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Stochastic Volatility With an Ornstein–Uhlenbeck Process: An Extension

RAINER SCHÖBEL and JIANWEI ZHU*
Eberhard-Karls-University Tübingen, College of Economics and Business Administration, Mohlstrasse 36, 72074 Tübingen, Germany. E-mail: rainer.schoebel@uni-tuebingen.de; jianwei.zhu@uni-tuebingen.de

Abstract. In this paper, we reexamine and extend the stochastic volatility model of Stein and Stein (S&S) (1991) where volatility follows a mean-reverting Ornstein–Uhlenbeck process. Using Fourier inversion techniques we are able to allow for correlation between instantaneous volatilities and the underlying stock returns. A closed-form pricing solution for European options is derived and some numerical examples are given. In addition, we discuss the boundary behaviour of the instantaneous volatility at $v(t) = 0$ and show that S&S do not work with an absolute value process of volatility.

Key words: Ornstein–Uhlenbeck process, Fourier inversion, option pricing, mean-reversion, volatility smile

JEL Classification: G13

Modelling stochastic volatility is crucial to capture stock return variability. In order to describe the empirical leptokurtic distributions of stock returns, a number of different specifications of stochastic volatility have been suggested. Wiggins (1987) considers a case where the log-volatility $\ln v(t)$ follows a mean-reverting Ornstein–Uhlenbeck (OU) process. Hull and White (1987) solve a special option pricing problem with stochastic volatility by using a Taylor expansion technique. All models before 1990 including Scott (1987) do not present closed-form solutions and require an extensive use of numerical techniques. Stein and Stein (S&S) (1991) assume that the volatility follows a mean reversion OU process and develop the analytic density function of stock returns to evaluate option prices. But since volatilities are uncorrelated with stock returns, their solution is not complete. Heston (1993) provides a new approach to derive a closed-form solution for options

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where the squared volatility (variance rate) is specified as a square-root process. The idea is that while the probability that the stock price is greater (less) than the strike price can not be expressed analytically, the corresponding characteristic function can indeed be described analytically in many cases, at least in the case of a square-root process for the variance rate. The probability function is then obtained via inverse Fourier transformation. Using this approach, Bakshi, Cao and Chen (1997) develop and test a comprehensive closed-form option pricing formula including jump components of the stock price process, stochastic interest rates, and a square-root based stochastic volatility. Stochastic volatility option pricing models with closed-form solutions include also Bates (1996), Bakshi and Chen (1997) and Scott (1997). The first two are on currency options. The method used by Scott is somewhat different from the other papers. Instead of deriving the characteristic functions from a fundamental partial differential equation (PDE), he calculates them directly using martingale methods. Recently, Heston and Nandi (1997) suggest a GARCH option pricing model and derive a closed-form solution which allows for correlation between stock returns and variance and even admits multiple lags in the GARCH process.

As shown in Hull and White (1987) as well as in Ball and Roma (1994), if stock returns are uncorrelated with volatilities, the option price can be approximated very accurately by replacing the variance in the Black and Scholes (1973) formula by the expected average variance which is defined by

$$V = \mathbb{E}\left[\frac{1}{T-t}\int_t^T v^2(u)\,\mathrm{d}u\right]. \tag{1}$$

The mathematical background for this replacement is that the conditional distribution of the terminal stock price, given the expected average variance, is still lognormal. However, this argument fails at the presence of correlation between stock returns and volatilities. Heston (1993) shows that the correlation between volatility and spot returns is necessary to create negative skewness and excess kurtosis in a leptokurtic distribution of spot returns. Bakshi, Cao and Chen (1997), Heston and Nandi (1997) and Nandi (1998) demonstrate that non-zero correlation is important in eliminating the 'smile' or 'sneer' of implied volatilities and leads to significant improvements both in pricing and in hedging options. Their studies are carried out in Heston's framework, i.e., squared volatility follows a square-root process. Motivated by these findings, we apply the method inspired by Scott (1997) in this paper to extend the S&S model to the case of non-zero correlation. That is, we assume that the stochastic volatilities follow a mean-reverting OU process and are correlated with stock returns. Closed-form solutions for option prices and price sensitivities are obtained via inverse Fourier transformation. Furthermore, we correct Ball and Roma's (1994) fallacy that S&S work with the absolute value process of volatility $v(t)$, although they are right in pointing out that S&S do not model volatility as a reflected OU process. We will show that S&S use an unrestricted OU process.

The remainder of the paper is organized as follows: Section 1 examines how to construct characteristic functions (CF) in a European-style option pricing formula and then develops a closed-form solution for our extension of S&S. Section 2 presents some properties of our model associated with non-zero correlation, and compares it with S&S as well as with Heston's model. In Section 3 we discuss the boundary behaviour of the volatility process and its implications. Section 4 concludes. Appendix A provides a detailed derivation of the option pricing formula. Appendix B shows that S&S and we use an unrestricted mean-reverting OU process.

1. The Model

We consider that volatility follows a mean-reverting OU process. As assumed by S&S, the stochastic processes of the log of the stock price $x(t) = \ln S(t)$ and its instantaneous volatility $v(t)$ can be described by

$$\mathrm{d}x(t) = (r - \tfrac{1}{2}v^2(t))\,\mathrm{d}t + v(t)\,\mathrm{d}w_s(t), \tag{2}$$

and

$$\mathrm{d}v(t) = \kappa(\theta - v(t))\,\mathrm{d}t + \sigma\,\mathrm{d}w_v(t), \tag{3}$$

respectively. Both processes are interpreted under the risk-neutralized probability measure.[1] Here we suggest that $\mathrm{d}w_s$ and $\mathrm{d}w_v$ are possibly correlated, $\mathrm{d}w_s\,\mathrm{d}w_v = \rho\,\mathrm{d}t$ which extends the S&S model and should generate a leptokurtic distribution of stock returns.[2]

In order to get a closed-form solution using inverse Fourier transformation, we have two methods available. The pure PDE approach used by Heston (1993), who ends up with a system of ODEs, and the more elaborated approach used by Scott

[1] Because the volatility v is not a traded asset, this risk-adjusted martingale measure is not unique but depends on the market price of volatility risk λ which is (implicitly) determined by the market participants. A common way to specify λ is to assume $\lambda\,\mathrm{d}t = \gamma Cov[\mathrm{d}v, \mathrm{d}C/C]$, where γ and C are the relative-risk aversion parameter and consumption, respectively. From the Cox, Ingersoll and Ross (1985a) equilibrium model one can get a consumption process [see also equation (8) in Heston (1993)]: $\mathrm{d}C = \mu_C v(t)^2 C\,\mathrm{d}t + \sigma_C v(t) C\,\mathrm{d}w_C(t)$, where the investor is assumed to have log-utility, i.e. $\gamma = 1$. Consequently, the risk premium is proportional to v, $\lambda(v) = \lambda v$ with λ a constant. In S&S (1991), λ is assumed to equate zero. Both interpretations of λ do not change the form of the risk-neutralized process.

[2] As in the Vasicek (1977) model for interest rates, modelling stochastic volatility as a mean-reverting OU process arises a possibility that volatility could become negative. Since an OU process is Gaussian, the distribution of $v(T)$ conditional on $v(t)$ is normal with mean $a = \theta + (v(t) - \theta)e^{-\kappa(T-t)}$ and variance $b^2 = \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa(T-t)})$. Thus, the probability that $v(T)$ becomes negative is given by $prob = N(-a/b)$. For example, setting $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$, $T - t = 0.3$ and $v(t) = 0.2$ gives $prob = 1.50 \times 10^{-9}$. Setting $\kappa = 1$, $\theta = 0.2$, $\sigma = 0.1$, $T - t = 0.5$ and $v(t) = 0.2$ gives $prob = 1.87 \times 10^{-4}$. Note, that these probabilities are negligibly small for a wide range of reasonable parameter values.

(1997) who applies stochastic calculus to compute the characteristic functions (CF) directly. Given the above processes (2) and (3), we find that the PDE-approach is too cumbersome to get closed-form solutions for the CFs while the second method turns out to be much more elegant and straightforward.[3] It has the advantage that we have no need to guess a suitable form of the CFs. Instead these transformations can be perceived easily in following the calculations.

