}}(m)$	$\epsilon_{\mathrm{RT}}(m)$	$\epsilon^e_{\mathrm{RT}}(m)$
1	8	11.0891	13.6483	2.2053	0.3539
2	34	13.0461	13.2948	0.2483	0.0004
3	78	13.1864	13.2930	0.1080	0.0015
4	140	13.2336	13.2937	0.0608	0.0008
5	220	13.2555	13.2945	0.0390	0.0001
6	316	13.2673	13.2943	0.0271	0.0001
7	430	13.2745	13.2942	0.0200	0.0002
8	562	13.2791	13.2942	0.0153	0.0002

$n' = 0, \ldots, \tilde{n}$ as intermediate calculations. It is therefore sufficient to calculate a $2\lfloor\bar{\lambda}_m T\rfloor$ step tree and write out the parameters $\Phi_n^{(m)}$ as intermediate calculations. Computing prices in our model is then comparable to a trinomial model (and therefore to the CRR model) in terms of the computational burden. In order to compare both approaches properly depending on its complexity, we index calculations in the RT by $2\lfloor\bar{\lambda}_m T\rfloor$.

As explained on Figure 3, we adopt the discretizations $\Delta x_{m,0} = \Delta x_{m,1} = |\ln H/S_0|/m$ and $\Delta x_{m,2} = \Delta x_{m,3} = |\ln K/S_0|/m$ to place *all* nodes on critical "lines." Table I presents prices and errors. We see that RT yields better price approximations than CRR, however slightly worse than TA. Extrapolating the RT model we get very accurate price approximations, which outperform any other of the models we study here, and their respective extrapolations. This becomes even more apparent in the error picture 4. We see that prices with RT converge with order one and extrapolated prices seem to converge even with order two. Moreover we observe a gain in accuracy by extrapolation in comparison to the extrapolated TA model.

We now discuss the accuracy in a ruin setup ($V_i \equiv 0$). Table II and Figure 6 present values calculated for this case, adopting $\lambda = 0.1$. This choice is fairly arbitrary; the interpretation is that we expect the firm to get bankrupt after $1/0.1 = 10$ years. We also depict errors ϵ and extrapolated prices (errors) π^e (ϵ^e), calculated according to equation (15). We calculated 13.2944 as the price in a CRR model with a refinement of 100000. This is an estimation of the continuous time price. To infer the accuracy we look at Figure 2; there we saw that $1/\sqrt{n}$ was an appropriate upper bound for the error. With a tree of this size we have an upper bound for the error of $1/\sqrt{100000} = 0.0031$. We believe that the jump diffusion setup should have similar convergence properties to the pure diffusion setup. Therefore under this assumption we predict the error of our estimate being less than 0.0031. Here we see very impressively the slow convergence of the CRR model. Immediately

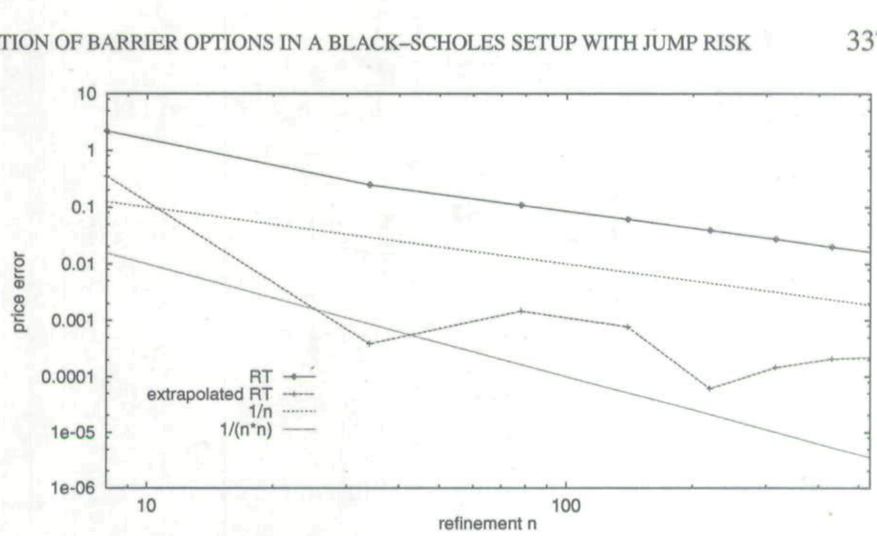

Figure 6. Error picture for the barrier option.

(with $m = 2$) we fall below this level with the extrapolated RT model. We do not perform a graphical analysis of the order, since we can not calculate sufficiently accurate values; iterating the extrapolated RT higher than $m = 2$ is even doubtful. Nevertheless we believe that these results are promising since they suggest that the remarkable convergence properties carry over from the Black–Scholes setup to the jump-diffusion case.

7. Conclusion

This paper constructed a randomized trinomial model and explained that this is a natural way to approximate jump-diffusions. The parameters were set such as to get consistency with the continuous-time processes. With increasing refinement approximated values therefore converge to their continuous counterpart. We also discussed how efficient numerical approximations for barrier options could result in the Black–Scholes and the jump-diffusion setup.

Appendix A: Proofs

Proof of Theorem 2. We apply the martingale central limit theorem, see, e.g., Ethier and Kurtz (1986), theorem 7.4.1 on p. 354. Denote $L_t^{(n)} = \eta_n(t)E[\bar{R}_{n,i}]$, $A_t^{(n)} = \eta_n(t)\mathrm{Var}(\bar{R}_{n,i}) + \zeta_t^{(n)}$, $M_t^{(n)} = \bar{X}_t^{(n)} - L_t^{(n)}$. Since the jump sizes of $\bar{R}_{n,i}$ vanish, the jump sizes of $L_t^{(n)}$, $M_t^{(n)}$, $A_t^{(n)}$ vanish on the set $[0, T]$. Moreover, by the conditions in (3) and the second in (4) we have $\sup_{0\le t\le T}(L_t^{(n)} - \mu t) \to 0$ and $\sup_{0\le t\le T}(\eta_n(t)\mathrm{Var}(\bar{R}_{n,i}) - \sigma^2 t) \to 0$. Together with the condition in (4) follows

$\sup_{0\le t\le T}(A_t^{(n)}-\sigma^2 t)\to 0$. It remains to check that $M^{(n)}$ and $\left(M^{(n)}\right)^2 - A^{(n)}$ are martingales. Since

$$\begin{aligned} E\left[\bar{X}_t^{(n)}\right] &= E\left[E\left[\bar{X}_t^{(n)}|N_t^{(n)}\right]\right] = \sum_{\nu=1}^{\infty} P[N_t^{(n)}=\nu]\cdot E[\bar{X}_t^{(n)}|N_t^{(n)}=\nu] \\ &= \sum_{\nu=1}^{\infty} P[N_t^{(n)}=\nu]\cdot \nu\cdot E\left[\bar{R}_{n,1}\right] \\ &= E\left[N_t^{(n)}\right] E\left[\bar{R}_{n,1}\right] = \eta_n(t) E\left[\bar{R}_{n,i}\right], \end{aligned}$$

$M^{(n)}$ is a martingale. By conditioning on N_t follows similarly

$$\begin{aligned} E\Big[\Big(\bar{X}_t^{(n)}\Big)^2\Big] &= E\Big[\Big(\sum_{i=1}^{N_t^{(n)}} \bar{R}_{n,i}\Big)^2\Big] \\ &= \eta_n(t)E[(\bar{R}_{n,i})^2] + \zeta_t^{(n)} + (\eta_n^2(t)-\eta_n(t))E[\bar{R}_{n,i}]^2, \end{aligned}$$

so that

$$E\Big[\Big(M_t^{(n)}\Big)^2\Big] = E\Big[\Big(\bar{X}_t^{(n)}\Big)^2\Big] - \Big(L_t^{(n)}\Big)^2 = \eta_n(t)\mathrm{Var}(\bar{R}_{n,i}) + \zeta_t^{(n)},$$

i.e., $(M^{(n)})^2 - A^{(n)}$ is a martingale.

Proof of Corollary 3. Since the renewal process is not random: $\eta_n(t) = N_t^{(n)} = \lfloor t/\Delta t_n \rfloor$. Therefore the conditions in (3) hold. Moreover, due to the serial independence of $(\bar{R}_{n,i})_i$, we have for all n, t $\zeta_t^{(n)} = N_t^{(n)}\left(N_t^{(n)}-1\right)E[\bar{R}_{n,i}]^2 - (\eta_n^2(t) - \eta_n(t))E[\bar{R}_{n,i}]^2 = 0$ and so $\sup_{0\le t\le T}\zeta_t^{(n)} = 0$ for any n.