We start from a fairly general representation of the pricing formula for European-style (call) options with stock price S and strike price K. The value of a call option is the expected terminal value of the option relative to the money market account

$$\begin{aligned} C(S,t,T) &= \mathbb{E}^{Q}\left[e^{-r(T-t)}(S(T)-K)\cdot \mathbf{1}_{\{S(T)>K\}}\right] \\ &= \mathbb{E}^{Q}\left[e^{-r(T-t)}S(T)\cdot \mathbf{1}_{\{x(T)>\ln K\}}\right] - e^{-r(T-t)}K\mathbb{E}^{Q}\left[\mathbf{1}_{\{x(T)>\ln K\}}\right], \end{aligned} \tag{4}$$

where Q denotes the risk-neutralized martingale measure. According to Geman, El Karoui and Rochet (1995), Björk (1997) and others, in order to simplify calculations, we change numeraire. For the first term we choose the stock price S as numeraire and switch from measure Q to Q_1. For the second term we use the T-forward measure to switch from Q to Q_2. The Radon–Nikodym derivatives are then given by

$$\frac{\mathrm{d}Q_1}{\mathrm{d}Q} = g_1(t,T) = \exp\left\{-\int_t^T r(s)\,\mathrm{d}s\right\}\frac{S(T)}{S(t)} \tag{5}$$

and

$$\frac{\mathrm{d}Q_2}{\mathrm{d}Q} = g_2(t,T) = \exp\left\{-\int_t^T r(s)\,\mathrm{d}s\right\}\frac{1}{B(t,T)}, \tag{6}$$

respectively, where Q_1 and Q_2 are again two martingale measures. $B(t,T)$ is the price of a zero-bond maturing at time T. Since the short rate of interest is constant here, we have immediately[4]

$$g_1(t,T) = \frac{e^{-r(T-t)}S(T)}{S(t)} \qquad \text{and} \qquad g_2(t,T) = 1. \tag{7}$$

[3] Heston (1993) wrote in the appendix of his paper: "*Although Stein and Stein (1991) assume the volatility process is uncorrelated with the spot asset, one can generalize this to allow $z_1(t)$ and $z_2(t)$ to have constant correlation. ......*". Following this suggestion, we found that it leads to a rather cumbersome procedure. His method does not lead to a decomposition of the PDE into several ordinary differential equations which can be solved successively.

[4] In fact, we can embody stochastic interest rates into this option pricing model by assuming that stochastic interest rates and stochastic volatility are mutually independent. Interest rates can be specified either as a mean-reverting OU process (Vasicek, 1977) or as a mean-reverting square-root process (Cox, Ingersoll and Ross, 1985b). The derivation of the corresponding CFs follows the same lines as shown in Appendix A.

Under the new measures Q_1 and Q_2,[5] the option price (4) can be restated as

$$\begin{aligned} C(S,t,T) &= S(t)\mathbb{E}^{Q_1}\left[\mathbf{1}_{\{x(T)>\ln K\}}\right] - e^{-r(T-t)}K\mathbb{E}^{Q_2}\left[\mathbf{1}_{\{x(T)>\ln K\}}\right] \\ &= S(t)F^{Q_1}(S(T)>K) - e^{-r(T-t)}KF^{Q_2}(S(T)>K). \end{aligned} \tag{8}$$

A powerful way to get closed-form solutions for the probabilities F^{Q_1} and F^{Q_2} is to derive their corresponding CFs, which are defined by

$$f_j(\phi) \equiv \mathbb{E}^{Q_j}\left[\exp\{i\phi x(T)\}\right] \qquad j=1,2. \tag{9}$$

Using the above two Radon–Nikodym derivatives, we obtain new expressions for the CFs $f_j(\phi)$ under the original martingale measure Q:

$$f_1(\phi) \equiv \mathbb{E}^{Q_1}\left[\exp\{i\phi x(T)\}\right] = \mathbb{E}^{Q}\left[\frac{e^{-r(T-t)}S(T)}{S(t)}\exp\{i\phi x(T)\}\right] \tag{10}$$

$$f_2(\phi) \equiv \mathbb{E}^{Q_2}\left[\exp\{i\phi x(T)\}\right] = \mathbb{E}^{Q}\left[\exp\{i\phi x(T)\}\right]. \tag{11}$$

With these new expressions we start calculating the CFs $f_j(\phi)$ under the martingale measure Q using (2) and (3). As shown in Appendix A, we obtain $f_1(\phi)$ which has the following form[6]

$$\begin{aligned} f_1(\phi) &= \mathbb{E}^{Q}\left[\exp\{-r(T-t) - x(t) + (1+i\phi)x(T)\}\right] \\ &= \exp\left\{i\phi\left(r(T-t)+x(t)\right) - \tfrac{1}{2}(1+i\phi)\rho\left[\sigma^{-1}v^2(t)+\sigma(T-t)\right]\right\} \\ &\quad \times \exp\left\{\tfrac{1}{2}D(t,T;s_1,s_3)v^2(t) + B(t,T;s_1,s_2,s_3)v(t)\right. \\ &\quad \left. + C(t,T;s_1,s_2,s_3)\right\} \end{aligned} \tag{12}$$

with

$$s_1 = -\tfrac{1}{2}(1+i\phi)^2\left(1-\rho^2\right) + \tfrac{1}{2}(1+i\phi)\left(1-2\kappa\rho\sigma^{-1}\right),$$

$$s_2 = (1+i\phi)\kappa\theta\rho\sigma^{-1},$$

5 Our choice of probability measure to calculate the CFs corresponds with Scott (1997). Depending on the two numeraires 'stock price' and 'default-free discount-bond price' the two functions g_1 in (5) and g_2 in (6) define two likelihood processes which are martingales themselves. Therefore we can switch from one measure to the other without violating the no arbitrage condition. For a constant interest rate, g_2 has a value of one. Thus in this case, the measure Q_2 is identical to the original measure Q. See Geman, El Karoui and Rochet (1995) for details.

6 In these calculations, two facts from stochastic calculus are employed. One is the decomposition of a standard Brownian motion. If two standard Brownian motions $\mathrm{d}w_1$ and $\mathrm{d}w_2$ are correlated with $\mathrm{d}w_1\,\mathrm{d}w_2 = \rho\,\mathrm{d}t$, so $\mathrm{d}w_1$ can be expressed as $\mathrm{d}w_1 = \rho\,\mathrm{d}w_2 + \sqrt{1-\rho^2}\,\mathrm{d}w$ where $\mathrm{d}w\,\mathrm{d}w_2 = 0$ and $\mathrm{d}w\,\mathrm{d}w_1 = \sqrt{1-\rho^2}\,\mathrm{d}t$. The second is the so-called Itô isometry which says $var[\int_0^t v(u)\,\mathrm{d}w(u)] = \mathbb{E}[\int_0^t v^2(u)\,\mathrm{d}u]$ for any Itô process $v(t)$.

$$s_3 = \tfrac{1}{2}(1 + i\phi)\rho\sigma^{-1}.$$

The functions $D(t,T)$, $B(t,T)$ and $C(t,T)$ are also derived in Appendix A. Similarly, $f_2(\phi)$ is given by

$$\begin{aligned} f_2(\phi) &= \mathbb{E}^Q\left[\exp\{i\phi x(T)\}\right] \\ &= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right) - \tfrac{1}{2}i\phi\rho\left[\sigma^{-1}v^2(t)+\sigma\left(T-t\right)\right]\right\} \\ &\quad \times \exp\left\{\tfrac{1}{2}D(t,T;\hat{s}_1,\hat{s}_3)v^2(t) + B(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)v(t)\right. \\ &\quad \left. + C(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)\right\} \end{aligned} \tag{13}$$

with

$$\hat{s}_1 = \frac{1}{2}\phi^2\left(1-\rho^2\right) + \frac{1}{2}i\phi\left(1-2\kappa\rho\sigma^{-1}\right),$$
$$\hat{s}_2 = i\phi\kappa\theta\rho\sigma^{-1},$$
$$\hat{s}_3 = \frac{1}{2}i\phi\rho\sigma^{-1}.$$

Given the CFs, the probability distribution functions F_1 and F_2 can be calculated using the Fourier inversion formula:[7]

$$F_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(f_j(\phi)\frac{\exp\{-i\phi\ln K\}}{i\phi}\right)\mathrm{d}\phi, \qquad j = 1, 2. \tag{14}$$

[7] Numerical integration of these real valued probability integrals is not trivial. Heston (1993) as well as Bakshi, Cao and Chen (1997) did not report their numerical procedures in detail. Bates (1996) evaluated the integrals in his formula using Gaussian quadrature software and obtained sufficiently accurate results except for extreme and implausible jump parameters. Scott (1997) compared the Fourier inversion method with Monte Carlo simulations and finite difference methods and reported that the Fourier inversion method is superior with respect to accuracy and computing time. Our experience with the numerical implementation of the Fourier inversion integral (14) confirms these results. Two remarks might be useful information:

First, depending on the frequency of the integrand in (14) we integrate over successive equally spaced intervals over the domain $[0, \phi_{\max}]$. $\phi_{\max}$ is not fixed but triggered by a local stopping criterion: If the contribution of the last strip becomes smaller than let's say 10^{-15} the integration is stopped. Given the decaying oscillatory behaviour of the integrand this procedure proved to be quite robust. We use a standard Gaussian quadrature routine with $n = 96$ weights and abscissas for each interval taken from Abramowitz and Stegun (1965), p. 919.

Second, we have to take care of the *complex logarithm* arising in our formula, which has to be handled carefully, because the complex logarithm is a *multivalued* function. If one uses the *principal value* of the complex logarithm only (similar to many commercial software packages like GAUSS, Mathematica, Mathcad and others) this results in a wrong integration of the CF. The uncontrolled usage of the principal value possibly leads to discontinuities in the value of the complex logarithm along the integration path. Therefore we implemented our formula carefully keeping track of the complex logarithm along the integration path. This leads to a smooth CF and to the correct values of the probabilities.

Note, that this numerical peculiarity is not specific to our model. The Heston model also has a complex logarithm in its formula for the CF. Not surprising, we would also get jumps in the CF along the integration path for arbitrary extreme parameter values in the CF of the Heston model. The same is true for other variants in the literature.

Finally, the call option pricing formula is given by

$$C(S, v, t, T) = S(t)F_1 - e^{-r(T-t)}KF_2. \tag{15}$$

The formula for European puts can easily be obtained by using put-call parity. Formula (15) is more general than the solution by S&S (1991) since here the volatilities are correlated with the stock prices. Furthermore, our closed-form solution has a clear structure and gives the two probabilities F_1 and F_2 explicitly. Hence, the hedge ratio Δ and other Greeks can be given analytically. This is certainly a positive feature for the application of the model. Some popular hedge ratios are:

$$\Delta_S(S, v, t, T) = \frac{\partial C}{\partial S} = F_1, \tag{16}$$

$$\Delta_v(S, v, t, T) = \frac{\partial C}{\partial v} = S(t)\frac{\partial F_1}{\partial v} - e^{-r(T-t)}K\frac{\partial F_2}{\partial v}, \tag{17}$$

$$\Gamma_S(S, v, t, T) = \frac{\partial^2 C}{\partial S^2} = \frac{\partial F_1}{\partial S}, \tag{18}$$

$$\Gamma_v(S, v, t, T) = \frac{\partial^2 C}{\partial v^2} = S(t)\frac{\partial^2 F_1}{\partial v^2} - e^{-r(T-t)}K\frac{\partial^2 F_2}{\partial v^2}, \tag{19}$$

where for $h = S, v$ and $j = 1, 2$

$$\frac{\partial F_j}{\partial h} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\partial f_j(\phi)}{\partial h}\frac{\exp(-i\phi\ln K)}{i\phi}\right)d\phi,$$

and

$$\frac{\partial^2 F_j}{\partial h^2} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\partial^2 f_j(\phi)}{\partial h^2}\frac{\exp(-i\phi\ln K)}{i\phi}\right)d\phi.$$

Our results on hedge ratios are similar to those in Bakshi, Cao and Chen (1997). While Heston (1993), Bates (1996) and Bakshi, Cao and Chen (1997) got $f_j(\phi)$ via two-dimensional partial differential equations, as shown in Appendix A, the key technique here is that we separate the volatility shocks in two components by decomposition of the Brownian motion $w_v(t)$: the shocks orthogonal to stock prices, and the shocks perfectly correlated with stock prices. Thanks to this manipulation, we only need to solve a one-dimensional PDE using the Feynman–Kač theorem after reducing the explicit form of $f_j(\phi)$ to a one-dimensional problem.

2. Properties and Calibration

In this section, we compare our model with the S&S model and with Heston's solution and illustrate some properties of our model by calibration. It is relatively difficult to exhibit the inherent connection between Heston's model and our model since he models variance instead of volatility. However, as assumed in his model, volatility follows an OU process with a mean-reversion level equal to zero (see equation (2) in Heston, 1993), that is

$$\mathrm{d}v(t) = -\beta v(t)\,\mathrm{d}t + \delta\,\mathrm{d}w. \tag{20}$$

Then, from Itô's Lemma, the squared volatility $y(t) = v^2(t)$, namely the variance of instantaneous stock returns, follows a square-root process

$$\mathrm{d}y(t) = \kappa_h(\theta_h - y(t))\,\mathrm{d}t + \sigma_h\sqrt{y(t)}\,\mathrm{d}w, \tag{21}$$

with

$$\beta = \frac{\kappa_h}{2}, \quad \delta = \frac{\sigma_h}{2}, \quad \theta_h = \frac{\delta^2}{\kappa_h}. \tag{22}$$

Hence, the only difference between equation (3) and equation (20) is the mean-reversion parameter θ. While θ in (3) generally differs from zero, in (20) it is always nil. Since θ gives the level of volatility in the long run, the process (20) seems not very reasonable. But in fact, Heston's solution is based on the process (21) not (20). Note that the parameters in (21) are overdetermined by (22). Hence, for a wide range of values for κ_h, σ_h and θ_h, the volatility process (21) can not be derived from (20). Therefore the two processes (20) and (21) are not mutually consistent for many parameter values. For the special case $\theta = 0$ our model is reduced to Heston's model by setting the following parameters:

$$\kappa = \frac{\kappa_h}{2}, \quad \sigma = \frac{\sigma_h}{2}, \quad \theta = 0, \quad \theta_h = \frac{\sigma^2}{\kappa_h}. \tag{23}$$

A favorite property of our model is that both volatility and the squared volatility perform mean-reversion, whereas in Heston's model, only the squared volatility exhibits mean-reversion.[8]

[8] Applying Itô's Lemma once again, we obtain the process for $v(t) = \sqrt{y(t)}$:

$$\mathrm{d}v(t) = [\tfrac{1}{2}(\kappa_h\theta_h - \tfrac{1}{4}\sigma_h^2)v^{-1}(t) - \tfrac{1}{2}\kappa_h v(t)]\,\mathrm{d}t + \tfrac{1}{2}\sigma_h\,\mathrm{d}w(t).$$

Obviously, if (22) is not satisfied, $\kappa_h\theta_h - \frac{1}{4}\sigma_h^2$ will not be zero. Hence the term $v^{-1}(t)$ will appear in the volatility process. As a consequence, the specification of $y(t) = v^2(t)$ such as (21) should also be examined carefully. If the volatility follows a mean-reverting OU process as in (3), the process for the squared volatility is

$$\mathrm{d}y(t) = [\sigma^2 + 2\kappa\theta\sqrt{y(t)} - 2\kappa y(t)]\,\mathrm{d}t + 2\sigma\sqrt{y(t)}\,\mathrm{d}w_v(t).$$

This is a mean-reverting double square-root process with the additional drift term $2\kappa\theta\sqrt{y(t)}$. Using a double square-root process, Longstaff (1989) developed a model for a non-linear term structure of interest rates.

Here we calibrate our model based on formula (15). For comparison, we shall choose suitable Black-Scholes (BS) option prices as benchmarks. The first possible benchmark is the BS price using the expected average variance AV in equation (1) as input. As shown in Ball and Roma (1994), if $v(t)$ follows a mean-reverting OU process, we have

$$\begin{aligned} AV &= \frac{\sigma^2}{2\kappa} + \theta^2 + \frac{1}{(T-t)}(1 - e^{-\kappa(T-t)}) \\ &\quad \times \left(\frac{2\theta}{\kappa}(v(t) - \theta) - \frac{\sigma^2 - 2\kappa(v(t) - \theta)^2}{4\kappa^2}(1 + e^{-\kappa(T-t)}) \right). \end{aligned} \tag{24}$$

We denote the BS prices evaluated in this manner by $AVBS$. A second possibility is to calculate the BS option prices according to the target volatility θ, as in S&S. The calculated BS prices called $SSBS$ are identical to these prices by using the BS formula with a constant volatility v being equal to θ.[9]

In Table I, we choose parameter values suggested by S&S. In order to show the impact of correlation on the option prices, we let ρ range from -1.0 to 1.0. Some observations are in order. First, options with different moneyness have different sensitivity to the correlation ρ. Overall the values of at-the-money (ATM) options do not change remarkably. However, the sensitivity of out-of-the-money (OTM) options to ρ is more conspicuous than that of in-the-money (ITM) options. For example, in Panel C the relative changes of the OTM option prices due to the correlation ρ for $K = 120$ is about $\pm 14\%$ of the S&S value which is here 2.635.