Proof of Corollary 4. Since $\eta_n(t) = \lambda_n t$, we only need to prove the first condition in (4): Conditioning on $N_t^{(n)}$ and then using the independence of $(N^{(n)})_n$ and $(\bar{R}_{n,i})_i$ proves

$$\begin{aligned} &\zeta_t^{(n)} + (\eta_n^2(t)-\eta_n(t))E[\bar{R}_{n,i}]^2 \\ &\quad = E\left[E\left[\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \bar{R}_{n,i}\,\bar{R}_{n,j}\,\middle|\, N_t^{(n)}\right]\right] \\ &\quad = \sum_{n=0}^{\infty} P[N_t^{(n)}=n] E\Big[\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \bar{R}_{n,i}\,\bar{R}_{n,j}\Big] \end{aligned}$$

$$= \sum_{n=0}^{\infty} P[N_t^{(n)} = n]n(n-1)E[\bar{R}_{n,i}^2] = (\lambda_n t)^2 \cdot E[(\bar{R}_{n,i})^2]$$

so that $\zeta_t^{(n)} = \lambda_n t \left(E[\bar{R}_{n,i}]\right)^2$. Together with Equation (6) and $\lambda_n \to^n \infty$ follows $\sup_{0\le t\le T} \zeta_t^{(n)} \to^n 0$.

Proof of Theorem 5. The condition $\exp\{r\Delta t_n\} = E[\exp \bar{R}_{n,1}]$ can be rewritten as

$$\begin{aligned}
&1 + r\Delta t_n + \mathcal{O}(\Delta t_n^2)\\
&= \exp(r\Delta t_n) = q_n \exp(\Delta x_n) + (1-q_n)\exp(-\Delta x_n)\\
&= q_n\left(1 + \Delta x_n + \Delta x_n^2/2\right) + (1-q_n)\left(1 - \Delta x_n + \Delta x_n^2/2\right) + \mathcal{O}(\Delta t_n^{3/2})\\
&= 1 + \Delta x_n(2q_n - 1) + \Delta x_n^2/2 + \mathcal{O}(\Delta t_n^{3/2}),
\end{aligned}$$

which is, using Equation (8), equivalent to $\left(r - \frac{\sigma^2}{2}\right)\Delta t_n = (2q_n - 1)\sigma\sqrt{\Delta t_n} + \mathcal{O}(\Delta t_n^{3/2}) = E[\bar{R}_{n,1}] + \mathcal{O}(\Delta t_n^{3/2})$. This and the converse – which follows similarly – prove the first two assertions. Then Corollary 3 proves $\bar{X}^{(n)} \stackrel{d}{\Rightarrow} X$ and the proof concludes, observing that the exponential function is continuous.

Proof of Theorem 7. Note that

$$\begin{aligned}
p_m + q_m &= \left(r - \frac{\sigma^2}{2}\right)\frac{(\Delta x_m^d)^2 - (\Delta x_m^u)^2}{\Delta x_m^u \Delta x_m^d(\Delta x_m^u + \Delta x_m^d)}\Delta t_m\\
&\quad + \sigma^2 \frac{\Delta x_m^u + \Delta x_m^d}{\Delta x_m^u \Delta x_m^d(\Delta x_m^u + \Delta x_m^d)}\Delta t_m\\
&= \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta x_m^d - \Delta x_m^u}{\Delta x_m^u \Delta x_m^d}\Delta t_m + \sigma^2 \frac{\Delta t_m}{\Delta x_m^u \Delta x_m^d}.
\end{aligned}$$

Using the definition of $\Delta t_m = (\Delta x_m/\sigma)^2$ we find

$$p_m + q_m = \left(\frac{r - \frac{\sigma^2}{2}}{\sigma^2}(\Delta x_m^d - \Delta x_m^u) + 1\right)\frac{\Delta x_m^2}{\Delta x_m^u \Delta x_m^d}$$

If $\Delta x_m = \Delta x_m^u = \Delta x_m^d$, we have $p_m + q_m = 1$. Otherwise, Δx_m^u and Δx_m^d are not both equal to Δx_m and so $0 < \frac{(\Delta x_m)^2}{\Delta x_m^u \Delta x_m^d} < 1$, since $\Delta x_m = \min_{0\le i\le L}\{\Delta x_{m,i}\}$. Increasing the refinement from m to m' changes all layer spacings by the same amount: $\Delta x_{m',i} = \Delta x_{m,i}\frac{m}{m'}$. Therefore, the quotient $\frac{(\Delta x_m)^2}{\Delta x_m^u \Delta x_m^d}$ remains constant when

m changes, and $\lim_{n\to\infty} \frac{r-\frac{\sigma^2}{2}}{\sigma^2}(\Delta x_m^d - \Delta x_m^u)+1 = 1$. This implies $0 < p_m + q_m \leq 1$, and proves that $0 \leq 1 - p_m - q_m < 1$, for sufficiently high refinements.