Second, a comparison of Panels A, B and C shows that the mean-reversion level θ is important for the pricing of options. Keeping other parameters unchanged, the differential $(\theta - v)$ (mean-reversion level minus current volatility) has a great impact on option values. From Panel A to C, $(\theta - v)$ is 0, -0.1 and 0.1 respectively, and the differences in option prices across these panels are mostly between 0.60$ and 1.50$. Since the expectation of the future spot price volatility approaches θ, the prices of options, especially options with a long-term maturity, should be mainly affected by θ.

Thirdly, as expected, we find the price differences between $AVBS$ and the model values with $\rho = 0$ are smallest for all panels. This confirms Ball and Roma's finding. The good fit of these two values is not surprising since $AVBS$ is by its very nature an approximation for the exact option value with $\rho = 0$. Panel D in Table 2 presents all $AVBS$ values corresponding to Panels E-G. We can see that the $AVBS$ values in Panel D agree very good with the option values in Panel F. However, if $\rho \neq 0$, the price biases between $AVBS$ values and the

[9] Heston (1993) uses BS prices with a volatility that equates the square root of the variance of the spot stock returns up to maturity. His benchmark then depends on the correlation ρ and makes comparisons remarkably complicated. Since Heston's BS benchmark is a function of ρ, every model option value has a corresponding Heston's benchmark. Thus, we would lose some clarity in comparison using Heston's benchmark although it is reasonable in some sense.

exact option values are significant. Thus, $AVBS$ is not a suitable approximation for the non-zero correlation case. It is not surprising either that the $SSBS$ values match our model values most closely for Panel A where $\theta = v$. S&S report an overall overvaluation relative to $SSBS$ due to stochastic volatility. This upward pricing bias should be caused by the zero correlation between volatilities and the underlying asset returns in S&S. For $\rho = 0$ the direction of the movement of $S(t)$ is not affected by stochastic volatility, any stochastic volatility raises additional uncertainty of $S(t)$, only. Consequently, the exact option prices (the italic numbers) and $AVBS$ values are greater than $SSBS$ values in Panel A.

Finally, ITM options and OTM options react to correlation just oppositely. Whereas ITM options ($K = 90, 95$) decrease in value with increasing correlation ρ, OTM option prices ($K = 105, 110, 115, 120$) go up. This finding is consistent with Hull's (1997, pp. 492–495) excellent intuitive explanation of how correlation affects option prices. It is also empirically evident that stock returns are inversely correlated with the underlying volatilities. Panels A to C show that a negative correlation ρ leads to ITM (OTM) option prices in our model that are greater (less) than the corresponding BS option prices. This feature caused by negative correlation is useful for explaining volatility 'sneers'. The 'smile' pattern of implied volatility is in most cases not so obvious and appears to be more of a 'sneer' pattern.[10] The monotonic downward sloping of the implied volatility with moneyness displayed in a 'sneer' implies that the market ITM (OTM) options are undervalued (overvalued) by the BS formula. This pricing bias in the BS formula is reported by a number of recent empirical studies as in Nandi (1998), Heston and Nandi (1997), Bakshi, Cao and Chen (1997). Figure 1 shows implied volatilities for Panel A where the downward sloping of implied volatilities for $\rho < 0$ is significant. Bakshi, Cao and Chen (1997) and Nandi (1998) reported that stochastic volatility is of first-order importance in eliminating the volatility inconsistence and leads to significant improvements in hedging performance, but only in presence of non-zero correlation. Their results should be also valid for our model.

In Table II, we examine how the option prices vary with the mean-reversion level θ. The finding that option prices are very sensible to θ is confirmed. Since θ indicates the long-run level of volatility, this sensitivity can be considered as the sensitivity of option prices to their volatilities in the long run. Furthermore, it seems to be that θ is more important than the spot volatility $v(t)$ for the pricing of options in the framework of a mean-reverting process. If the true process of volatility performs mean-reversion, and option prices are evaluated using the BS formula regardless of $AVBS$ or $SSBS$, a significant pricing bias will occur although the magnitude of the pricing bias for $AVBS$ is smaller than for $SSBS$. All prices in Panel F correspond to the case $\rho = 0$ of S&S. The numbers in italics in Panels D, E, F and G are option prices under the restricted Heston model in the sense

[10] Sneers sometimes are referred to as smirks. The symmetric implied volatility 'smile' could be explained by zero-correlation as discussed in Ball and Roma (1994), Hull and White (1987), and Scott (1987). The smile arises also in Figure 1 for $\rho = 0$.

Table I. The impact of ρ on option prices

ρ \ K	90	95	100	105	110	115	120
$SSBS$	15.118	11.342	8.142	5.584	3.658	2.293	1.377
$AVBS$	15.152	11.391	8.203	5.650	3.722	2.349	1.422
−1.00	15.416	11.617	8.307	5.576	3.468	1.966	0.995
−0.75	15.355	11.562	8.275	5.585	3.525	2.063	1.110
−0.50	15.292	11.503	8.243	5.595	3.582	2.156	1.218
−0.25	15.225	11.443	8.210	5.606	3.638	2.246	1.321
0.00	*15.155*	*11.379*	*8.176*	*5.617*	*3.694*	*2.333*	*1.420*
0.25	15.081	11.313	8.141	5.628	3.749	2.416	1.514
0.50	15.003	11.243	8.106	5.640	3.803	2.497	1.605
0.75	14.919	11.169	8.070	5.652	3.856	2.576	1.693
1.00	14.828	11.092	8.034	5.665	3.909	2.653	1.777

A: $\theta = 0.2$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.2$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

ρ \ K	90	95	100	105	110	115	120
$AVBS$	14.501	10.348	6.829	4.132	2.282	1.150	0.530
−1.00	14.730	10.600	6.976	4.057	1.977	0.739	0.179
−0.75	14.679	10.541	6.936	4.066	2.051	0.853	0.279
−0.50	14.626	10.480	6.894	4.075	2.121	0.957	0.371
−0.25	14.572	10.415	6.849	4.085	2.189	1.053	0.458
0.00	*14.515*	*10.346*	*6.803*	*4.093*	*2.254*	*1.144*	*0.542*
0.25	14.456	10.271	6.753	4.102	2.316	1.230	0.621
0.50	14.395	10.191	6.701	4.110	2.376	1.311	0.698
0.75	14.330	10.103	6.645	4.117	2.433	1.389	0.773
1.00	14.261	10.005	6.587	4.125	2.489	1.463	0.845

B: $\theta = 0.1$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.2$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

ρ \ K	90	95	100	105	110	115	120
$AVBS$	16.111	12.666	9.711	7.264	5.305	3.786	2.646
−1.00	16.357	12.846	9.777	7.186	5.084	3.449	2.235
−0.75	16.298	12.800	9.755	7.198	5.133	3.531	2.340
−0.50	16.236	12.751	9.732	7.210	5.182	3.611	2.441
−0.25	16.172	12.702	9.710	7.223	5.230	3.690	2.540
0.00	*16.106*	*12.651*	*9.687*	*7.237*	*5.279*	*3.768*	*2.635*
0.25	16.037	12.598	9.665	7.251	5.328	3.844	2.728
0.50	15.964	12.544	9.642	7.265	5.377	3.919	2.819
0.75	15.889	12.489	9.620	7.280	5.426	3.993	2.908
1.00	15.810	12.432	9.598	7.296	5.475	4.065	2.994

C: $\theta = 0.3$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.2$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

The italic numbers correspond to the S&S model.

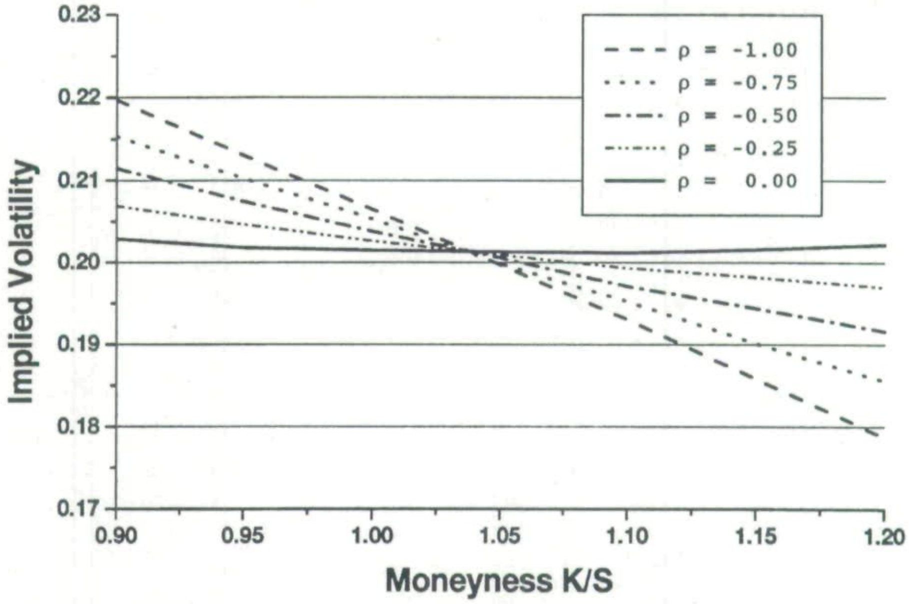

Figure 1. Volatility smile and sneer pattern for zero and negative correlation. Calibrated from Panel A in Table 1.

of equations (20) and (21). The implied zero level of mean-reversion leads to an overall undervaluation of options compared with the $SSBS$ on top of Table II.

Table III demonstrates the impact of ρ on option deltas Δ_S which is of first-order importance for hedging purposes whenever stochastic volatility models are used. First, for the given parameters almost all of the deltas are decreasing with correlation ρ except a few deep-ITM and deep-OTM options across Panels H, I and J. Second, the changes of values of near-ATM options relative to the correlation ρ are more sensitive than these of deep-ITM and deep-OTM options. The differences between Δ_S in our model and the BS model ($AVBS$ and $SSBS$) for near-ATM options should not be neglected. For a negative correlation, using Δ_S of the BS formula or the S&S model seems to cause a severe underhedging for near-ATM options and some OTM options. Furthermore, the long-term level of volatility θ is important for hedging. Keeping other parameters unchanged, the greater θ is, the smaller (greater) Δ_S will be for ITM (OTM) options. The sensitivity of Δ_S to θ is remarkable and can be studied more detailed by the second derivative $\Delta_{S\theta}$. We conclude that an unbiased estimate of θ is crucial for delta-hedging.

3. Is there a Boundary at $v(t) = 0$?

S&S claimed that assuming volatility to be driven by an OU process is "*equivalent to putting a reflecting barrier at $v = 0$ in the volatility process, since v enters everywhere else in squared fashion*" (Stein and Stein, 1991, p. 729). Ball and Roma

Table II. The impact of θ on option prices

θ \ K	90	95	100	105	110	115	120
$SSBS$	14.515	10.374	6.867	4.175	2.322	1.180	0.550
0.0	*14.194*	*9.510*	*5.277*	*2.213*	*0.654*	*0.132*	*0.018*
0.1	14.333	9.990	6.282	3.499	1.708	0.728	0.272
0.2	14.864	10.962	7.662	5.060	3.155	1.859	1.037
0.3	15.757	12.213	9.187	6.707	4.755	3.278	2.200

D: Approximate solution for $\rho = 0$ suggested by Ball and Roma (1994). The $AVBS$ values are calculated using the expected average volatility from (24) as input. Other parameter values are: $\kappa = 4$, $\sigma = 0.1$, $v = 0.15$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

θ \ K	90	95	100	105	110	115	120
0.0	*14.189*	*9.459*	*5.141*	*2.173*	*0.763*	*0.244*	*0.075*
0.1	14.262	9.842	6.132	3.466	1.812	0.895	0.425
0.2	14.723	10.804	7.551	5.044	3.239	2.014	1.220
0.3	15.608	12.082	9.109	6.703	4.828	3.414	2.376

E: $\rho = 0.5$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.15$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

θ \ K	90	95	100	105	110	115	120
0.0	*14.200*	*9.527*	*5.269*	*2.170*	*0.645*	*0.150*	*0.030*
0.1	14.351	9.997	6.254	3.451	1.679	0.731	0.292
0.2	14.871	10.952	7.632	5.022	3.124	1.845	1.040
0.3	15.754	12.198	9.161	6.676	4.727	3.259	2.192

F: $\rho = 0.0$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.15$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

θ \ K	90	95	100	105	110	115	120
0.0	*14.225*	*9.600*	*5.372*	*2.152*	*0.504*	*0.061*	*0.004*
0.1	14.441	10.130	6.361	3.435	1.529	0.545	0.155
0.2	15.003	11.084	7.710	5.003	3.004	1.660	0.842
0.3	15.887	12.306	9.213	6.652	4.626	3.097	1.995

G: $\rho = -0.5$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.15$, $T - t = 0.5$, $S = 100$, $r = 0.0953$. The italic numbers correspond to the restricted Heston model.

Table III. The impact of ρ on option deltas Δ_S

ρ \ K	90	95	100	105	110	115	120
$SSBS$	0.8755	0.7795	0.6582	0.5249	0.3950	0.2807	0.1890
$AVBS$	0.8731	0.7773	0.6571	0.5253	0.3968	0.2836	0.1922
−1.00	0.8751	0.7949	0.6895	0.5644	0.4300	0.3002	0.1883
−0.75	0.8751	0.7916	0.6825	0.5547	0.4204	0.2941	0.1881
−0.50	0.8751	0.7881	0.6751	0.5449	0.4113	0.2889	0.1883
−0.25	0.8752	0.7844	0.6673	0.5350	0.4027	0.2843	0.1887
0.00	*0.8754*	*0.7802*	*0.6591*	*0.5251*	*0.3945*	*0.2802*	*0.1891*
0.25	0.8756	0.7757	0.6504	0.5153	0.3868	0.2765	0.1896
0.50	0.8759	0.7707	0.6413	0.5055	0.3795	0.2732	0.1900
0.75	0.8761	0.7649	0.6317	0.4957	0.3725	0.2701	0.1904
1.00	0.8762	0.7584	0.6216	0.4862	0.3660	0.2673	0.1907

H: $\theta = 0.2$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.2$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

ρ \ K	90	95	100	105	110	115	120
$AVBS$	0.9344	0.8401	0.6937	0.5166	0.3440	0.2047	0.1093
−1.00	0.9246	0.8490	0.7302	0.5684	0.3816	0.2048	0.0763
−0.75	0.9267	0.8479	0.7229	0.5554	0.3694	0.2025	0.0867
−0.50	0.9293	0.8467	0.7151	0.5424	0.3584	0.2012	0.0947
−0.25	0.9323	0.8456	0.7068	0.5293	0.3485	0.2004	0.1012
0.00	*0.9359*	*0.8445*	*0.6978*	*0.5163*	*0.3395*	*0.1996*	*0.1066*
0.25	0.9403	0.8433	0.6881	0.5033	0.3311	0.1989	0.1111
0.50	0.9458	0.8421	0.6774	0.4903	0.3234	0.1982	0.1149
0.75	0.9530	0.8404	0.6656	0.4773	0.3163	0.1975	0.1182
1.00	0.9629	0.8379	0.6526	0.4645	0.3095	0.1968	0.1211

I: $\theta = 0.1$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.2$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

ρ \ K	90	95	100	105	110	115	120
$AVBS$	0.8225	0.7358	0.6373	0.5341	0.4334	0.3410	0.2606
−1.00	0.8317	0.7552	0.6646	0.5644	0.4606	0.3599	0.2682
−0.75	0.8300	0.7512	0.6584	0.5568	0.4532	0.3541	0.2652
−0.50	0.8282	0.7469	0.6519	0.5492	0.4459	0.3487	0.2626
−0.25	0.8264	0.7424	0.6452	0.5416	0.4389	0.3437	0.2604
0.00	*0.8243*	*0.7376*	*0.6383*	*0.5340*	*0.4322*	*0.3390*	*0.2584*
0.25	0.8221	0.7324	0.6311	0.5263	0.4256	0.3347	0.2566
0.50	0.8196	0.7269	0.6237	0.5187	0.4193	0.3306	0.2550
0.75	0.8169	0.7210	0.6162	0.5112	0.4133	0.3267	0.2536
1.00	0.8137	0.7147	0.6084	0.5038	0.4074	0.3231	0.2522

J: $\theta = 0.3$, $\kappa = 4$, $\sigma = 0.1$, $v = 0.2$, $T - t = 0.5$, $S = 100$, $r = 0.0953$.

The italic numbers correspond to the S&S model.

(1994) argued that S&S did not model volatility as a reflected but as the absolute value of a mean-reverting process, since "'*the volatility*, v, *enters only as* v^2, *so it may be taken to be the positive root* $|v|$" (Ball and Roma, 1994, p. 604), and discussed the difference of an absolute value process of $v(t)$ and a reflected process of $v(t)$. Unfortunately, their claim is not correct although they are right in pointing out that S&S did not work with a reflected process of $v(t)$. Both S&S's solution and our model utilize the backward equation of a normal OU process without imposing any boundary condition at $v = 0$ (see (B3) in S&S (1991) and (A1) in this paper).