It remains to prove positivity of p_m and q_m. Using the definition of $\Delta t_m = (\Delta x_m/\sigma)^2$ we find

$$p_m = \left(r - \frac{\sigma^2}{2}\right) \frac{\Delta x_m^d \Delta x_m}{\Delta x_m^u(\Delta x_m^u + \Delta x_m^d)} \frac{\Delta x_m}{\sigma} + \frac{\Delta x_m^2}{\Delta x_m^u(\Delta x_m^u + \Delta x_m^d)}.$$

Note that $\frac{\Delta x_m^d \Delta x_m}{\Delta x_m^u(\Delta x_m^u + \Delta x_m^d)}$ is bounded with changing refinement, as all layers change by the same proportional amount. Therefore, increasing the refinement m, $\frac{\Delta x_m^d \Delta x_m^2}{\Delta x_m^u(\Delta x_m^u + \Delta x_m^d)}$ can be made arbitrarily small, whereas $\frac{\Delta x_m^2}{\Delta x_m^u(\Delta x_m^u + \Delta x_m^d)}$ remains positive and bounded away from zero. Therefore p_m is positive for sufficiently high refinements. The same arguments apply to conclude that $q_m > 0$.

Corollary 3 together with condition (14) is sufficient to deduce $\bar{X}^{(m)} \overset{d}{\Rightarrow} X$ and $\bar{G}^{(m)} \overset{d}{\Rightarrow} G$.

Proof of Theorem 8. Note that the equations for the transition probabilities are identical to those in Theorem 7, after $\lambda_m = (\sigma/\Delta x_m)^2$ and $\Delta t_m = (\Delta x_m/\sigma)^2$ have been introduced. Therefore the proof that $p_m, q_m, 1 - p_m - q_m \in [0, 1]$ follows along the lines of the proof of Theorem 7.

Corollary 4 implies $\bar{X}^{(m)} \overset{d}{\Rightarrow} X$, which in turn proves $\bar{G}^{(m)} \overset{d}{\Rightarrow} G$, since the exponential function is continuous.

Proof of Lemma 9. Denote Ψ the characteristic function of the process $\sum_{i=1}^{N} U_i$, Ψ^1 that of the process $\sum_{i=1}^{N^1} U_i^1$, and Ψ^2 that of $\sum_{i=1}^{N^2} U_i^2$. According to Bertoin (1996), p. 12:

$$\Psi(x) = \lambda \int (1 - e^{ixy})\nu(dy),\ \ \Psi^1(x) = \lambda_1 \int (1 - e^{ixy})\nu^1(dy),$$

$$\Psi^2(x) = \lambda_2 \int (1 - e^{ixy})\nu^2(dy),$$

where ν, ν^1, ν^2 denote the probability laws of U, U^1, U^2. Due to the independence,

$$\begin{aligned}
\exp(t\Psi(x)) &= E\left[\exp\left(ix\sum_{i=1}^{N_t} U_i\right)\right] \\
&= E\left[\exp\left(ix\sum_{i=1}^{N_t^1} U_i^1\right)\right] \cdot E\left[\exp\left(ix\sum_{i=1}^{N_t^2} U_i^2\right)\right] \\
&= \exp(t\Psi^1(x)) \cdot \exp(t\Psi^2(x)).
\end{aligned}$$

Therefore, for any x, $\Psi(x) = \Psi^1(x) + \Psi^2(x)$, and so

$$\begin{aligned}\lambda \int (1 - e^{ixy})\nu(dy) &= \lambda_1 \int (1 - e^{ixy})\nu^1(dy) + \lambda_2 \int (1 - e^{ixy})\nu^2(dy) \\ &= (\lambda_1 + \lambda_2) \int (1 - e^{ixy}) \frac{\lambda_1 \nu^1(dy) + \lambda_2 \nu^2(dy)}{\lambda_1 + \lambda_2},\end{aligned}$$

which implies $\nu(dy) = \frac{\lambda_1 \nu^1(dy) + \lambda_2 \nu^2(dy)}{\lambda_1 + \lambda_2}$.

Proof of Theorem 10. Following the proof of Theorem 8 we get

$$\sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i} \overset{d}{\Rightarrow} \left(r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]\right) t + \sigma W_t \,.$$

Denote by h the function $h : x \mapsto x + \sum_{i=1}^{N} \ln(V_i)$ on the space of processes. Since h is continuous, VI.1.23 and VI.3.8 (ii) of Jacod and Shiryaev (1987) imply that

$$h\left(\sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i}\right) \overset{d}{\Rightarrow} h\left(\left(r - \frac{\sigma^2}{2} - \lambda E[V_i - 1]\right) t + \sigma W_t\right) = Y \,.$$

The proof concludes applying Lemma 9 (Equation (22)):

$$\bar{Y}_t^{(m)} = \sum_{i=1}^{\bar{N}_t^{(m)}} \bar{R}'_{m,i} \overset{d}{=} \sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i} + \sum_{i=1}^{N_t} \ln(V_i) = h\left(\sum_{i=1}^{N_t^{(m)}} \bar{R}_{m,i}\right).$$

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