Volatility enters the option pricing formulae both in the S&S model and in our model *explicitly* not only as $v^2(t)$, but also as $v(t)$. This is obvious in equation (B6) for any $M_1 \neq 0$ in S&S (1991). In our model, setting $\rho = 0$ does not let the function $B(t, T)$ in (12) and (13) vanish. However, if we set $\theta = 0$ the function $B(t, T)$ vanishes. In this special case volatility enters the option pricing formula (15) only via $v^2(t)$, and we do get the same option prices for $v(t)$ as well as for $-v(t)$. Figure 2 illustrates this symmetry. But for any other target value $\theta \neq 0$ this is no longer true. We obtain different option prices for $v(t)$ and $-v(t)$ applying both the original S&S formula and our extended pricing formula. This asymmetry holds regardless of zero or non-zero correlation for $\theta \neq 0$ as shown in Figure 3. Only in the symmetric case, option prices can be treated as if they are reflected at $v(t) = 0$ and have the same values for $v(t)$ and $-v(t)$, no matter which model we are using: the S&S solution, Ball and Roma's approximative formula or our solution. This indicates that the reflected process and the absolute value process of $v(t)$ might be identical in the case of $\theta = 0$.

Nevertheless, an unrestricted OU process is different from these two special processes independent of the value of θ. There are at least two reasons for this difference. First, the support of an unrestricted OU process is the whole real axis while the two restricted processes are defined on the non-negative real axis only. Since all solutions including S&S as well as Ball and Roma (1994) admit a negative $v(t)$, S&S's solution can not be derived from any one of these two special processes. Second, in contrast to these two restricted processes, the unrestricted OU process used in S&S and in our model has no boundary condition at $v(t) = 0$.

In Appendix B we investigate the boundary behavior of an OU process more rigorously. We show that (i) for arbitrary values of the long-term mean $\theta \neq 0$, the absolute value OU process, the reflected OU process and the unrestricted OU process are substantially different and the absolute value OU process does not satisfy a unique Kolmogorov forward equation. This result applies for $\rho = 0$ (the S&S case) as well as for $\rho \neq 0$ (our case). However, (ii) only for the special (symmetric) case $\theta = 0$ the absolute value process coincides with the reflecting process at the origin since their boundary conditions are identical in this case. In both cases Ball and Roma's claim is disproved; (iii) the density function derived by applying the same backward equation as in S&S and in our model is identical to the one for the unrestricted OU process and does not satisfy the boundary condition which is necessary for an absolute value process. As a consequence, we find that

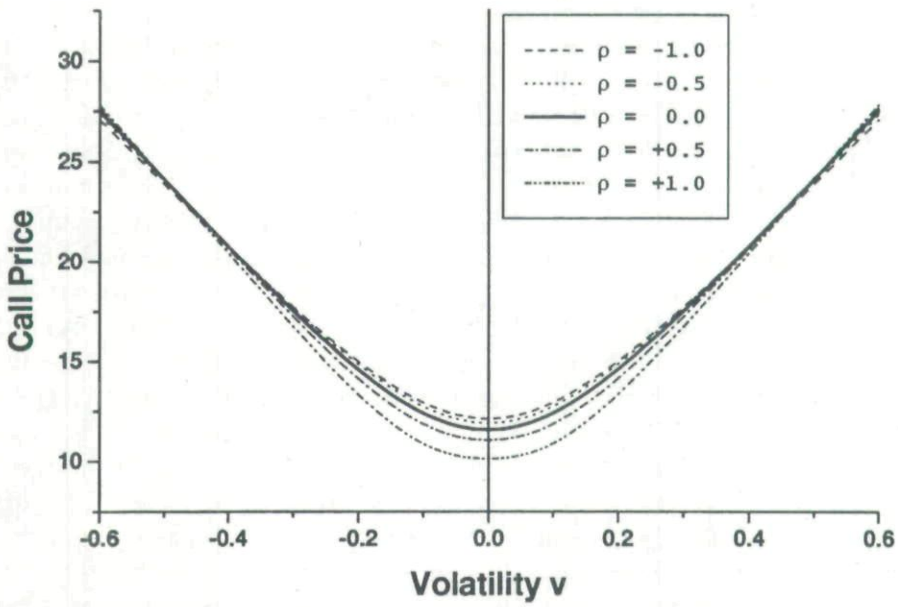

Figure 2. Call prices as a function of instantaneous volatility v. Symmetric case with $\theta = 0$. Other parameter values are: $\kappa = 0.5$, $\sigma = 0.1$, $T - t = 3$, $S = 100$, $K = 120$, $r = 0.0953$.

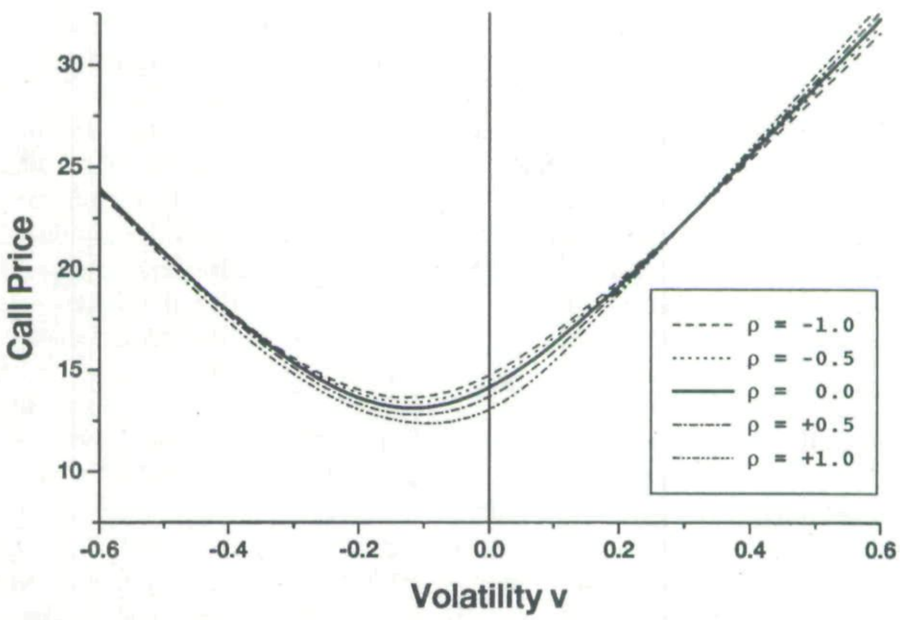

Figure 3. Call prices as a function of instantaneous volatility v: Asymmetric case with $\theta \neq 0$. Parameter values are: $\theta = 0.2$, $\kappa = 0.5$, $\sigma = 0.1$, $T - t = 3$, $S = 100$, $K = 120$, $r = 0.0953$.

not only $v^2(t)$ but also $v(t)$ play a crucial role in a mean-reverting OU volatility model. Hence, S&S did in fact solve the unrestricted case without dealing with a reflected or an absolute value process, and so do we in our extension allowing for non-zero correlation.

4. Conclusions

Stochastic volatility option pricing models provide us with new insights into derivative security markets. Stochastic volatilities have been specified at least by two classes of stochastic processes. The first specification is the mean-reverting square-root process in the line with the famous interest rate process of Cox, Ingersoll and Ross (1985b). The closed-form pricing formula for options with squared volatility (variance) following such a process is given by Heston (1993). The advantage of the square-root process might be obvious: Squared volatilities never become negative.

The second specification is a mean-reverting OU process. In this paper, we derived a closed-form pricing formula for the general case where volatility is allowed to display arbitrary correlation with the underlying stock price. This solution includes the well-known Stein and Stein (1991) model as a special case. Calibrating this model illustrates that it is very promising to eliminate 'smiles' and 'sneers' of implied volatility in the Black and Scholes (1973) model by incorporating the correlation between stock returns and volatilities.

In addition, we discuss the boundary behaviour of the instantaneous volatility at $v(t) = 0$ and show that Stein and Stein (1991) work with an unrestricted mean-reverting OU process as we do here. Consequently, there is a positive probability for volatilities to become negative in the S&S model as well as in our model. Since in a diffusion context a negative sign of the diffusion coefficient only means that upward moves of the driving Brownian motion become downward moves of the stock price and vice versa, we believe that this is not a severe theoretical restriction and suggest this new closed-form pricing formula as an alternative to Heston's solution: Not surprisingly, squared volatilities never become negative in our model either.

Certainly it is interesting to study the empirical evidence of our model and compare its performance with the Heston model and its generalizations. This is left for future research.

Appendix A: Derivation of Characteristic Functions

$f_1(\phi)$ can be calculated as follows (The expansion of $f_2(\phi)$ follows the same way.):

$$
\begin{aligned}
f_1(\phi) &= \mathbb{E}^Q[\exp\{-r(T-t) - x(t) + (1+i\phi)x(T)\}] \\
&= \mathbb{E}^Q\Bigg[\exp\Bigg\{-r(T-t) - x(t) + (1+i\phi) \\
&\quad \times \left(x(t) + \int_t^T r\,\mathrm{d}u - \frac{1}{2}\int_t^T v^2(u)\,\mathrm{d}u + \int_t^T v(u)\,\mathrm{d}w_s(u)\right)\Bigg\}\Bigg] \\
&= \exp\{i\phi(r(T-t) + x(t))\} \\
&\quad \times \mathbb{E}^Q\left[\exp\left\{(1+i\phi)\left(-\frac{1}{2}\int_t^T v^2(u)\,\mathrm{d}u + \int_t^T v(u)\,\mathrm{d}w_s(u)\right)\right\}\right] \\
&= \exp\{i\phi\,(r\,(T-t) + x(t))\} \\
&\quad \times \mathbb{E}^Q\Bigg[\exp\Bigg\{(1+i\phi)\Bigg(-\frac{1}{2}\int_t^T v^2(u)\,\mathrm{d}u + \rho\int_t^T v(u)\,\mathrm{d}w_v(u) \\
&\quad + \sqrt{1-\rho^2}\int_t^T v(u)\,\mathrm{d}w(u)\Bigg)\Bigg\}\Bigg].
\end{aligned}
$$

Note that $\mathrm{d}w$ is uncorrelated with $\mathrm{d}w_v$:

$$
\begin{aligned}
f_1(\phi) &= \exp\{i\phi\,(r\,(T-t) + x(t))\} \\
&\quad \times \mathbb{E}^Q\Bigg[\exp\Bigg\{(1+i\phi)\Bigg(-\frac{1}{2}\int_t^T v^2(u)\,\mathrm{d}u + \rho\int_t^T v(u)\,\mathrm{d}w_v(u)\Bigg) \\
&\quad + \frac{1}{2}(1+i\phi)^2(1-\rho^2)\int_t^T v^2(u)\,\mathrm{d}u\Bigg\}\Bigg] \\
&= \exp\{i\phi\,(r\,(T-t) + x(t))\} \\
&\quad \times \mathbb{E}^Q\Bigg[\exp\Bigg\{\frac{1}{2}(1+i\phi)\big((1+i\phi)(1-\rho^2) - 1\big)\int_t^T v^2(u)\,\mathrm{d}u \\
&\quad + (1+i\phi)\rho\int_t^T v(u)\,\mathrm{d}w_v(u)\Bigg\}\Bigg] \\
&= \exp\{i\phi\,(r\,(T-t) + x(t))\} \\
&\quad \mathbb{E}^Q\Bigg[\exp\Bigg\{\frac{1}{2}(1+i\phi)\big((1+i\phi)(1-\rho^2) - 1\big)\int_t^T v^2(u)\,\mathrm{d}u \\
&\quad + (1+i\phi)\frac{\rho}{2\sigma}\Bigg(v^2(T) - v^2(t) - \sigma^2(T-t)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\kappa\theta \int_t^T v(u)\,\mathrm{d}u + 2\kappa \int_t^T v^2(u)\,\mathrm{d}u\Big)\Big\}\Big]\Big] \\
&= \exp\{i\phi\,(r\,(T-t) + x(t))\} \\
&\quad \times \mathbb{E}^Q\Big[\exp\Big\{\frac{1}{2}(1+i\phi)\Big((1+i\phi)(1-\rho^2) - 1 + \frac{2\kappa\rho}{\sigma}\Big)\int_t^T v^2(u)\,\mathrm{d}u \\
&\quad - (1+i\phi)\frac{\rho}{2\sigma}\big(v^2(t) + \sigma^2(T-t)\big) + (1+i\phi)\frac{\rho}{2\sigma}v^2(T) \\
&\quad - (1+i\phi)\frac{\kappa\theta\rho}{\sigma}\int_t^T v(u)\,\mathrm{d}u\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
f_1(\phi) &= \exp\Big\{i\phi\,[x(t) + r\,(T-t)] - (1+i\phi)\,\frac{\rho}{2\sigma}\big(v^2(t) + \sigma^2(T-t)\big)\Big\} \\
&\quad \times \mathbb{E}^Q\Big[\exp\Big\{-s_1\int_t^T v^2(u)\,\mathrm{d}u - s_2\int_t^T v(u)\,\mathrm{d}u + s_3 v^2(T)\Big\}\Big].
\end{aligned}
$$

Now we have to calculate the expectation

$$
\begin{aligned}
y(v,t,T) &= \mathbb{E}^Q\Big[\exp\Big\{-s_1\int_t^T v^2(u)\,\mathrm{d}u - s_2\int_t^T v(u)\,\mathrm{d}u + s_3 v^2(T)\Big\}\Big] \\
&= \mathbb{E}^Q\Big[\exp\Big(\int_t^T (-s_1 v^2(u) - s_2 v(u))\,\mathrm{d}u\Big)\exp(s_3 v^2(T))\Big]
\end{aligned}
$$

for arbitrary complex numbers s_1, s_2 and s_3 and $s_1 v^2(u) + s_2 v(u)$ is lower bounded. According to the Feynman-Kač formula, y satisfies the following (backward) differential equation (see Karlin and Taylor, 1981, or Øksendal, 1995)

$$
\frac{1}{2}\sigma^2\frac{\partial^2 y}{\partial v^2} + \kappa(\theta - v)\frac{\partial y}{\partial v} - (s_1 v^2 + s_2 v)y + \frac{\partial y}{\partial t} = 0 \tag{A1}
$$

with terminal condition

$$
y(v,T,T) = \exp(s_3 v^2).
$$

It can be shown that the above differential equation has a solution of the form

$$
\begin{aligned}
y(v,t,T) &= \exp\big\{\tfrac{1}{2}A(t,T)v^2(t) + B(t,T)v(t) + C(t,T) + s_3 v^2(t)\big\} \\
&= \exp\big\{\tfrac{1}{2}(A(t,T) + 2s_3)v^2(t) + B(t,T)v(t) + C(t,T)\big\} \\
&= \exp\big\{\tfrac{1}{2}(D(t,T)v^2(t) + B(t,T)v(t) + C(t,T)\big\}
\end{aligned}
$$

with $D(t,T) = A(t,T) + 2s_3$. Substituting this into the differential equation, we obtain a system of three ordinary differential equations that determine $D(t,T)$, $B(t,T)$ and $C(t,T)$.

$$D_t = -\sigma^2 D^2 + 2\kappa D + 2s_1$$

$$B_t = \left(\kappa - \sigma^2 D\right) B - \kappa\theta D + s_2$$

$$C_t = -\tfrac{1}{2}\sigma^2 B^2 - \kappa\theta B - \tfrac{1}{2}\sigma^2 D$$

where $D(T,T) = 2s_3$ and $B(T,T) = C(T,T) = 0$. Solving these equations is straightforward but tedious. We get

$$D(t,T) = \frac{1}{\sigma^2}\left(\kappa - \gamma_1 \frac{\sinh\{\gamma_1(T-t)\} + \gamma_2\cosh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}}\right)$$

$$B(t,T)$$

$$= \frac{1}{\sigma^2\gamma_1}\left(\frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3) + \gamma_3\left(\sinh\{\gamma_1(T-t)\} + \gamma_2\cosh\{\gamma_1(T-t)\}\right)}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}} - \kappa\theta\gamma_1\right)$$

$$\begin{aligned} C(t,T) = {} & -\tfrac{1}{2}\ln\left(\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}\right) + \tfrac{1}{2}\kappa(T-t) \\ & + \frac{\left(\kappa^2\theta^2\gamma_1^2 - \gamma_3^2\right)}{2\sigma^2\gamma_1^3}\left(\frac{\sinh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}} - \gamma_1(T-t)\right) \\ & + \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\,\gamma_3}{\sigma^2\gamma_1^3}\left(\frac{\cosh\{\gamma_1(T-t)\} - 1}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}}\right) \end{aligned}$$

with

$$\gamma_1 = \sqrt{2\sigma^2 s_1 + \kappa^2}, \quad \gamma_2 = \frac{1}{\gamma_1}\left(\kappa - 2\sigma^2 s_3\right), \quad \gamma_3 = \kappa^2\theta - s_2\sigma^2.$$

Using the time dependent functions $D(t,T)$, $B(t,T)$ and $C(t,T)$, we obtain closed-form solutions for $f_j(\phi)$.

Appendix B: Boundary Behavior of an OU Process

In this appendix, we prove that (i) for arbitrary values of the long-term mean $\theta \neq 0$, the absolute value OU process, the reflected OU process and the unrestricted OU process are substantially different and the absolute value OU process does not satisfy the Kolmogorov forward equation; however, (ii) for the special (symmetric) case $\theta = 0$ the absolute value process coincides with the reflecting barrier process and does satisfy the Kolmogorov forward equation; (iii) the density function derived by applying a similar backward equation as in S&S and in our model is identical to the one for the unrestricted OU process and does not satisfy the boundary condition which is necessary for an absolute value process.

Suppose that volatility follows a mean-reverting process

$$\mathrm{d}v = \kappa\,(\theta - v)\,\mathrm{d}t + \sigma\,\mathrm{d}w. \tag{B1}$$

If $q(v, t; u, T)$ is the transition density of a diffusion process from state (v, t) to state (u, T), then this transition density for the unrestricted OU process reads

$$q\,(v, t; u, T) = \varphi\left(u - ve^{-\kappa(T-t)} - \theta\left(1 - e^{-\kappa(T-t)}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right) \tag{B2}$$

where φ is the Gaussian kernel

$$\varphi\,(z, \tau) = \frac{1}{\sqrt{2\pi\tau}}\exp\left\{-\frac{z^2}{2\tau}\right\}. \tag{B3}$$

The Kolmogorov forward equation satisfied by q is accordingly

$$\frac{1}{2}\sigma^2\frac{\partial^2 q}{\partial u^2} - \frac{\partial}{\partial u}\{\kappa\,(\theta - u)\,q\} + \frac{\partial q}{\partial t} = 0. \tag{B4}$$

If a reflecting barrier at zero is assumed, the transition density must also satisfy the reflecting boundary condition at zero:

$$\left[\frac{1}{2}\sigma^2\frac{\partial q}{\partial u} - \kappa\,(\theta - u)\,q\right]_{u=0} = 0. \tag{B5}$$

See, i.e. Cox and Miller (1965, p. 223). Obviously, for the density (B2) of the unrestricted process this is not the case. As shown in Ball and Roma (1994), the corresponding density $q^A(v, t; u, T)$ for the absolute value process is given by

$$q^A(v, t; u, T) = q(v, t; u, T) + q(v, t; -u, T), \tag{B6}$$

with $v, u \geqq 0$. Differentiating (B6) by the forward variable u yields

$$\left[\frac{\partial q(v, t; u, T)}{\partial u} - \frac{\partial q(v, t; -u, T)}{\partial u}\right]_{u=0} = \left[\frac{\partial q^A}{\partial u}\right]_{u=0} = 0. \tag{B7}$$

This means that the first derivative of the density function q^A with respect to the forward variable u at the origin equates zero. Hence, (B7) is a necessary (not a sufficient) boundary condition that a density for an absolute value process should satisfy, and differs from the boundary condition (B5) for the reflected process. However, albeit q^A satisfies the necessary condition (B7) it does no longer satisfy the Kolmogorov forward equation (B4) for any $\theta \neq 0$. In fact, while the first density $q(v,t;u,T)$ satisfies (B4) for any θ, the second density $q(v,t;-u,T)$ does not. To see this, we calculate the partial derivatives for both terms $q(v,t;\pm u,T) = \varphi(z,\tau)$ of the density q^A

$$\frac{\partial q}{\partial u} = \varphi_z z_u \quad \frac{\partial^2 q}{\partial u^2} = \varphi_{zz} z_u^2 + \varphi_z z_{uu} \quad \frac{\partial q}{\partial t} = \varphi_\tau \tau_t + \varphi_z z_t \tag{B8}$$

using transformed variables z and τ

$$\begin{aligned} z &= \pm u - v e^{-\kappa(T-t)} - \theta\left(1 - e^{-\kappa(T-t)}\right) & \tau &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right) \\ z_u &= sign(u) & \tau_t &= -\sigma^2 e^{-2\kappa(T-t)} \\ z_{uu} &= 0 & z_t &= -\kappa v e^{-\kappa(T-t)} + \theta\kappa e^{-\kappa(T-t)}. \end{aligned}$$

Hence, putting (B8) into (B4) leads to the following expression

$$\begin{aligned} &\frac{1}{2}\sigma^2\varphi_{zz} + \kappa\left(\pm u - v e^{-\kappa(T-t)} \mp \theta + \theta e^{-\kappa(T-t)}\right)\varphi_z + \kappa\varphi - \varphi_\tau \sigma^2 e^{-2\kappa(T-t)} \\ &= \frac{1}{2}\sigma^2\varphi_{zz} + \kappa\begin{pmatrix} z \\ z + 2\theta \end{pmatrix}\varphi_z + \kappa\varphi - \varphi_\tau\sigma^2 e^{-2\kappa(T-t)} \\ &= \begin{pmatrix} 0 \\ 2\kappa\theta\varphi_z \end{pmatrix}, \end{aligned}$$

which means that q^A does not satisfy the forward equation (B4) which is valid for the unrestricted as well as for the reflected OU process. However, for the symmetric case $\theta = 0$ we can proceed with $z = \pm u - v e^{-\kappa(T-t)}$ to get the identity

$$\tfrac{1}{2}\sigma^2\varphi_{zz} + \kappa z\varphi_z + \kappa\varphi - \varphi_\tau\sigma^2 e^{-2\kappa(T-t)} = \kappa\tau\varphi_{zz} + \kappa z\varphi_z + \kappa\varphi = 0,$$

where we have used the fact that the Gaussian kernel $\varphi(z,\tau)$ satisfies the standard diffusion equation $\frac{1}{2}\varphi_{zz} - \varphi_\tau = 0$. Hence, for the special case $\theta = 0$, q^A does satisfy (B4). Additionally, since the conditions (B5) and (B7) are identical for $\theta = 0$ at the boundary $u = 0$, it is also proved that the density of the reflecting OU process and the density of the absolute value OU process are identical for $\theta = 0$.

Summing up, we arrive at the following conclusions: For the symmetric case $\theta = 0$, imposing a reflecting barrier at $v = 0$ is equivalent to dealing with the absolute value $|v|$ of the mean-reverting process. For $\theta \neq 0$, the transition density q^A of the absolute value process does not satisfy the forward equation of an OU process.

So far, we only worked with the forward equation since the boundary conditions for these two restricted processes are associated with the forward variable u. To verify that the S&S model and our extended model do not work with $|v(t)|$ or with $v^2(t)$, respectively, we only need to show that the density function implied in the S&S model and our extended model does not generally satisfy (B6) and (B7). To see why, we consider a function

$$\widehat{y}(v,t,T) = \mathbb{E}^Q\left[\exp\left\{-s_1\int_t^T v^2(\zeta)\,\mathrm{d}\zeta - s_2\int_t^T v(\zeta)\,\mathrm{d}\zeta + s_3 v^2(T) + s_4 v(T)\right\}\right], \tag{B9}$$

obviously, the functions $y(v,t,T)$ in Appendix A as well as $I(\lambda t)$ in S&S (see (B3) in S&S, 1991) are two special cases of this function. To compute the function $\widehat{y}(v,t,T)$, we have to use the backward equation as done in this paper as well as in S&S. To demonstrate the procedure just for the one dimensional case under consideration in this appendix, we set $s_1 = s_2 = s_3 = 0$, $s_4 = i\phi$, and obtain $\widehat{y}(v,t,T) = g(\phi) = \mathbb{E}^Q[i\phi v(T)]$ as the CF of $v(T)$. It is not difficult to get the solution for $g(\phi)$ which takes the following form

$$g(\phi) = \exp\left\{A^*(t,T)v(t) + B^*(t,T)\right\} \tag{B10}$$

with

$$A^*(t,T) = i\phi e^{-\kappa(T-t)}$$

$$B^*(t,T) = \frac{\sigma^2\phi^2}{4\kappa}(1 - e^{-2\kappa(T-t)}) - i\phi\theta(1 - e^{-\kappa(T-t)}).$$

The transition density $q^*(v,t;u,T)$ derived via inverse Fourier transformation from $g(\phi)$ is exactly identical to $q(v,t;u,T)$ in (B2). Consequently, neither (B6) nor (B7) are satisfied by $q^*(v,t;u,T)$. The method to derive $q^*(v,t;u,T)$ is identical to the method used by S&S and us previously. (B9) and (B10) serve to illustrate that the methods to derive functions $g(\phi)$, $y(v,t,T)$ in Appendix A as well as $I(\lambda t)$ in S&S coincide with each other. Hence, we conclude once again that S&S and we do not work with an absolute value OU process, but with an unrestricted OU process.

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